"I don’t see why you are talking about this problem when either of you is capable of sitting down and solving it." – H. Bethe, to his students

Several problems involving images.

1) Jackson 2.1 [15 points] This should just be a review... (a)-3, (b)-2 (c)-3, (d)-2, (e)-2, (f)-3

2) Jackson 2.7 [20 points] Not quite identical to the previous one. (a)-3, (b)-3, (c)-4, (d)-10

3) Jackson 2.11 [20 points] Do this one before trying the next one! (a+b)-12, (c)-5, (d) 3.

4) Jackson 2.8 parts a-b only [20 points]. I found it hard to get the answer quoted in the text. Here are two potentially useful hints:
   Hint 1: If $x = \ln z$ and $\cosh^{-1} y = x$, then $y = (z + 1/z)/2$.
   Hint 2: You probably have two equations giving $d$ as a function of $a$, $b$, and the offsets of the image wires from the centers of the cylinders (call them $d_1$ and $d_2$). Try multiplying them together to give $d^2$. If you don’t understand this, and can work the problem in some other way, ignore the hint! (The reason why Jackson’s answer for the capacitance is desirable is that it only depends on simple parameters, the radii of the two cylinders and their separation.)
2.1 The iconic image problem - point charge

\[ E^2 = x^2 + y^2 \]

**Image Charge**

\[ \Phi = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{1}{\sqrt{E^2 + (z - d^2)^2}} - \frac{1}{\sqrt{E^2 + (z + d)^2}} \right\} \]

a) \[ \sigma = -\frac{q}{4\pi \varepsilon_0} d^2 \left\{ \frac{z - d}{\sqrt{E^2 + (z - d)^2}}^{3/2} - \frac{z + d}{\sqrt{E^2 + (z + d)^2}}^{3/2} \right\} \]

\[ \sigma = -\frac{\varepsilon_0}{2\pi} \left[ \frac{1}{E^2 + d^2} \right]^{3/2} \]

b) \[ F = -\frac{q^2}{4\pi \varepsilon_0 (2\pi d)^2} \text{ from the image - attractive, naturally} \]

c) \[ F = \frac{q}{2\pi \varepsilon_0} \cdot 2\pi \varepsilon d \frac{d}{2\pi} = \frac{q^2}{2\pi \varepsilon_0} \frac{d}{\pi} \int_0^\infty \frac{e^{de}}{[E^2 + d^2]^2} \]

\[ = \left( \frac{q^2 d^2}{4\pi \varepsilon_0} \right) \times \frac{1}{4d^4} = \frac{q^2}{4\pi \varepsilon_0 (2\pi d)^2} \]

This is the force on the plane, equal and opposite to the result in (b), as expected.
d) Because the force is attractive, positive work must be done to take the point charge to infinity:

\[ W = \frac{1}{4\pi \varepsilon_0} \int \frac{q^2}{4\pi \varepsilon_0} \, dx = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{4d} \]

e) Potential energy \( U = \frac{q^2}{4\pi \varepsilon_0} \frac{1}{2d} \) between the charge and the image. The work to remove the charge is not \( q^2/8\pi \varepsilon_0 d^2 \) because the image moves for free. (d) is correct!

f) \( W = \frac{q^2}{4\pi \varepsilon_0} \frac{1}{4d} \) so \( \frac{q^2}{4\pi \varepsilon_0} \frac{1}{4d} \) in Volts gives

in eV. In MKS this is

\[ \left[ 1.6 \times 10^{-19} \text{ C} \right] \left[ \frac{9 \times 10^9}{4\pi \varepsilon_0} \times 10^{-7} \right] \]

\[ \frac{4 \times 10^{-19} \text{ m}^2}{4 \times 10^{-10} \text{ m}} = \frac{9 \times 1.6}{4} \text{ eV} = 3.6 \text{ eV} \]

