Set 1—due 6 September

"You won’t become a good pianist by listening to good concerts." — J. Wess

For this week, we have many very short problems, while we build tools: First some (rather formal) Green’s theorem manipulations...

1) [5 points] Jackson 1.10: note that this is only true if there is no charge inside the sphere!

2) [5 points] Jackson 1.12: this is needed for

3) [10 points] Jackson 1.13

And now for a sequence of problems on variational principles: the idea is that any guess for a potential function, which satisfies the boundary conditions of the true potential, can be used to give a variational upper bound on the capacitance of an object (and thus the energy stored in it).

4) [10 points] Jackson 1.17; (a)-5 (b)-5

5) [10 points] Jackson 1.19. For this problem, also consider the cases $b/a = 10, 100, 1.1$ and $1.01$. Compare the functional form of your result to the analytic result, at small $(b/a - 1)$. The point of this exercise is that there is a limit where your variational guess is nearly exact, and you can quantify its non-exactness—a very useful thing to know.
\[ \Phi(x) = \frac{1}{4\pi \epsilon_0} \int \frac{d^3x'}{r} \frac{\epsilon(x')}{|x-x'|} + \frac{1}{4\pi} \int_{\mathcal{S}} \frac{dA'}{2\pi} \left[ \frac{\partial \Phi}{\partial n'} - \frac{\Phi}{\epsilon(x') R} \right] \]

and if \( \epsilon(x') = 0 \) the first term is zero, of course. So let \( S \) be a sphere of radius \( R \), let \( x \) be an interior point. In fact, go to the center of the sphere, call it \( x = 0 \). Then \( \hat{n}' \) and \( \hat{x}' \) are parallel and \( r' = |x'| = R \).

\[ \frac{\partial \Phi}{\partial n'} = -E' \cdot \hat{n}' \quad \text{and} \quad \frac{\partial}{\partial r'} \left( \frac{1}{r'} \right)_{r'=R} = -\frac{1}{R^2} \]

\[ \frac{\partial}{\partial n'} = \hat{n}' \cdot \nabla' = \frac{\partial}{\partial n'} \]. Putting everything together,

\[ \Phi(0) = -\frac{1}{4\pi R} \int dA' \quad E' \cdot \hat{n}' + \frac{1}{4\pi R^2} \int dA' \quad \Phi(x') \]

The first term vanishes due to Gauss' law,

\[ \int dA' E' \cdot \hat{n}' = \text{Poynting flux} = \frac{\Phi(x')}{\epsilon_0} = 0 \]

leaving \( \Phi(0) = -\frac{1}{4\pi R^2} \int dA' \quad \Phi(x') \)

which is the average value (or value averaged over the surface of the sphere) of the potential on the surface.
2) \( E \) is the potential due to \( \sigma \) and \( \varepsilon \).

We are asked to show
\[
\int_V \psi \nabla^2 \phi \, d^3x + \int_S \psi \nabla \phi \cdot d\mathbf{A} = \int_V \phi \nabla^2 \psi \, d^3x + \int_S \phi \nabla \psi \cdot d\mathbf{A}.
\]

To prove this, set \( \phi = \psi \) and \( \psi = \psi' \) in Green's 2nd identity, eq. 1.35. Also use
\[
\begin{align*}
\psi &= \psi' = -E_0 V^2 \psi, \\
\psi' &= -E_0 V^2 \psi', \\
\frac{\partial \psi}{\partial n} &= -E_0 = -\frac{E}{E_0}.
\end{align*}
\]

There is an annoying sign flip—\( \frac{\partial \psi}{\partial n} \) is an outward normal \( \vec{n} \) while in 1.35 \( \vec{n} \) is into the integration region, an inner normal.

Eq 1.35 is
\[
\int_V d^3x \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) = \int_S d\mathbf{A} \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right]
\]

or \(-\frac{1}{E_0} \int_V d^3x \left[ \psi \nabla \cdot \nabla \psi' - \psi' \nabla \cdot \nabla \psi \right] = -\frac{1}{E_0} \int_S d\mathbf{A} \left[ -E_0 \psi' + \psi' \right] \]

by the definition of normal.

We are done after a rearrangement,
\[
\int_V \psi \nabla^2 \phi \, d^3x + \int_S \psi \nabla \phi \cdot d\mathbf{A} = \int_V \phi \nabla^2 \psi \, d^3x + \int_S \phi \nabla \psi \cdot d\mathbf{A}.
\]

This is called "Green's reciprocity theorem."
3.12 - now to apply a

Unprimed system

\[ x = \delta \]

Primed system

\[ x = \delta \]

\[ E_x' = \frac{\sigma_0}{\varepsilon_0} \]

\( E_x = \frac{\sigma_0}{\varepsilon_0} \) - note \( \downarrow \)

\[ \overrightarrow{\sigma} = \sigma (\partial) \text{ on the top} \]

\( \overrightarrow{\sigma} = \pm \sigma_0 \)

\( \sigma = \sigma_0 \)

Notice that both systems are infinite in the transverse directions. If they weren’t, we couldn’t solve the primed system. Now plug in:

\[ g \int \int \int (x-x_0) (y-y_0) (z-z_0) \left[ \frac{\partial x}{\varepsilon_0} \right] + \int \int \sigma_0 \cdot D \cdot dA \]

\[ + \int \int \sigma (\partial) \cdot \sigma_0 \frac{d}{\varepsilon_0} dA = \int \int \overrightarrow{\sigma}_0 d^2x + \int \sigma_0 \overrightarrow{d} dA = 0 \]

\( \overrightarrow{D} = g \sigma_0 \frac{x_0}{\varepsilon_0} + \sigma_0 \frac{d}{\varepsilon_0} \int \int \sigma (\partial) dA \)

