Set 11 – due 15 November

"It is not enough to be wrong. One must also be polite." – N. Bohr

1) [10 points] Jackson 6.15 (a)–5, (b)–5.

2) [20 points] Jackson 7.2 (a)–17 (b)–3. Set all $\mu^\prime$s = $\mu_0$. It will be convenient to express your answer in terms of the reflection coefficients at the two interfaces,

$$ r_{12} = \frac{n_2 - n_1}{n_2 + n_1} \quad (1) $$

and

$$ r_{23} = \frac{n_3 - n_2}{n_3 + n_2} \quad (2) $$

(although it will still not be very compact). Keep a copy of your solution; we will revisit this problem next week.

3) Jackson 7.16 [20 points] (a)–3, (b)–12, (c)–5. Yet another set of Fresnel equations! This was a given as a problem back when we had written comprehensive exams.
6.15 - Symmetries and the Hall effect.

a) If $H = 0$ we can write $\vec{J} = \sigma \vec{E}$ or $\vec{E} = \rho_0 \vec{J}$ where $\sigma$ is the conductivity and $\rho_0$ the resistivity. If $H \neq 0$, there could be more terms. We are interested in the static limit so we do not include derivatives. $\vec{E}$ and $\vec{J}$ are polar vectors while $H$ is an axial vector, so through $O(\mathbf{H}^2)$ the possible terms with the correct spatial symmetry are $\vec{H} \times \vec{J}$, $(\vec{H} \cdot \vec{J}) \vec{H}$, and $\vec{H}^2 \vec{J}$.

b) $\vec{E}$ is even under time reversal ($\equiv T$) while $\vec{J}$ and $\vec{H}$ are T-odd so $\vec{J} = \sigma \vec{E}$ is inconsistent with time reversal symmetry. Ohm's law cannot be a true microscopic relation. It is about dissipation in the material. One way to think about it, $\vec{E}$ is proportional to a force, while $\vec{J}$ is proportional to a velocity, so Ohm's law, $\vec{J} = \sigma \vec{E}$ is like $\vec{V} = \vec{F}$, not $\frac{d\vec{V}}{dt} = \vec{F}$. It is like the equation for a heavily damped system,

$$m \ddot{\vec{v}} + b \vec{v} = F_{\text{ext}}$$

which is $b \vec{v} = F_{\text{ext}} + m \ddot{\vec{v}}$, if $m \ddot{\vec{v}} \ll b \vec{v}$.
Normal incidence on a coated lens. As in QM for 1-dimensional barrier penetration, there are transmitted and reflected waves in regions 1 & 2, but only an outgoing wave in region 3.

\[ E_i \rightarrow \begin{array}{c} 1 \rightarrow 2 \rightarrow 3 \end{array} \]

Write \( E_i = \chi E_i \)
\( B_i = \gamma B_i \)
\( \lambda_i = \lambda_i \)
\( C_B = n_i k x E_i, B_i = \mu_i H_i \)

\[ \begin{cases} E_{1x}(z,t) = E_i e^{i(k_1 z - \omega t)} + E_{1x} e^{i(-k_1 z - \omega t)} \\ c B_{1y}(z,t) = n_i \left[ E_{1x} e^{i(k_1 z - \omega t)} - E_{1x} e^{i(-k_1 z - \omega t)} \right] \end{cases} \]

Drop the common factor \( \exp(-i\omega t) \) henceforth.

**Region 2**: \( E_{2x} = E_{2x} e^{i k_2 z} + E_{2x} e^{-i k_2 z} \)
\( c B_{2y} = n_2 \left[ E_{2x} e^{i k_2 z} - E_{2x} e^{-i k_2 z} \right] \)

**Region 3**: \( E_{3x} = E_{3x} e^{i k_3 z} \)
\( c B_{3y} = n_3 E_{3x} e^{i k_3 z} \)

In each region \( \omega c = n i k_i \).

**Boundary conditions**: \( E_{1i} \) continuous at \( z=0, z=L \)

1) \( E_i + E_{1i} = E_{2i} + E_{2i} \)
2) \( E_{2i} + E_{2i} e^{-2i k_2 z} = E_{3i} e^{i (k_3 - k_2) z} \)

\( H_{1i} \) is continuous at \( z=0, z=L \) and we set \( \mu_2 = \mu_0 \) for all \( i \) match \( C_B_{2y} \)'s

3) \( n_1 (E_i - E_{1i}) = n_2 (E_{2i} - E_{2i}) \)
4) \( n_2 (E_{2i} - E_{2i} e^{-2i k_2 z}) = n_3 E_{3i} e^{i (k_3 - k_2) z} \)
It is best to solve for \( E_{ir} \) \( = E_{3\theta} \) and use \( T+R=0 \).

Messy algebra— but notice, if \( x + y = A(x - y) \) then
\[
(A-1)x = (A+1)y, \quad \text{so}
\]
\[
y = \frac{A-1}{A+1} x.
\]

Eq.2+4 \implies \( E_{2\theta} + E_{2r} e^{-2ik_2d} = E_{3\theta} e^{-i(k_2-k_2)d} \)
\[
= \frac{n_2}{n_3} (E_{2\theta} - E_{2r} e^{-2ik_2d})
\]
\[
\text{or} \quad E_{2r} e^{-2ik_2d} = \frac{n_2-n_3}{n_2+n_3} E_{2\theta} \equiv -y_{23} E_{2\theta}
\]

\(E_b\) : 
\[
E_{\theta} + E_{ir} = E_{2\theta} \left[ 1 - e^{-2ik_2d} \right]
\]
\[
\frac{n_2}{n_1} (E_{\theta} - E_{ir}) = E_{2\theta} \left[ 1 + e^{-2ik_2d} \right]
\]