In CGS it is \( \frac{1}{4\pi} \frac{e^2}{\hbar c} \)

\[ \frac{1}{4\pi} \times \frac{1}{137} \times 1970 \text{ eV} - 9 = 3.6 \text{ eV} \] (again!)
2) Jackson 2.7 is different from 2.1 because $\Phi_0$ specified on the conducting surface. This is an
first Dirichlet problem. In 2.1 $\Phi = 0$ on the boundary so $\Phi = G$, images gave us $G$. Here we need
Green's theorem.

a) $G(x, x') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

b) $\Phi = \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial n^2} = -\frac{\partial G}{\partial n}$

$\Phi(x) = -\frac{1}{4\pi} \int \frac{\Phi(x') \partial G}{\partial n'} \, dA'$

Using prob 1b, this is

$\Phi = \frac{\nabla \Phi}{4\pi} \int_0^{2\pi} d\phi' \int_0^a e'^3 \, d\phi' \cdot \frac{2\pi}{\sqrt{R^2 + z^2}} \frac{3}{2}$. 

R^2 = c^2 + c'^2 - 2cc' \cos \Phi_0'$, setting $\Phi = 0$
c) If \( e = 0 \), the integral is elementary

\[
\Phi(2, e = 0) = \frac{\sqrt{2}}{4\pi} \int_0^a \frac{e'de'}{[e'^2 + a^2]^{3/2}}
\]

\[
= \sqrt{1 - \frac{2}{\sqrt{e^2 + a^2}}}
\]

d) looks evil! Try expanding in \((e^2 + a^2)\)...

\[
e' \]

\[
\left[ e^{12} + e^{12} - 2ee' e^{12} \cos q' + a^2 \right]^{3/2}
\]

\[
= \frac{e'}{[e^2 + a^2]^{3/2}} \left[ 1 + \frac{e'^{12}}{e^2 + a^2} \left( 1 - 2e' \cos q' \right) \right]^{3/2}
\]

\[
= \frac{e'}{(e^2 + a^2)^{3/2}} \left[ 1 - \frac{3}{2} \frac{e'^{12}}{e^2 + a^2} \left( 1 - 2e' \cos q' \right) \right.
+ \frac{15}{8} \left( \frac{e'^{12}}{e^2 + a^2} \right)^2 \left[ 1 - 2e' \cos q' \right]^2 + \ldots
\]

This looks nasty, but the \( \cos q' \) terms integrate to zero and \( \langle \cos^2 q' \rangle = \frac{1}{2} \) so

\[
\Phi(2, e) = \frac{2\sqrt{2}}{4\pi} \left\{ \int_0^a \frac{e'de'}{[e^2 + a^2]^{3/2}}
\right.
\]

\[
- \frac{3}{2} \int_0^a \frac{e'^3 de'}{[e^2 + a^2]^{5/2}} + \frac{15}{8} \int_0^a \frac{e'^5 de'}{[e^2 + a^2]^{7/2}} \left( 1 + \frac{4e'^2}{e^2} \right) + \ldots
\]

courage!
\[ \vec{\Omega}(3, \theta) = \frac{\sqrt{2}}{(2\pi)^{3/2}} \left\{ \frac{a^2}{2} - \frac{3}{2 \cdot 4} \frac{a^4}{(e^2+2^2)^{3/2}} + \frac{15}{8} \frac{1}{(e^2+2^2)^2} \left( \frac{a^6}{6} + 2a^4e^2 \right) + \ldots \right\} \]

\[ = \frac{\sqrt{a^2e^2}}{2 \left[ (e^2+2^2)^{3/2} \right]} \left\{ 1 - \frac{3}{4} \frac{a^2}{e^2+2^2} + \frac{5}{8} \frac{a^4 + 2a^2e^2}{(e^2+2^2)^2} \right\} \]

This looks awful—we will develop techniques to do better in a few days.

