Set 7—due 18 October

The midterm will be Monday, October 21, 7-830 PM, in G-130.

"The best discoveries always seem to be made in the small hours of the morning, when most people are asleep, where there are no disturbances and the mind becomes most contemplative. You’re out in a lonely spot somewhere, looking at the numbers on reams of paper spewing out of a computer. You look and look, and suddenly you see some numbers that aren’t like the rest—a spike in the data. You apply some statistical tests and look for errors but no matter what you do, the spike’s still there. It’s real. You’ve found something. There’s just no feeling like it in the world." (Leon Lederman, 1922-2018)

1) Jackson 4.9. [20 points] (a)–10 points, (b)–4 points, (c)–6 points.

2) Jackson 4.13 [15 points]

3) Jackson 4.10 [10 points].

4) [15 points] A sphere of radius $R$ is made of a uniaxial dielectric with dielectric constant $\varepsilon$ along two principal axes and dielectric constant $\varepsilon_3$ along the third. The sphere is placed in a uniform electric field, with arbitrary orientation. Find the torque on the sphere.
4.9) the Point charge outside a dielectric sphere.

Put the charge on the z-axis for simplicity, so

\[ \frac{1}{\sqrt{x^2+y^2}} = \sum_{\ell=0}^{\infty} \frac{R^\ell}{\ell!} \frac{P\ell(\cos\theta)}{e^{\frac{r}{l+1}}} \]

where \( \vec{x}_1 = r \cos\theta \), \( \vec{x}_1 = \vec{r}' \) (\( = d \))

Add a homogeneous solution, to account for the dielectric sphere. Then

\[ \begin{align*}
 r < R & \quad \Phi_{\text{inside}} = \frac{1}{4\pi\varepsilon_0} \sum_{\ell} A_{\ell} R^\ell \frac{P\ell(\cos\theta)}{e^{\frac{r}{l+1}}} \\
 r > R & \quad \Phi_{\text{outside}} = \frac{1}{4\pi\varepsilon_0} \sum_{\ell} \left[ B_{\ell} R^\ell e^{\frac{r}{l+1}} - \frac{B_{\ell} R^\ell}{e^{l+1}} \right] \frac{P\ell(\cos\theta)}{e^{\frac{r}{l+1}}} 
\end{align*} \]

The boundary conditions at \( r = R \) are

\[ \Phi_{\text{in}} = \Phi_{\text{out}} \Rightarrow A_{\ell} R^\ell = \frac{B_{\ell} R^\ell e^{\frac{R}{l+1}}}{e^{l+1}} + \frac{B_{\ell} R^\ell}{e^{l+1}} \]

\( \vec{D} \cdot \vec{n} = \text{continua}: \quad \varepsilon \frac{\partial \Phi}{\partial r} \bigg|_{r=R} = \varepsilon_0 \frac{\partial \Phi}{\partial r} \bigg|_{r=R} \)

\[ \begin{align*}
 &\varepsilon \ell A_{\ell} R^{\ell-1} = \varepsilon_0 \left[ \ell \frac{B_{\ell} R^\ell}{e^{l+1}} - (l+1) \frac{B_{\ell} R^\ell}{e^{l+2}} \right] \\
 &\varepsilon \frac{\ell}{\varepsilon_0} A_{\ell} = \ell \frac{B_{\ell}}{e^{l+1}} - (l+1) \frac{B_{\ell} R^\ell}{R^{2l+1}} \\
 &\varepsilon \frac{\ell}{\varepsilon_0} \left[ \frac{B_{\ell}}{R^{2l+1}} + \frac{\ell}{e^{l+1}} \right] = \ell \frac{B_{\ell}}{e^{l+1}} - (l+1) \frac{B_{\ell} R^\ell}{R^{2l+1}} \\
 &\frac{B_{\ell}}{R^{2l+1}} \left[ \frac{\ell}{\varepsilon_0} + l + 1 \right] = \frac{B_{\ell}}{e^{l+1}} \left( 1 - \frac{\varepsilon}{\varepsilon_0} \right) 
\end{align*} \]
\[
B_e = -\frac{\varepsilon}{\varepsilon_0} e_l R^{2e+1} \frac{\left( \frac{\varepsilon}{\varepsilon_0} - 1 \right)}{l + \varepsilon \left( \frac{\varepsilon}{\varepsilon_0} + 1 \right)}
\]

\[
A_e = \frac{B}{e_l^{e+1}} \left[ 1 - \varepsilon \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \right] = \frac{B}{e_l^{e+1}} \frac{2e+1}{1 + \varepsilon \left( \frac{\varepsilon}{\varepsilon_0} + 1 \right)}
\]