\text{or} \quad \frac{E_{2\theta}}{E_i} = \frac{E_{\theta} + E_{ir}}{E_{2\theta}} = \frac{n_1}{n_2} \frac{n_2}{n_1} \frac{(E_{\theta} - E_{ir})}{(E_{2\theta})} \left[ 1 - e^{-2ik_2d} \right]
\]

\text{set} \quad b = \frac{y_{23}}{E_{2\theta}} e^{2ik_2d} \quad \text{and evaluate}
\[
E_{\theta} + E_{ir} = \frac{n_1}{n_2} \left( \frac{1-b}{1+b} \right) (E_{\theta} - E_{ir})
\]

\[
\frac{E_{ir}}{E_i} = \frac{n_1}{n_2} \frac{(1-b)}{(1+b)} = \frac{n_1 c_1 - b c_2}{n_1 c_1 + n_2 c_2}
\]
\[
= \frac{n_1 - n_2}{n_1 n_2 - b (n_1 + n_2)} = \frac{n_1 c_1 - b c_2}{n_1 c_1 + n_2 c_2}
\]
\[
i = \frac{n_1 c_1 - b c_2}{n_1 n_2 - b (n_1 + n_2)} = \frac{n_1 - n_2}{n_1 + n_2} - b
\]
\[
= - \left[ \frac{r_{12} + b}{1 + b r_{12}} \right] \quad \text{where} \quad r_{12} = \frac{n_2 - n_1}{n_2 + n_1}
\]
\[
= - \left[ \frac{r_{12} + y_{23}}{1 + y_{23} r_{12}} e^{i k_2 d} \right] \quad \text{where} \quad d = 2 k_2 d
\]
The reflection coefficient is 
\[ R = \frac{|E_r|^2}{|E_i|^2} = \frac{n_2^2 + r_{23}^2 + 2r_{12}r_{23}\cos\alpha}{1 + 2r_{12}r_{23}\cos\alpha + r_{12}^2r_{23}^2} \]

\[ T = 1 - R \text{ (note that is not really the transmitted energy of } n_3 \neq n_1 \text{!)} \]

\[ T = (1 - r_{12}^2)(1 - r_{23}^2) \]

\[ \frac{1}{1 + 2r_{12}r_{23}\cos\alpha + r_{12}^2r_{23}^2} \]

Jackson listed 3 cases:

<table>
<thead>
<tr>
<th>case</th>
<th>n_1</th>
<th>n_2</th>
<th>n_3</th>
<th>\gamma_{12}</th>
<th>\gamma_{23}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>\frac{1}{3}</td>
<td>\frac{1}{5}</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>-\frac{1}{5}</td>
<td>-\frac{1}{3}</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>\frac{1}{3}</td>
<td>-\frac{3}{5}</td>
</tr>
</tbody>
</table>

\[ \text{cases 1 and 2} \]

\[ \text{case 3} \]
b) We want $R = 0$ for $\omega = \omega_0$, where $d = 2 n_2 \frac{\omega_0 d}{c}$

\[ \frac{y_{12}^2 + y_{23}^2}{y_{12}^2 + y_{23}^2} + 2 y_{12} y_{23} \cos d = 0 \]

so \[ \cos d = -\frac{1}{2} \left[ \frac{y_{12}}{y_{23}} + \frac{y_{23}}{y_{12}} \right]. \] All this \[ f(x) = -\frac{1}{2} \left( x + \frac{1}{x} \right) \]

We plot $f(x)$ and noticing that $-1 \leq \cos d \leq 1$, the only solutions are at $x = \pm 1$ where $f(x) = \pm 1$.

If $\cos d = 1$, \[ \frac{y_{12}}{y_{23}} = -1 \Rightarrow n_1 = n_3 : \quad n_1 \mid n_2 \mid n_1 \]

This is not appropriate for an air-film-glass situation, so we need $\cos d = -1$ and $y_{12} = y_{23}$

\[ \frac{n_3 - n_2}{n_3 + n_2} = \frac{n_2 - n_1}{n_2 + n_1} \Rightarrow n_3 n_2 - n_2^2 + n_3 n_1 - n_2 n_1, \]

\[ n_3 n_2 + n_2^2 - n_3 n_1 - n_2 n_1, \]

or $n_2^2 = n_3 n_1$ or $n_2 = \sqrt{n_3 n_1} = \sqrt{n_3} \sqrt{n_1} = 1$.

Let's think about the film?

\[ 2 k_2 d = 2 \omega_0 n_2 d = \pi \]

\[ k_2 = \frac{2\pi}{\lambda_2} \Rightarrow \frac{4\pi d}{\lambda_2} = \pi \Rightarrow d = \frac{\lambda_2}{4}. \]

This is a famous result, called the "Greute waveplate." $d = \frac{1}{4} \lambda$ which makes $\lambda$ is the wavelength of light in the medium, including $n_2$. 
Begin with \( \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \) and \( \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \).

So \( \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} \left( \vec{\nabla} \times \vec{B} \right) = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} \).

If \( D(x,t) = e^{i\omega t} \cdot \vec{D} \), the RHS = \( \mu_0 \omega^2 \vec{D} \).

If \( \vec{E} = \vec{E}_0 e^{i(k \cdot r - \omega t)} \) the LHS is

\( \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}_0) = \mu_0 \omega^2 \vec{D} \) or \( \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot (\mu_0 \omega^2 \vec{D}) \).

(b) In general, if \( D_i = E_i \vec{E} \), then \( \vec{D} \) and \( \vec{E} \) are not parallel. To simplify things, choose coordinates so \( E_i \) is diagonal. Call the unit vectors along these directions \( n_i = k_i / |k_i| \). The ith component of eq. (1) is

\( n_i (\nabla \cdot \vec{E}) - E_i + \frac{\omega^2}{k^2} \mu_0 \varepsilon_i E_i = 0 \)

Define \( \frac{k}{\omega} = \frac{1}{v} \) and \( \frac{v^2}{\mu_0} = \frac{1}{\varepsilon_i} \) to convert this to

\( n_i (\nabla \cdot \vec{E}) - E_i + \frac{v^2}{\mu_0} E_i = 0 \)

or \( E_i = \frac{n_i (\nabla \cdot \vec{E})}{1 - \frac{v^2}{\mu_0}} \).

