Set 9 – due 3 November

"The nation that controls magnetism will control the universe" – Dick Tracy (1935)

1) Jackson 5.27 [10 points]

2) Jackson 5.33 [10 points] (a)–5, (b)–5.

3) Jackson 5.34 [20 points] (a)–3: Use the formula given in Problem 5.10b as the start. (b)–7; (c)–7; (d)–3: No discussion of Prob. 5.18 is needed.

4) Jackson 6.8 [20 points]

The hard part of this problem is the start. $\vec{P}$ always follows $\vec{E}$, so $\vec{P}$ points along $\hat{x}$. You need the surface magnetic pole density $\sigma_M = \vec{M} \cdot \hat{n}$ to source Φ_M. Once you have it, the problem comes apart in your hands.

There are (at least) three ways to begin. First, you could use the surface current density $\vec{K}_M$ and surface magnetization $\vec{M}$, $\vec{K}_M = \vec{M} \times \hat{n}$ where $\hat{n}$ is an outward normal to the surface. The surface current density comes from the surface polarization density $\vec{K} = \sigma_P \vec{v}$ where σ_P is the surface polarization charge density, and $\vec{v} = \vec{\omega} \times \vec{r}$. $\vec{K} = \vec{M} \times \hat{n}$ so $\vec{M} = \hat{k}\omega P_0 x$ where P_0 is the magnitude of the polarization vector.

Second, you could look at the volume magnetization M and find the volume current $\vec{J}_M = \vec{\nabla} \times \vec{M}$. You imagine a little dipole whose head and tail are separated by a small difference, so $\vec{J} = Nq(\vec{v}_+ - \vec{v}_-)$. This is nice, but wrong by a sign – the dipole remains oriented along $\hat{x}$, so the charge hops from dipole to dipole in the *opposite* direction to what you have found. You can find $\vec{M}$ from $\vec{J}_M = \vec{\nabla} \times \vec{M}$, you discover $\vec{\nabla} \cdot \vec{M} = 0$ and construct σ_M.

The third way is to look arouhd Jackson Eq. 6.100: a material in bulk motion acquires an effective magnetization $\vec{M}_{eff} = \vec{P} \times \vec{v}$. The derivation is awful, it is fiddling along the lines of Eqs. 6.93-6.96.

5.27

$$\Phi_B = L \cdot I$$

$$W = \frac{1}{2} L I^2 = \frac{1}{2}\int d^3x\, \vec{B}\cdot\vec{H}$$

a) Ampere's law gives us $\vec{H} = \hat{\varphi}\dfrac{I}{2\pi r}$ for $a > r > b$

$$= \hat{\varphi} I \left(\frac{\pi r^2}{\pi b^2}\right)\frac{1}{2\pi r}$$ for $0 < r < b$

$= 0$ if $r > a$

The conductor has permeability μ, so the energy per unit length stored in the inductor is

$$W = \frac{1}{2}\mu_0\left[\frac{I}{2\pi}\right]^2 \cdot 2\pi \cdot \left\{\int_b^a \frac{r\,dr}{r^2} + \frac{\mu}{\mu_0}\int_0^b r\,dr \cdot \frac{r^2}{b^4}\right\}$$

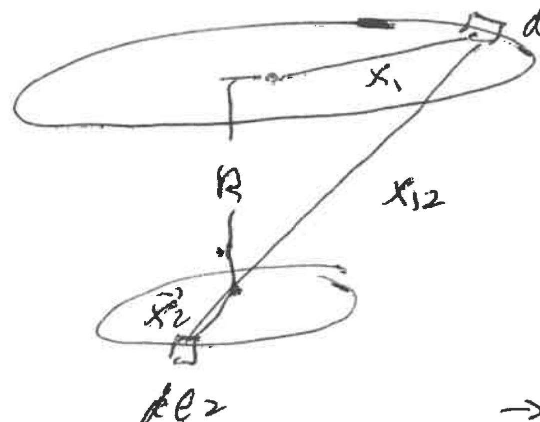
$$= \frac{\mu_0 I^2}{4\pi}\left\{\ln\frac{a}{b} + \frac{1}{4}\frac{\mu}{\mu_0}\frac{b^4}{b^4}\right\}$$

$= \frac{1}{2} L' I^2$ where L' is the self inductance per unit length, $L' = \dfrac{\mu_0}{4\pi}\left\{2\ln\dfrac{a}{b} + \dfrac{1}{2}\dfrac{\mu}{\mu_0}\right\}$

b) if the inner conductor is hollow, $B = H = 0$ for $r < b$ and L' is just $\dfrac{\mu_0}{2\pi}\ln\dfrac{a}{b}$.