b) near \( r = 0 \)
\[
\Phi = A_1 \ \text{rad} \theta = A_1 \frac{z}{4\pi 6_0} \Rightarrow E = -\frac{A_1}{2} E_2
\]
with \( E_2 = \frac{-A_1}{4\pi 6_0} = \frac{-3B}{4\pi 6_0} \left( \frac{1}{2 + \varepsilon 6_0} \right) \) from part (a)

\[c) \text{as} \ \varepsilon \to \infty \ \ A_e \to 0 \quad B_e \to -\varepsilon \frac{R^{2e+1}}{e_l^{e+1}} \quad \text{if} \ l \neq 0,\]
and if \( l = 0 \) \( A_0 = \frac{B}{e_l} \) and \( B_0 = 0 \). In that case
\[
\Phi_{\text{in}} = \frac{1}{4\pi 6_0} \frac{B}{e_l} > \Phi_{\text{out}} = \frac{1}{4\pi 6_0} \left[ \frac{B}{l^{e+1}} - \sum_{e=1}^{\infty} \frac{R^{2e+1}}{e_l^{e+1} \ l^{e+1}} \right]
\]
which we can re-write (adding and subtracting the \( l = 0 \) term)
\[
\Phi_{\text{out}} = \frac{B}{4\pi 6_0} \left[ \frac{1}{l^{e+1}} - \frac{R}{e_l} \sum_{e=0}^{\infty} \frac{(R^2 \ l^e)}{e \ l^e} \right]
\]
The middle term is an image charge \( \Phi' = -\frac{B R}{e_l} \)
located at \( y' = R^2 / l \). The last term is an image charge at the origin, which maintains the sphere's neutrality,
\[
\Phi = \frac{1}{4\pi 6_0} \left[ \frac{B}{l^{e+1}} - \frac{B}{e_l} \frac{R}{l^{e+1}} \frac{1}{l^{e+1}} + \frac{B R}{e_l} \frac{1}{l^{e+1}} \right]
\]
Compute the energy $W$ stored in the capacitor when the height of the oil is $h$. The force of attraction will be

$$F_e = -\frac{\partial W}{\partial h}$$

This is balanced by gravity. The gravitational potential energy associated with the height of the column is $U$, and $F_g = -\frac{\partial U}{\partial h}$. $F_g = F_e$ gives $h$.

First, gravity. To raise the column of oil from $z$ to $z+dz$ costs $dU = [\pi \rho (b^2-a^2)z] g dz$, where $\rho$ is the density, the mass of oil is density times volume,

$$U = \int_0^h dU = \pi \rho (b^2-a^2) g \frac{h^2}{2}$$

$$\frac{\partial U}{\partial h} = \pi \rho (b^2-a^2) \rho g h.$$ 

Then electrostatics. $W = \frac{1}{2} CV^2$ so we need $C$ vs $h$.

Gauss' law gives the (radial) $D = \frac{\lambda}{2\pi r}$ for $J = \text{I or II}$, $\lambda$: charge $/m$ for $\text{II}$. Above the oil $E_I = \frac{\lambda}{2\pi \varepsilon_0 r}$. Below the oil $E_{\Pi} = \frac{\lambda}{2\pi \varepsilon_0 r}$. $E_{\text{III}}$ is continuing, and there are $E_{\text{IV}}$ etc, so $E_I = E_{\Pi} \approx \frac{\lambda}{\varepsilon_0}$
\( \vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E} \rightarrow 20 \)

\( 1 + \chi = \varepsilon / \varepsilon_0 \) and \( \frac{\lambda_I}{\lambda_{II}} = \frac{1}{1 + \chi} \)

The total \( \Phi = \lambda_I (l - h) + \lambda_{II} h \)

\[ = \lambda_I \left[ (l - h) + (1 + \chi) h \right] = \lambda_I (l + \chi h) \]

\[ \therefore \lambda_I = \frac{\Phi}{l + \chi h} \]

Integrate \( E \rightarrow V_I = \frac{\lambda_I}{2 \pi \varepsilon_0} \ln \frac{b}{a} = \frac{\Phi}{2 \pi \varepsilon_0} \frac{1}{[l + \chi h]} \ln \frac{b}{a} \]

This equals \( V_{II} \). And \( CV = \Phi \) so \( C = \frac{\Phi}{V} \)

\[ C = \frac{2 \pi \varepsilon_0 (l + \chi h)}{\ln b/a} \]

\[ W = \frac{1}{2} CV^2 \rightarrow F_e = \frac{1}{2} V^2 \frac{\partial C}{\partial h} = \frac{\pi \varepsilon_0 b X V^2}{\ln b/a} \]