In the meantime, to compare with (c), set \( e = 0 \)

\[ \vec{\Omega}(3, 0) = \frac{\sqrt{a}}{2} \left( \frac{a}{2} \right)^2 \left\{ 1 - \frac{3}{4} \frac{a^2}{2^2} + \frac{5}{8} \left( \frac{a}{2} \right)^4 \right\} \]

which is the Taylor expansion \( \sqrt{1 - \frac{1}{\sqrt{1 + \left( \frac{a}{2} \right)^2}}} \)

\[ \sqrt{\left[ 1 - \frac{1}{\sqrt{1 + \left( \frac{a}{2} \right)^2}} \right]} = \sqrt{\left[ \frac{1}{2} \left( \frac{a}{2} \right)^2 \right.} \]

\[ - \frac{3}{8} \left( \frac{a}{2} \right)^4 \]

\[ + \frac{5}{16} \left( \frac{a}{2} \right)^6 + \ldots \]

2.11) Line charge and conducting cylinder. \( r \) observe at \((r, \theta)\). Image at \( r_+\).

A distance \( r \) from a line charge,
\[
\Phi = -\frac{\kappa}{2\pi \epsilon_0} \ln r + \phi, \quad \text{where} \quad \phi \text{ is a constant.}
\]

To force \( \Phi \to 0 \) as \( r \to \infty \), we hide an image line charge \( \phi \) a distance \( d \) from the origin, so
\[
\Phi(r, \theta) = \frac{\kappa}{2\pi \epsilon_0} \ln \frac{r}{r_+} \quad \text{(adding the true + image charges)}
\]

To find \( d \), note \( \Phi \) constant at \( r = b \), on the cylinder.
This means \( \frac{r}{r_+} = \text{constant.} \) Here: call it \( k = \frac{b}{r} \).

\[
b = \sqrt{d^2 - 2bd \cos \theta} \]
\[
= k^2 \left[ b^2 + R^2 - 2bd \cos \theta \right] \quad \text{(law of triangles)}
\]

\[
\Rightarrow b^2 + d^2 - k^2 (b^2 + R^2) = 0 - 2bd \cos \theta = 0.
\]

In order that the coefficient of \( \cos \theta \) vanish, \( k^2 = \frac{d}{R} \).

Then
\[
b^2 + d^2 - k^2 (b^2 + R^2) = 0 = b^2 + d^2 - \frac{d}{R} (b^2 + R^2)
\]
\[
(d - R) (d - \frac{b^2}{R}) = 0 \quad \text{(look at it!)}
\]

so \( \left[ \text{part (a)}: \quad d = \frac{b^2}{R} \right] \)
b) \( \Phi (r, \theta) = \frac{c}{4\pi \varepsilon_0} \ln \frac{r^2 + \left( \frac{b}{R} \right)^2 - 2rb \cos \theta}{r^2 + R^2 - 2rR \cos \theta} \)

\[ \Phi (r, \theta) = \frac{c}{4\pi \varepsilon_0} \ln \frac{1 - \frac{b^2}{R^2} \cos \theta + \left( \frac{b}{rR} \right)^2}{1 - \frac{b^2}{R^2} \cos \theta + \frac{R^2}{r^2}} \]

As \( r \to \infty \), \( r \theta \to \infty \)

\( \Phi (r, \theta) \to \frac{c}{4\pi \varepsilon_0} \ln \left\{ 1 + \frac{2}{r} \cos \theta \left( R - \frac{b^2}{R^2} \right) + \ldots \right\} \)

\[ \Phi (r, \theta) \approx \frac{c}{2\pi \varepsilon_0} \frac{1}{r} \left( R - \frac{b^2}{R^2} \right) \cos \theta + \ldots \]

There must be an easier path to this nice answer!