To groom this, multiply both sides by \( n_i \) and sum over \( i \)

\[ \sum_i n_i E_i = \nabla \cdot \vec{E} = (\nabla \cdot \vec{E}) \sum_i \frac{n_i^2}{1 - \frac{v^2}{\mu_0}} \]
or \[ 1 = \sum_{i} \frac{n_i^2}{1 - \frac{V_i^2}{V_e^2}}. \]

Re-arrange this using \( \sum_{i} n_i^2 = 1 \):

\[ 0 = \sum_{i} n_i^2 \left[ \frac{1}{1 - \frac{V_i^2}{V_e^2}} - 1 \right] = \sum_{i} \frac{n_i^2 V_i^2}{V_e^2 - V_i^2}. \]

Drop the overall \( V_i^2 \):

\[ \sum_{i} \frac{n_i^2}{V_e^2 - V_i^2} = 0 = \frac{n_1^2}{V_1^2 - V_e^2} + \frac{n_2^2}{V_2^2 - V_e^2} + \frac{n_3^2}{V_3^2 - V_e^2}. \]

This is a quadratic equation for \( V \). (Multiply it out if you don't believe me!) and it has two solutions, we can call them \( V = V_a \) or \( V = V_b \). For each of them (look at \( \tilde{n} \cdot \tilde{E}_a \)) \( \tilde{n}_i (\tilde{n} \cdot \tilde{E}_a) - \kappa_i + \mu_0 V_i^2 D_i = 0 \).

If \( \tilde{n} \) points along a principal axis, \( D_i = \epsilon_i \tilde{E}_i \) (and \( \tilde{D}_i = \epsilon_i \tilde{E}_i \)). Multiply by \( \tilde{D}_b \) sum i

\[ 0 = \sum_{i} (\tilde{n} \cdot \tilde{D}_b) \left( \tilde{n} \cdot \tilde{E}_a \right) - \sum_{i} \epsilon_i \tilde{E}_i \tilde{E}_i \tilde{n}_i + \mu_0 V_e^2 \tilde{D}_a \cdot \tilde{D}_b. \]

\[ \nabla \cdot \tilde{D}_b = 0 = \tilde{k} \cdot \tilde{D}_b / |\tilde{k}| = \tilde{n} \cdot \tilde{D}_b = \tilde{n}_i \tilde{D}_b \]

Hillo the first term leaving

\[ 0 = -\sum_{i} \epsilon_i \tilde{E}_i \tilde{E}_i + \mu_0 V_e^2 \tilde{D}_a \cdot \tilde{D}_b. \]

exchange a \& b labels

\[ 0 = -\sum_{i} \epsilon_i \tilde{E}_i \tilde{E}_i + \mu_0 V_e^2 \tilde{D}_a \cdot \tilde{D}_b. \]
and subtract $\rightarrow$

$$(V_a^2 - V_b^2) \vec{D}_a \cdot \vec{D}_b = 0$$

Because $V_a \neq V_b$, it must be that $\vec{D}_a \cdot \vec{D}_b = 0$, which is what we were asked to show.

Take a look at the Wikipedia article "Bi-refrangement."
Birefringence

Birefringence is the optical property of a material having a refractive index that depends on the polarization and propagation direction of light. These optically anisotropic materials are said to be birefringent (or birefractive). The birefringence is often quantified as the maximum difference between refractive indices exhibited by the material. Crystals with non-cubic crystal structures are often birefringent, as are plastics under mechanical stress.

Birefringence is responsible for the phenomena of double refraction whereby a ray of light, when incident upon a birefringent material, is split by polarization into two rays taking slightly different paths. This effect was first described by the Danish scientist Ramus Bertholin in 1669, who observed it in calcite, a crystal having one of the strongest birefringences. However, it was not until the 19th century that Augustin-Jean Fresnel described the phenomenon in terms of polarization, understanding light as a wave with field components in transverse polarizations (perpendicular to the direction of the wave vector).

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Explanations

A mathematical description of wave propagation in a birefringent medium is presented below. Following is a qualitative explanation of the phenomenon.

Uniaxial materials

The simplest type of birefringence is described as uniaxial, meaning that there is a single direction governing the optical anisotropy whereas all directions perpendicular to it (or at a given angle to it) are optically equivalent. Thus, rotating the material around this axis does not change its optical behavior. This special direction is known as the optic axis of the material. Light propagating parallel to the optic axis (whose polarization is always perpendicular to the optic axis) is governed by a refractive index \( n_0 \) (for "ordinary") regardless of its specific polarization. For rays with any other propagation direction, there is one linear polarization that would be perpendicular to the optic axis, and a ray with that polarization is called an ordinary ray and is governed by the same refractive index value \( n_0 \). However, for a ray propagating in the same direction but with a polarization perpendicular to that of the ordinary ray, the polarization direction will be partly in the direction of the optic axis, and this extraordinary ray will be governed by a different, direction-dependent refractive index. Because the index of refraction depends on the polarization, when unpolarized light enters a uniaxial birefringent material, it is split into two beams travelling in different directions, one having the polarization of the ordinary ray and the other the polarization of the extraordinary ray. The ordinary ray will always experience a refractive index of \( n_0 \), whereas the refractive index of the extraordinary ray will be in between \( n_0 \) and \( n_e \), depending on the ray direction as described by the index ellipsoid. The magnitude of this difference is quantified by the birefringence:

\[ \Delta n = n_e - n_0. \]

The propagation (as well as reflection coefficient) of the ordinary ray is simply described by \( n_0 \) as if there were no birefringence involved. However, the extraordinary ray, as its name suggests, propagates unlike any wave in an isotropic optical material. Its refraction (and reflection) at a surface can be understood using the effective refractive index \( n_e \). However, its power flow (given by the Poynting vector) is not exactly in the direction of the wave vector. This causes an additional shift in that beam, even when launched at normal incidence, as is popularly observed using a crystal of calcite as photographed above. Rotating the calcite crystal will cause one of the two images, that of the extraordinary ray, to rotate slightly around that of the ordinary ray, which remains fixed.