Putting it all together

\[ \frac{\pi \varepsilon_0 X V^2}{\ln b/a} = \pi (b^2 - a^2) \varepsilon g h \]

\[ X = \left[ (b^2 - a^2) \ln \frac{b}{a} \right] \frac{\varepsilon g h}{\varepsilon_0 V^2} \]
\[ E \] is radial because \( E \propto \frac{1}{r^2} \),

\[ D_I : \varepsilon_0 E_I = E \]
\[ D_{II} = \varepsilon \varepsilon_{II} = \varepsilon E \]

Note \( D_I \) is discontinuous on the boundary,
but \( \nabla \cdot \hat{n} \) boundary = 0 on both sides...

Call \( E(r = a) = E \). The free charge is
\[ \sigma_F = \hat{D} \cdot \hat{n} = D \]
\[ \sigma_F^I = D_I = \varepsilon_0 E \]
\[ \sigma_F^{II} = D_{II} = \varepsilon E \]

In region II there is a polarization charge
\[ D_{II} = \varepsilon_0 E_{II} + P_{II} \]
\[ P_{II} = (\varepsilon - \varepsilon_0) E \]

The bound or polarization charge is \( \sigma_b = \hat{P} \cdot \hat{n} \).
\( \hat{n} \) points away from the dielectric, which is \( \hat{n} = -\hat{r} \) on the inner sphere. Here
\[ \sigma_b = -(\varepsilon - \varepsilon_0) E(a) \]

Because \( E \) is radial, \( \sigma \) is uniform over
the sphere. So
\[ \sigma_F^I = \sigma_F^{II} + \sigma_b \] at \( r = a \).

The total free charge at \( r = a \)
\[ Q = 2\pi a^2 \left( \sigma_F^I + \sigma_F^{II} \right) = 2\pi a^2 (\varepsilon_0 + \varepsilon) E(a) \]

\[ E = \frac{Q}{2\pi a^2} \frac{1}{\varepsilon_0 + \varepsilon} \]
\[ \sigma_b = - \left( \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} \right) \frac{Q}{2\pi a^2} = \sigma_{II}^b \]

\[ \sigma_F^I = \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon} \frac{Q}{2\pi a^2} \]
\[ \sigma_F^{II} = \frac{\varepsilon}{\varepsilon_0 + \varepsilon} \frac{Q}{2\pi a^2} \]
\[ \sigma_F^I + \sigma_b = \sigma_F^{II} \]
4) A dielectric sphere in an external $E$ field.

For an isotropic dielectric, $E_{in} = \frac{3}{K+2} E_{out}$

$$\vec{D}_{in} = \frac{3e}{K+2} E_{out}, \quad K = \frac{\varepsilon}{\varepsilon_0}.$$

Choose your coordinates so the unique axis is aligned along $z$ (so $E_z = E_3$). Let the external $E$ field lie in the $x-z$ plane, and use superposition to consider each component separately, so $K_z = \varepsilon_0/\varepsilon_0^z$, $K_z = E_z/\varepsilon_0$.

$\vec{E}_{in} = (E_x, E_y, E_z)$

$$\vec{E}_{in} = \left( \frac{3}{K+2} E_0 \sin \theta, 0, \frac{3}{K+2} E_0 \cos \theta \right).$$

$$\vec{D}_{in} = (D_x, D_y, D_z) = \left( \frac{3e}{K+2} E_0 \sin \theta, 0, \frac{3e}{K+2} E_0 \cos \theta \right).$$

The polarization $\vec{P}$ is the dipole moment $\vec{p}^2$ per unit volume, so $\vec{p}^2 = \left( \frac{4\pi R^3}{3} \right) \vec{P}$ and $\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$.

Thus $\vec{p}^2 = \frac{4\pi R^3}{3} \varepsilon_0 \left[ 3 \left( \frac{K-1}{K+2} \right) \sin \theta \cos \theta, 3 \left( \frac{K_z-1}{K_z+2} \right) \cos \theta \right].$

The torque is $\vec{\tau} = \vec{p} \times \vec{E} = \left| \begin{array}{ccc} \varepsilon_0 & 0 & p_z \\ 0 & \varepsilon_0 & p_x \\ p_y & p_z & \varepsilon_0 \end{array} \right|$

$$\vec{\tau} = \int 4\pi R^3 \varepsilon_0 \sin \theta \cos \theta E_0^2 \left[ \frac{K_z-1}{K_z+2} - \frac{K-1}{K+2} \right].$$

Sanity check: this is zero if $K_z = K_z^0$ or $\theta = 0^\circ$.

or $\theta = 90^\circ$—think about it!