c) \( \sigma = -\varepsilon_0 \left( \frac{\partial \Phi}{\partial r} \right) \bigg|_{r=b} \)

\[ \sigma \bigg|_{r=b} = -\frac{c}{4\pi} \left\{ \frac{2r - 2b \cos \theta}{r^2 + b^2 - 2rb \cos \theta} - \frac{2r - 2b^2}{r^2 + \left( \frac{b^2}{R^2} \right)^2 - 2r \left( \frac{b^2}{R^2} \right) \cos \theta} \right\} \]

\[ \sigma \bigg|_{r=b} = \frac{c}{2\pi} \left\{ \frac{2b - 2b \cos \theta}{b^2 + R^2 - 2bR \cos \theta} - \frac{2b - 2b^2}{b^2 + \left( \frac{b^2}{R^2} \right)^2 - 2b \left( \frac{b^2}{R^2} \right) \cos \theta} \right\} \]

\[ \sigma \bigg|_{r=b} = \frac{c}{4\pi b} \left\{ \frac{2b - 2b \cos \theta}{b^2 + R^2 - 2bR \cos \theta} - \frac{2b - 2b^2}{b^2 + \left( \frac{b^2}{R^2} \right)^2 - 2b \left( \frac{b^2}{R^2} \right) \cos \theta} \right\} \]

\[ \sigma \bigg|_{r=b} = \frac{c}{2\pi b} \left( \frac{b^2 - R^2}{b^2 + R^2 - 2bR \cos \theta} \right) \]
Writing \( x = \frac{\text{radius of cylinder}}{\text{distance to charge}} = \frac{b}{R} \)

\[ \theta = -\frac{2}{2n} \cdot \frac{1-x^2}{1-2x \cos \theta + x^2} \]

\[ \sigma \leq \frac{\frac{\pi}{2n}}{2 \pi b} \]

The smaller \( x \) is, the less charge piles up on one side.

a) From Gauss' law, for a line charge

\[ \vec{E} = -\frac{\rho}{2 \pi \varepsilon_0} \frac{1}{r} \hat{r} \]

So

\[ \frac{F}{l} = -\frac{\rho^2}{2 \pi \varepsilon_0} \frac{1}{R-d} \]

and

\[ d = \frac{b^2}{\pi R} \]
4) Jackson 2.8. Part (a) is at the end, it 2.8.1 is identical to what we did in 2.11

Gauss' Law says that the charge on the wire is the charge on the cylinder. The surfaces of the 2 cylinders are equipotentials, so the ratio $r_1/r_0$ is a constant going around the cylinder. Work this out for the left hand cylinder. The law of cosines says

$$r_1^2 = x^2 + a^2 - 2ax \cos \theta$$  \(\text{Eq. 1}\)

$$r_0^2 = d_1^2 + a^2 - 2a d_1 \cos \theta$$  \(\text{Eq. 2}\)

Look at (1) and (2). If the cylinder is an equipotential, the ratio of cosine terms in (1) and (2) must be equal to the ratio of non cosine terms. Then

3) \(\frac{x^2 + a^2}{d_1^2 + a^2} = \frac{x_1}{d_1}\)  or 4) \(x d_1 = a^2\)

Check: \(\frac{x^2 + a^2}{\left(\frac{a^2}{x}\right)^2 + a^2} = \frac{x^2}{a^2} = \frac{x^2}{a^2} \left(\frac{x^2 + a^2}{x^2 + a^2}\right) = \frac{1}{a^2} (x^2 + a^2)\) or recall 2.11 - Eqs. 4 say the surface radius is the geometric mean of the separators. In the original figure, then

$$d_1 (d - d_2) = a^2 \text{ or } d_2 (d - d_2) = b^2$$  \(\text{Eq. 5}\)
The potentials on the two surfaces are

\[ \phi_1 = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r(1)}{r'(1)} \quad \text{and} \quad \phi_2 = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r(2)}{r'(2)} \]

\[ V = \phi_2 - \phi_1 \] is the voltage difference between the two cylinders and the capacitance is \( C = \frac{\varepsilon_0}{\lambda} V \).

On each cylinder, \( r'/r \) is a constant.