When the light propagates either along or orthogonal to the optic axis, such a lateral shift does not occur. In the first case, both polarizations are perpendicular to the optic axis and see the same effective refractive index, so there is no extraordinary ray. In the second case, the extraordinary ray propagates at a different phase velocity (corresponding to \( n_e \)), but still has the power flow in the direction of the wave vector. A crystal with its optic axis in this orientation, parallel to the optical surface, may be used to create a waveplate, in which there is no distortion of the image but an intentional modification of the state of polarization of the incident wave. For instance, a quarter-wave plate is commonly used to create circular polarization from a linearly polarized source.
Biaxial materials

The case of so-called biaxial crystals is substantially more complex. These are characterized by three refractive indices corresponding to three principal axes of the crystal. For most ray directions, both polarizations would be classified as extraordinary rays but with different effective refractive indices. Being extraordinary waves, however, the direction of power flow is not identical to the direction of the wave vector in either case.

The two refractive indices can be determined using the index ellipsoids for given directions of the polarization. Note that for biaxial crystals the index ellipsoid will not be an ellipsoid of revolution ("sphere") but is described by three unequal principal refractive indices \( n_\alpha, n_\beta, \) and \( n_\gamma \). Thus there is no axis around which a rotation leaves the optical properties invariant (as there is with uniaxial crystals whose index ellipsoid is a sphere).

Although there is no axis of symmetry, there are two optical axes or bisectrices which are defined as directions along which light may propagate without birefringence, i.e., directions along which the wavelength is independent of polarization. For this reason, birefringent materials with three distinct refractive indices are called biaxial. Additionally, there are two distinct axes known as optical ray axes or bisectrices along which the group velocity of the light is independent of polarization.

Double refraction

When an arbitrary beam of light strikes the surface of a birefringent material, the polarizations corresponding to the ordinary and extraordinary rays generally take somewhat different paths. Unpolarized light consists of equal amounts of energy in any two orthogonal polarizations, and even polarized light (except in special cases) will have some energy in each of these polarizations. According to Snell's law of refraction, the angle of refraction will be governed by the effective refractive index which is different between these two polarizations. This is clearly seen, for instance, in the Wollaston prism which is designed to separate incoming light into two linear polarizations using a birefringent material such as calcite.

The different angles of refraction for the two polarization components are shown in the figure at the top of the page, with the optic axis along the surface (and perpendicular to the plane of incidence), so that the angle of refraction is different for the ordinary polarization (the "ordinary ray" in this case, having its electric vector perpendicular to the optic axis) and the s polarization (the "extraordinary ray" with a polarization component along the optic axis). In addition, a distinct form of double refraction occurs in cases where the optic axis is not along the refracting surface (not exactly normal to it); in this case the dielectric polarization of the birefringent material is not exactly in the direction of the wave's electric field for the extraordinary ray. The direction of power flow (given by the Poynting vector) for this inhomogeneous wave is at a finite angle from the direction of the wave vector resulting in an additional separation between these beams. So even in the case of normal incidence, where the angle of refraction is zero (according to Snell's law, regardless of effective index of refraction), the energy of the extraordinary ray may be propagated at an angle. This is commonly observed using a piece of calcite cut appropriately with respect to its optic axis, placed above a paper with writing, as in the above two photographs.

Terminology

Much of the work involving polarization preceded the understanding of light as a transverse electromagnetic wave, and this has affected some terminology in use. Isotropic materials have symmetry in all directions and the refractive index is the same for any polarization direction. An anisotropic material is called "birefringent" because it will generally refract a single incoming ray into two directions, which we now understand correspond to the two different polarizations. This is true of either a uniaxial or biaxial material.

In a uniaxial material, one ray behaves according to the normal law of refraction (corresponding to the ordinary refractive index), so an incoming ray at normal incidence remains normal to the refracting surface. However, as explained above, the other polarization can deviate from normal incidence, which cannot be described using the law of refraction. This then became known as the extraordinary ray. The terms "ordinary" and "extraordinary" are still applied to the polarization components perpendicular to and not perpendicular to the optic axis respectively, even in cases where no double refraction occurs.

A material is termed uniaxial when it has a single direction of symmetry in its optical behavior, which we term the optic axis. It also happens to be the axis of symmetry of the index ellipsoid (a sphere in this case). The index ellipsoid could still be described according to the refractive indices, \( n_\alpha, n_\beta, \) and \( n_\gamma, \) along three coordinate axes, however in this case two are equal. So if \( n_\alpha = n_\beta \) corresponding to the \( x \) and \( y \) axes, then the extraordinary index is \( n_\gamma \) corresponding to the \( z \) axis, which is also called the optic axis in this case.

However, materials in which all three refractive indices are different are termed biaxial and the origin of this term is more complicated and frequently misunderstood. In a uniaxial crystal, different polarization components of a beam travel at different phase velocities, except for rays in the direction of what we call the optic axis. Thus the optic axis has the particular property that rays in that direction do not exhibit birefringence, with all polarizations in such a beam experiencing the same index of refraction. It is very different when the three principal refractive indices are all different; then an incoming ray in any of those principal directions will still encounter two different refractive indices. But it turns out that there are two special directions (at an angle to all of the 3 axes) where the refractive indices for different polarizations are again equal. For this reason, these crystals were designated as biaxial, with the two "axes" in this case referring to ray directions in which propagation does not experience birefringence.