\( -\frac{x_0}{d} g = \int \int \sigma (\partial) dA = \text{induced charge in } S^+ \)

We’ll do this a different way later in this term, more methodically (and with a special function identity we have to look up).
4.1.17) We have a region \( V \) bounded by many conductors. All but one are grounded at \( V = 0 \) on that one \( C(V) \) (simplifies algebra).

a) 1.54 says

\[
W = \frac{\varepsilon_0}{2} \int_V \nabla^2 \phi \, d^3x
\]

1.62 says

\[
W = \frac{1}{2} \sum_{i,j} V_i V_j C_{ij} = \frac{1}{2} C
\]

(with \( V \)'s as stated) so

\[
C = \varepsilon_0 \int_V \nabla^2 \phi \, d^3x
\]

The potential \( \phi \) must satisfy our boundary conditions on all surfaces, \( \phi = V_i \) on the \( i \)th conductor.

b) For any \( \phi \) satisfying the b.c. define

\[
C(\phi) = \varepsilon_0 \int_V \nabla^2 \phi \, d^3x.
\]

We want to show that \( C \) is minimized when \( \phi \) is equal to the true \( \phi_0 \) that is \( \nabla^2 \phi_0 = 0 \) inside \( V \) and \( \phi = V_i \) on the \( i \)th conductor. To prove this, redefine the true \( \phi \) as \( \phi_0 + \phi \)

No \( \phi \) on all surfaces. Also

\[
\nabla^2 (\phi_0 + \phi) = \nabla^2 \phi + 2 \nabla \nabla \phi + \phi
\]

Use Gauss' theorem on the middle term to write it as

\[
2 [\nabla \cdot (\phi_0 \nabla \phi) - \phi_0 \nabla^2 \phi] + 2 \varepsilon_0 \int_V \nabla^2 \phi \, d^3x.
\]

Then

\[
C(\phi) = \varepsilon_0 \int_V \nabla^2 \phi \, d^3x + 2 \varepsilon_0 \int_V \nabla^2 \phi \, d^3x + 2 \varepsilon_0 \int_V \nabla \nabla \phi \, d^3x + 2 \varepsilon_0 \int_V \nabla \phi \, d^3x \cdot \nabla \phi + 2 \varepsilon_0 \int_V \nabla^2 \phi \, d^3x.
\]
The first two terms vanish: \( \nabla^2 \Psi_0 = 0 \) inside, and \( \Psi_0 = 0 \) on the surface.

Thus \( C(\Psi) \leq \varepsilon_0 \int d^3 x \left( \left[ \nabla \Psi_0 \right]^2 + \left[ \nabla' \Psi_0 \right]^2 \right) \geq \varepsilon_0 \int d^3 x \left[ \nabla' \Psi_0 \right]^2 \)

which is our variational principle.
1.19 is an application. Take a trial \( \psi \)

\[ \psi(r) = \frac{b-r}{b-a} \]

so \( \psi = 1 \) at \( r = a \)
\( \psi = 0 \) at \( r = b \)

Before doing the integral, let's find the exact solution. This is an undergraduate problem

\[ \int \vec{E} \cdot \hat{n} \, dA = 2\pi r \, E(r) \cdot \hat{r} = \text{Perimeter} = \frac{AL}{\varepsilon_0} \]

where \( \lambda = \text{charge per unit length on the inner surface} \)

then \( E(r) = \frac{\lambda}{2\pi \varepsilon_0 r} \)

\[ V_{ab} = -\int_a^b E_r \cdot d\ell = -\frac{\lambda}{2\pi \varepsilon_0} \ln \frac{b}{a} = -\frac{\Phi}{L} \frac{1}{2\pi \varepsilon_0} \ln \frac{b}{a} \]

\[ CV = \Phi \cdot 2\pi \frac{C}{L} = \frac{2\pi \varepsilon_0}{\ln b/a} \]

Now for the variational calculation

\[ \nabla \psi = \psi \frac{\partial \psi}{\partial r} = \hat{n} \left( -\frac{1}{b-a} \right) \]

\[ C \text{trind} \frac{1}{L} \]

\[ = \varepsilon_0 \int_a^b 2\pi r \, dr \frac{1}{(b-a)^2} = \frac{2\pi \varepsilon_0}{2} \left[ \frac{b^2-a^2}{(b-a)^2} \right] \]

\[ = 2\pi \varepsilon_0 \frac{1}{2} \left( \frac{b/a + \frac{1}{2}}{b/a - 1} \right) \]
\[
X: \frac{b}{a} \quad \frac{1}{\ln(b/a)} \quad \frac{1}{2} \left( \frac{b}{a} + 1 \right) \quad \frac{C_{\text{approx}}}{C_{\text{true}}} \\
1.5 \quad 2.466 \quad 2.5 \quad 1.053 \\
2 \quad 1.443 \quad 1.5 \quad 1.039 \\
3 \quad 0.910 \quad 1.0 \quad 1.099 \\
10 \quad 0.434 \quad 0.611 \quad 1.467 \\
100 \quad 0.257 \quad 0.510 \quad 2.35 \\
1.05 \quad 10.492 \quad 10.5 \quad 1.00076 \\
1.005 \quad 100.499 \quad 100.5 \quad 1.00001
\]

Why are the last 2 cases so accurate?

\[
\log x = 2 \left[ \frac{x-1}{x+1} + \frac{1}{3} \left( \frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left( \frac{x-1}{x+1} \right)^7 + \cdots \right]
\]

The variational calculation is the first term in the expansion. As \( x \to 1 \) it is a better and better approximation to the full expression. The correction from the next term is

\[
\frac{2}{3} \left( \frac{x-1}{2} \right)^3 = \frac{(x-1)^3}{12}.
\]