From (1), (2), (3), (4), on the LH cylinder,

\[ \frac{r^2}{r'^2} = \frac{a^2}{x^2} \quad \text{and} \quad x = d - d' \]

so

\[ \phi_1 = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r(1)}{r'(1)} = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{a}{d-d'} = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{d}{a} \]

\[ \phi_2 = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r(2)}{r'(2)} = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b}{d-d'} = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{d}{d} \]

\[ V = \phi_2 - \phi_1 = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{d'd}{ab} \]

\[ C = \frac{2\pi\varepsilon_0}{\ln \left( \frac{d'd}{ab} \right)} \]

Now for the hints. Call \( \ln \frac{d'd}{ab} = x = \cosh y \)

\[ y = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \left( \frac{d'd}{ab} + \frac{ab}{d'd} \right) \]

Write \( d^2 = dx \cdot dx \) (?), use (5): \( d = \frac{x^2}{d'}, \) and \( d = \frac{b^2}{a^2} + d' \)
\[ d^2 = \left( \frac{a^2 + d_1d_2}{d_1} \right) \left( \frac{b^2 + d_1d_2}{d_2} \right) \]

\[ = \frac{a^2b^2}{d_1d_2} + a^2 + b^2 + d_1d_2 \]

\[ d^2 - a^2 - b^2 = \frac{a^2b^2}{d_1d_2} + d_1d_2 = ab \left[ \frac{a b + d_1d_2}{d_1d_2} \right] \]

\[ \alpha^2 = \frac{d^2 - a^2 - b^2}{2ab} = \frac{1}{2} \left[ \frac{d_1d_2 + ab}{ab} \right] \]

which gives the beautiful though improbable answer

\[ C = \frac{2\pi\alpha^2}{\cosh^{-1} \left[ \frac{d^2-a^2-b^2}{2ab} \right]} \]

It is also useful - \( d \), \( a \), and \( b \) are easy to measure.

\[ \begin{array}{c}
  a \\
  \text{ } \\
  \text{ } \\
  b
\end{array} \]
Part as - show \( \Phi \) is constant on a circle of radius \( c \) offset by \( d_i \) from one of the line charges.

\[
\Phi = \frac{1}{4\pi \epsilon_0} \ln \frac{\frac{r_2^2}{r_1^2}}{r_2^2}
\]

and

\[
r_1^2 = d_1^2 + c^2 - 2d_1c \cos \theta
\]

\[
r_2^2 = (d_1 + R)^2 + c^2 - 2(c)(d_1 + R) \cos \theta
\]

From previous discussion, if \( \Phi \) is an equipotential

\[
d_1^2 + c^2 = \frac{d_1}{(d_1 + R)^2 + c^2}
\]

The solution to this is \( d_1(d_1 + R) = c^2 \). To check

\[
d_1^2 + c^2 = \frac{d_1^2}{(d_1 + R)^2 + c^2} - \frac{d_1^2}{d_1} = \frac{d_1^2}{d_1 + R} = \frac{d_1}{d_1 + R}
\]

Again the physics is that \( c \) is the geometric mean of \( d_1 \) and \( d_1 + R \). To express in terms of \( V \)

\[
\exp \left[ \frac{4\pi \epsilon_0 \Phi}{\lambda} \right] = \frac{r_2^2}{r_1^2} = \frac{(d_1 + R)^2 + c^2 - 2(c)(d_1 + R) \cos \theta}{d_1^2 + c^2 - 2(d_1c) \cos \theta}
\]

Use \( d_1(c + c) = c^2 \)

\[
\left[ \frac{d_1 + R}{d_1} \right] \left[ \frac{d_1^2 + c^2}{d_1^2 + c^2 - 2(d_1c) \cos \theta} \right]
\]

\[
\frac{d_1 + R}{d_1} \left\{ \frac{d_1^2 + c^2}{d_1^2 + c^2 - 2(d_1c) \cos \theta} \right\}
\]

give \( d_1 \) in terms of \( \Phi, \lambda, \) and \( R \)