Fast and slow rays

In a birefringent material, a wave consists of two polarization components which generally are governed by different effective refractive indices. The so-called slow ray is the component for which the material has the higher effective refractive index (slower phase velocity), while the fast ray is the one with a lower effective refractive index. When a beam is incident on such a material from air (or any material with a lower refractive index), the slow ray is thus refracted more towards the normal than the fast ray. In the figure at the top of the page, it can be seen that refracted ray with a polarization (with its electric vibration in the direction of the optic axis, thus the extraordinary ray) is the slow ray in this case.

Using a thin slab of that material at normal incidence, one would implement a waveplate. In this case there is essentially no spatial separation between the polarizations, however the phase of the wave in the parallel polarization (the slow ray) will be retarded with respect to the perpendicular polarization. These directions are thus known as the slow axis and fast axis of the waveplate.

Positive or negative

Uniaxial birefringence is classified as positive when the extraordinary index of refraction \( n_\gamma \) is greater than the ordinary index \( n_\alpha \). Negative birefringence means that \( \Delta n = n_\gamma - n_\alpha \) is less than zero. In other words, the polarization of the fast (or slow) wave is perpendicular to the optic axis when the birefringence of the crystal is positive (or negative, respectively). In the case of uniaxial crystals, all three of the principal axes have different refractive indices, so this designation does not apply. But for any defined ray direction one can just as well designate the fast and slow ray polarizations.

Sources of optical birefringence

While birefringence is usually obtained using an anisotropic crystal, it can result from an optically isotropic material in a few ways:

- Stress birefringence results when isotropic materials are stressed or deformed (i.e., stretched or bent) causing a loss of physical isotropy and consequently a loss of isotropy in the material's permittivity tensor.
- Circular birefringence in liquids where there is an enantiomeric excess in a solution containing a molecule which has stereo isomers.
Common birefringent materials

The best characterized birefringent materials are crystals. Due to their specific crystal structures, their refractive indices are well defined. Depending on the symmetry of a crystal structure (as determined by one of the 32 possible crystallographic point groups), crystals in that group may be forced to be isotropic (not birefringent), to have uniaxial symmetry, or neither in which case it is a biaxial crystal. The crystal structures permitting uniaxial and biaxial birefringence are noted in the two tables, below, listing the two or three principal refractive indices (at wavelength 590 nm) of some better known crystals.\(^\text{[6]}\)

Many plastics are birefringent because their molecules are "frozen" in a stretched conformation when the plastic is molded or extruded.\(^\text{[7]}\) For example, ordinary cellulose is birefringent. Polymers are routinely used to detect stress in plastics such as polystyrene and polycarbonate.

Cotton fibers are birefringent because of high levels of cellulose material in the fiber's secondary cell wall.

Polarized light microscopy is commonly used in biological tissue, as many biological materials are birefringent. Collagen, found in cartilage, tendon, bone, cornea, and several other areas in the body, is birefringent and commonly studied with polarized light microscopy.\(^\text{[8]}\) Some proteins are also birefringent, exhibiting form birefringence.\(^\text{[9]}\)

Invisible manufacturing imperfections in optical fiber leads to birefringence, which is one cause of pulse broadening in fiber-optic communications. Such imperfections can be geometrical (lack of circular symmetry), due to stress applied to the optical fiber and/or due to bending of the fiber. Birefringence is intentionally introduced (for instance, by making the cross-section elliptical) in order to produce polarization maintaining optical fibers.

In addition to anisotropy in the electric polarizability (electric susceptibility), anisotropy in the magnetic polarizability (magnetic permeability) can also cause birefringence. However, at optical frequencies, values of magnetic permeability for natural materials are not measurably different from \(\mu_0\), so this is not a source of optical birefringence in practice.

### Material

| Uniaxial crystals, at 590 nm\(^{[6]}\) | Biaxial crystals, at 590 nm\(^{[6]}\) |
| Material | Crystal system | \(n_1\) | \(n_2\) | \(\Delta n\) | Material | Crystal system | \(n_{\parallel}\) | \(n_{\perp}\) | \(n_V\) |
| barium borate BaB\(_2\)O\(_4\) | Trigonal | 1.6776 | 1.6534 | -0.0242 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| beryl Be\(_3\)Al\(_2\)(SiO\(_4\)) | Hexagonal | 1.602 | 1.557 | -0.045 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| calcite CaCO\(_3\) | Trigonal | 1.658 | 1.486 | -0.172 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| ice H\(_2\)O | Hexagonal | 1.309 | 1.313 | +0.004 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| lithium niobate LiNbO\(_3\) | Trigonal | 2.272 | 2.187 | -0.085 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| magnesium fluoride MgF\(_2\) | Tetragonal | 1.380 | 1.385 | +0.006 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| quartz SiO\(_2\) | Trigonal | 1.544 | 1.553 | +0.009 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| ruby Al\(_2\)O\(_3\) | Trigonal | 1.770 | 1.762 | -0.008 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| rutile TiO\(_2\) | Tetragonal | 2.616 | 2.903 | +0.287 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| sapphire Al\(_2\)O\(_3\) | Trigonal | 1.768 | 1.760 | -0.008 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| silicon carbide 3C | Hexagonal | 2.847 | 2.693 | +0.154 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| tourmaline (complex silicate) | Trigonal | 1.699 | 1.638 | -0.061 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| zircon, high ZrSiO\(_4\) | Tetragonal | 1.960 | 2.015 | +0.055 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| zircon, low ZrSiO\(_4\) | Tetragonal | 1.920 | 1.967 | +0.047 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| borax Na\(_2\)(B\(_2\)O\(_4\))\(_3\)·3H\(_2\)O | Monoclinic | 1.477 | 1.469 | 1.472 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| epsom salt MgSO\(_4\)·7H\(_2\)O | Monoclinic | 1.433 | 1.455 | 1.461 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| mica, biotite K(Mg,Fe\(_3\))(Al\(_2\)Si\(_2\)O\(_10\))\(_{(x\text{-OH})}\) | Monoclinic | 1.595 | 1.640 | 1.640 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| mica, muscovite KAl\(_2\)(Al\(_2\)Si\(_2\))\(_{10}\)\(_{(x\text{-OH})}\) | Monoclinic | 1.563 | 1.596 | 1.601 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| olivine (Mg,Fe\(_2\))Si\(_2\)O\(_4\) | Orthorhombic | 1.640 | 1.660 | 1.680 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| perovskite CaTiO\(_3\) | Orthorhombic | 2.300 | 2.340 | 2.380 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| topaz Al\(_2\)Si\(_4\)O\(_10\)\(_{(x\text{-OH})}\) | Orthorhombic | 1.618 | 1.620 | 1.627 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |
| ulexite NaCa\(_2\)B\(_2\)O\(_5\)·4H\(_2\)O | Triclinic | 1.490 | 1.510 | 1.520 | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; |

**Measurement**

Birefringence and other polarization-based optical effects (such as optical rotation and linear or circular dichroism) can be measured by measuring the changes in the polarization of light passing through the material. These measurements are known as polarimetry. Polarized light microscopes, which contain two polarizers that are at 90° to each other on either side of the sample, are used to visualize birefringence. The addition of quarter-wave plates permit examination of circularly polarized light. Birefringence measurements have been made with phase-modulated systems for examining the transient flow behavior of fluids.\(^\text{[10]}\)^\text{[11]}\]

Birefringence of lipid bilayers can be measured using dual polarization interferometry. This provides a measure of the degree of order within these fluid layers and how this order is disrupted when the layer interacts with other biomolecules.

**Applications**

Birefringence is used in many optical devices. Liquid-crystal displays, the most common use of flat panel display, cause their pixels to become lighter or darker through rotation of the polarization (circular birefringence) of linearly polarized light as viewed through a sheet polarizer at the screen's surface. Similarly, light modulators modulate the intensity of light through electrically induced birefringence of polarized light followed by a polarizer. The Lyot filter is a specialized narrowband spectral filter employing the wavelength dependence of birefringence. Wave plates are thin birefringent sheets widely used in certain optical equipment for modifying the polarization state of light passing through it.

Birefringence also plays an important role in second-harmonic generation and other nonlinear optical components, as the crystals used for this purpose are almost always birefringent. By adjusting the angle of incidence, the effective refractive index of the extraordinary ray can be tuned in order to achieve phase matching, which is required for efficient operation of these devices.

**Medicine**
Birefringence is utilized in medical diagnostics. One powerful accessory used with optical microscopes is a pair of crossed polarizing filters. Light from the source is polarized in the x direction after passing through the first polarizer, but above the specimen is a polarizer (a so-called analyzer) oriented in the y direction. Therefore, no light from the source will be accepted by the analyzer, and the field will appear dark. However, areas of the sample possessing birefringence will generally couple some of the x-polarized light into the y polarization; those areas will then appear bright against the dark background. Modifications to this basic principle can differentiate between positive and negative birefringence.

For instance, needle aspiration of fluid from a gouty joint will reveal negatively birefringent monosodium urate crystals. Calcium pyrophosphate crystals, in contrast, show weak positive birefringence. Urate crystals appear yellow, and calcium pyrophosphate crystals appear blue when their long axes are aligned parallel to that of a red compensator filter, or a crystal of known birefringence is added to the sample for comparison.

Birefringence can be observed in amyloid plaques such as are found in the brains of Alzheimer's patients when stained with a dye such as Congo Red. Modified proteins such as immunoglobulin light chains abnormally accumulate between cells, forming fibrils. Multiple fibrils of these fibers line up and take on a beta-pleated sheet conformation. Congo red dye intercalates between the fibrils and, when observed under polarized light, causes birefringence.

In ophthalmology, binocular retinal birefringence screening of the Henle fibers (photoreceptor axons that go radially outward from the fovea) provides a reliable detection of strabismus and possibly also of anisotropic amyloidosis. Furthermore, scanning laser polarimetry utilizes the birefringence of the optic nerve fiber layer to indirectly quantify its thickness, which is of use in the assessment and monitoring of glaucoma.

Birefringence characteristics in sperm heads allow the selection of spermatozoa for intracytoplasmic sperm injection. Likewise, zona imaging uses birefringence on oocytes to select the ones with highest chances of successful pregnancy. Birefringence of particles biopsied from pulmonary nodules indicates silicosis.

Dermatologists use dermatoscopes to view skin lesions. Dermatoscopy uses polarized light, allowing the user to view crystalline structures corresponding to dermal collagen in the skin. These structures may appear as shiny white lines or rosette shapes and are only visible under polarized dermoscopy.

**Stress-induced birefringence**

Color pattern of a plastic box with "frozen in" mechanical stress placed between two crossed polarizers

Isotropic solids do not exhibit birefringence. However, when they are under mechanical stress, birefringence results. The stress can be applied externally or it is "frozen in" after a birefringent plastic wafer is cooled after it is manufactured using injection molding. When such a sample is placed between two crossed polarizers, color patterns can be observed, because polarization of a light ray is rotated after passing through a birefringent material and the amount of rotation is dependent on wavelength. The experimental method called photoelasticity used for analyzing stress distributions in solids is based on the same principle. There has been recent research on using stress-induced birefringence in a glass plate to generate an Optical vortex and full Pointcare beams (optical beams that have every possible polarization states across its cross-section).

**Other cases of birefringence**

Birefringence is observed in anisotropic elastic materials. In these materials, the two polarizations split according to their effective refractive indices, which are also sensitive to stress.

The study of birefringence in shear waves traveling through the solid Earth (the Earth's liquid core does not support shear waves) is widely used in seismology.

Birefringence is widely used in mineralogy to identify rocks, minerals, and gemstones.

**Theory**

In an isotropic medium (including free space) the so-called electric displacement \( \mathbf{D} \) is just proportional to the electric field \( \mathbf{E} \) according to \( \mathbf{D} = \varepsilon \mathbf{E} \) where the medium's permittivity \( \varepsilon \) is just a scalar (and equal to \( \varepsilon = \varepsilon_0 n^2 \) where \( n \) is the index of refraction). However, in an anisotropic material exhibiting birefringence, the relationship between \( \mathbf{D} \) and \( \mathbf{E} \) must now be described using a tensor equation:

\[
\mathbf{D} = \varepsilon \mathbf{E}
\]

where \( \varepsilon \) is now a \( 3 \times 3 \) permittivity tensor. We assume linearity and no magnetic permeability in the medium: \( \mu = \mu_0 \). The electric field of a plane wave of angular frequency \( \omega \) can be written in the general form:

\[
\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

where \( \mathbf{r} \) is the position vector, \( t \) is time, and \( \mathbf{E}_0 \) is a vector describing the electric field at \( t = 0, \mathbf{r} = 0 \). Then we shall find the possible wave vectors \( \mathbf{k} \). By combining Maxwell's equations for \( \nabla \times \mathbf{E} \) and \( \nabla \times \mathbf{H} \), we can eliminate \( \mathbf{H} \) to obtain:

\[
\mathbf{J} = \varepsilon_0 \mathbf{E}
\]
\[- \nabla \times \nabla \times \mathbf{E} = \rho_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \]  

(3a)

With no free charges, Maxwell's equation for the divergence of \( \mathbf{D} \) vanishes:

\[ \nabla \cdot \mathbf{D} = 0 \]  

(2a)

We can apply the vector identity \( \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \) to the left hand side of eq. 3a, and use the spatial dependence in which each differentiation in \( x \) (for instance) results in multiplication by \( ik_x \) to find:

\[- \nabla \times \nabla \times \mathbf{E} = (k_x \cdot \mathbf{E})k_x - (k_x \cdot k_x)\mathbf{E} \]  

(3c)

The right hand side of eq. 3a can be expressed in terms of \( \mathbf{E} \) through application of the permittivity tensor \( \mathbf{\varepsilon} \) and noting that differentiation in time results in multiplication by \( -i\omega \), eq. 3a then becomes:

\[-(k_x \cdot \mathbf{E})k_x + (k_x \cdot k_x)\mathbf{E} = -\mu_0 \omega^2 (\mathbf{e}\mathbf{E}) \]  

(4a)

Applying the differentiation rule to eq. 3b we find:

\[ k \cdot \mathbf{D} = 0 \]  

(4b)

Eq. 4b indicates that \( \mathbf{D} \) is orthogonal to the direction of the wavevector \( k \), even though that is no longer generally true for \( \mathbf{E} \) as would be the case in an isotropic medium. Eq. 4b will not be needed for the further steps in the following derivation.

Finding the allowed values of \( k \) for a given \( \omega \) is easiest done by using Cartesian coordinates with the \( x, y \) and \( z \) axes chosen in the directions of the symmetry axes of the crystal (or simply choosing \( z \) in the direction of the optic axis of a uniaxial crystal), resulting in a diagonal matrix for the permittivity tensor \( \mathbf{\varepsilon} \):

\[ \varepsilon = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \]  

(4c)

where the diagonal values are squares of the refractive indices for polarizations along the three principal axes \( x, y \) and \( z \). With \( \varepsilon \) in this form, and substituting in the speed of light \( c \) using \( c^2 = \frac{1}{\varepsilon_0 \mu_0} \), eq. 4a becomes:

\[-(k_x^2 - k_y^2 - k_z^2) E_x + \omega^2 n_x^2 E_y + \omega^2 n_y^2 E_z = -\frac{n_x^2 \omega^2}{c^2} E_x \]  

(5a)

where \( E_x, E_y, E_z \) are the components of \( \mathbf{E} \) (at any given position in space and time) and \( k_x, k_y, k_z \) are the components of \( k \). Rearranging, we can write (and similarly for the \( y \) and \( z \) components of eq. 4a)

\[-(k_x^2 - k_y^2 + \frac{\omega^2 n_y^2}{c^2}) E_x + k_y k_x E_y + k_x k_y E_x = 0 \]  

(5b)

\[-k_x k_y E_x + k_x k_y + \left( -k_x^2 - k_y^2 + \frac{\omega^2 n_x^2}{c^2} \right) E_y = 0 \]  

(5c)

\[-k_x k_z E_x + k_x k_z E_x + \left( -k_x^2 - k_z^2 + \frac{\omega^2 n_z^2}{c^2} \right) E_z = 0 \]  

(5d)

This is a set of linear equations in \( E_x, E_y, E_z \) so it can have a nontrivial solution (that is, one other than \( \mathbf{E} = \mathbf{0} \)) as long as the following determinant is zero:

\[\begin{vmatrix} -(k_x^2 - k_y^2 + \frac{\omega^2 n_y^2}{c^2}) & k_x k_y & k_x k_z \\ k_x k_y & -(k_x^2 - k_y^2 + \frac{\omega^2 n_x^2}{c^2}) & k_y k_z \\ k_x k_z & k_y k_z & -(k_x^2 - k_z^2 + \frac{\omega^2 n_z^2}{c^2}) \end{vmatrix} = 0 \]  

(6)

Evaluating the determinant of eq. 6, and rearranging the terms, we obtain:

\[ \frac{\omega^2}{c^2} \left( k_x^2 + k_y^2 + k_z^2 \right) \left( \frac{k_x^2 + k_y^2}{n_x^2} + \frac{k_z^2}{n_z^2} \right) + \frac{k_x^2 + k_y^2}{n_x^2} + \frac{k_y^2 + k_z^2}{n_y^2} + \frac{k_z^2 + k_x^2}{n_z^2} \left( k_x^2 + k_y^2 + k_z^2 \right) = 0 \]

In the case of a uniaxial material, choosing the optic axis to lie in the \( x \) direction so that \( n_x = n_y = n_0 \) and \( n_z = n_\perp \), this expression can be factored into:

\[ \left( \frac{k_x^2 + k_y^2}{n_0^2} + \frac{k_z^2}{n_\perp^2} - \frac{\omega^2}{c^2} \right) \left( \frac{k_y^2 + k_z^2}{n_0^2} + \frac{k_x^2 + k_z^2}{n_\perp^2} - \frac{\omega^2}{c^2} \right) = 0 \]  

(6b)
Setting either of the factors in eq. 8 to zero will define an ellipsoidal surface in the space of wavevectors k that are allowed for a given o. The first factor being zero defines a sphere; this is the solution for so-called ordinary rays, in which the effective refractive index is exactly np regardless of the direction of k. The second defines a spherical symmetric about the z axis. This solution corresponds to the so-called extraordinary rays in which the effective refractive index is in between np and na, depending on the direction of k. Therefore, for any arbitrary direction of propagation (other than in the direction of the optical axis), two distinct wavevectors k are allowed corresponding to the polarizations of the ordinary and extraordinary rays.

For a biaxial material a similar but more complicated condition on the two waves can be described (193) the locus of allowed k vectors (the wavevector surface) is a 4th-degree two-sheeted surface, so that in a given direction there are generally two permitted k vectors (and their opposites) (193). By inspection one can see that eq. 6 is generally satisfied for two positive values of a. Or, for a specified optical frequency co and direction normal to the wavefronts k, it is satisfied for two wavevectors (or propagation constants) |k| (and thus effective refractive indices) corresponding to the propagation of two linear polarizations in that direction.

When those two propagation constants are equal then the effective refractive index is independent of polarization, and there is consequently no birefringence encountered by a wave traveling in that particular direction. For a uniaxial crystal, this is the optic axis, the z direction according to the above construction. But when all three refractive indices (or permittivities), np, nπ, and nγ are distinct, it can be shown that there are exactly two such directions, where the two sheets of the wavevector surface touch (193) these directions are not at all obvious and do not lie along any of the three principal axes (x, y, z according to the above convention). Historically that accounts for the use of the term "biaxial" for such crystals, as the existence of exactly two such special directions (considered "axes") was discovered well before polarization and birefringence were understood physically. However these two special directions are generally not of particular interest; biaxial crystals are rather specified by their three refractive indices corresponding to the three axes of symmetry.

A general state of polarization launched into the medium can always be decomposed into two waves, one in each of those two polarizations, which will then propagate with different wave numbers |k|. Applying the different phase of propagation to those two waves over a specified propagation distance will result in a generally different net polarization state at that point; this is the principle of the waveplate for instance. However with a waveplate, there is no spatial displacement between the two rays as their k vectors are still in the same direction. That is true when each of the two polarizations is either normal to the optic axis (the ordinary ray) or parallel to it (the extraordinary ray).

In the more general case, there is a difference not only in the magnitude but the direction of the two rays. For instance, the photograph through a calcite crystal (top of page) shows a shifted image in the two polarizations; this is due to the optic axis being neither parallel nor normal to the crystal surface. And even when the optic axis is parallel to the surface, this will occur for waves launched at non-normal incidence (as depicted in the explanatory figure). In these cases the two k vectors can be found by solving eq. 6 constrained by the boundary condition which requires that the components of the two transmitted waves k vectors, and the k vector of the incident wave, as projected onto the surface of the interface, must all be identical. For a uniaxial crystal it will be found that there is not a spatial shift for the ordinary ray (hence its name) which will refract as if the material were non-birefringent with an index the same as the two axes which are not the optic axis. For a biaxial crystal neither ray is deemed "ordinary" nor generally be refracted according to a refractive index equal to one of the principal axes.

See also

- Cotton-Mouton effect
- Crystal optics
- Dichroism
- Ice crystal
- John Kerr
- Periodic poling
- Pleochroism

Notes

1. Although related, note that this is not the same as the index ellipsoid.

References


2. See:
   - Erasmus Bartholin, Experientia crystalli islandici disdiasticici quibus minis & infatla refractione detectatur [https://books.google.com/books?id=FTRAAAAMAAJ&pg=P11#v=onepage&q=false] [Experiments on birefringent Icelandis crystal through which it is detected a remarkable and unique refraction] (Copenhagen, Denmark: Daniel Paufl, 1665).
   - Erasmus Bartholin (January 1, 1673) "An account of sundry experiments made and communicated by that learned mathematician, Dr. Erasmus Bartholin, upon a crystal-like body, sent to him out of Iceland," (https://books.google.com/books?id=Z6F5AAAAAMAAJ&pg=PA214#v=onepage&q=false) Philosophical Transactions of the Royal Society of London, 5: 204-486.


4. Born & Wolf, 2002, pp. 807-8. (In 19th-century terminology, the ordinary ray is said to be polarized in the plane of the optic axis, but this "plane of polarization" is the plane perpendicular to the vibration; cf. Fresnel, 1827, tr. Hobson, p. 318.)


Bibliography


External links

- Stress Analysis Apparatus (based on Birefringence theory) (http://www.photostress.com)
- Video of stress birefringence in Polymermethacrylate (PMMA or Plexiglas). (https://www.youtube.com/watch?v=RECYQbG7U)
- Artist Austin Wood Comarow employs birefringence to create kinetic figurative images. (http://www.austin.com)
- The Birefringence of Thin Ice (Tom Wagner, photographer) (http://lowatom.wooly.com)


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