5.33

$$F_{12} = -\mu_0 I_1 I_2 \oint\oint d\vec{l}_1 \cdot d\vec{l}_2 \left(\frac{\vec{x}_{12}}{x_{12}^3} \right)$$



$$\vec{x}_{12} = \vec{R} + \vec{x}_1 - \vec{x}_2$$

$$\frac{\vec{x}_{12}}{x_{12}^3} = -\vec{\nabla}_R \frac{1}{(\vec{R} + \vec{x}_1 - \vec{x}_2)}$$

so $\vec{F} = I_1 I_2 \vec{\nabla}_R M_{12}(\vec{R})$ where

a) $$M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{(\vec{R} + \vec{x}_1 - \vec{x}_2)}$$

b) $$\nabla_R^2 M_{12}(\vec{R}) = \frac{\mu_0}{4\pi} \oint\oint dl_1 \cdot dl_2 \left[-4\pi \, \delta^3(\vec{R} + \vec{x}_1 - \vec{x}_2) \right]$$

This is zero unless $\vec{x}_{12} = 0$. But that only happens if the two current loops coincide everywhere in space. That would mean that the loops are identical - the force would be a self-force. This situation is not even well defined!

So for all practical purposes

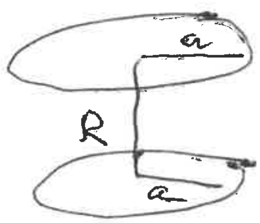
$$\nabla_R^2 M_{12}(\vec{R}) = 0.$$

5.34) $W_{12} = M_{12} I_1 I_2 = I_1 \Phi_{12} = \int \vec{J}_1 \cdot \vec{A}_2 \, d^3x$

so $M_{12} = \frac{1}{I_1 I_2} \int \vec{J}_1 \cdot \vec{A}_2 \, d^3x = \frac{1}{I_2} \oint \vec{A}_2 \cdot d\vec{\ell}$.

To start, lift the formula from problem 5.10b:

$$\vec{A}_2(\rho, z) = \hat{\varphi} \, \mu_0 I_2 \frac{a}{2} \int_0^\infty dk \, e^{-k|z|} J_1(ka) J_1(k\rho)$$



so (a) $M_{12} = \frac{\mu_0 a}{2} \int_0^{2\pi} a \, d\varphi \int_0^\infty dk \, e^{-kR} J_1(ka)^2$

Next, expand $J_1(x)$ for small x, square it, integrate term by term –

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

$x = ka$, $dx = a \, dk$, $kR = x\left(\frac{R}{a}\right)$ so

$$M_{12} = \frac{\mu_0}{2} \cdot 2\pi a \int_0^\infty dx \, e^{-\frac{R}{a}x} \frac{x^2}{4}\left(1 - \frac{x^2}{8} + \frac{x^4}{12 \cdot 16} + \cdots\right)^2$$

$$= \frac{\mu_0 \pi a}{4} \int_0^\infty dx \, e^{-\frac{R}{a}x} x^2 \left[1 - \frac{x^2}{4} + x^4\left(\frac{1}{64} + \frac{2}{12 \cdot 16}\right) + \cdots\right]$$

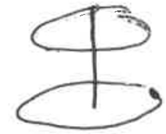
$$\frac{1}{64} + \frac{2}{12 \cdot 16} = \frac{1}{16}\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{1}{32}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{6 \cdot 32}$$

and $\int_0^\infty dx \, x^n \exp\left(-\frac{R}{a}x\right) = n!\left(\frac{a}{R}\right)^{n+1}$.

$$M_{12} = \mu_0 \frac{\pi a}{4}\left[2\left(\frac{a}{R}\right)^3 - \frac{24}{4}\left(\frac{a}{R}\right)^5 + \frac{5\cdot 6!}{6\cdot 32}\left(\frac{a}{R}\right)^7 + \cdots\right]$$

$$\frac{5\cdot 6!}{32\cdot 6} = \frac{25\cdot 4\cdot 3\cdot 2}{32} = \frac{75}{4} \quad \text{and}$$

$$M_{12} = \mu_0 \frac{\pi a}{2}\left[\left(\frac{a}{R}\right)^3 - 3\left(\frac{a}{R}\right)^5 + \frac{75}{8}\left(\frac{a}{R}\right)^7 + \cdots\right] \quad (1)$$

- Eq (1) is for orientation .

c) Recall $\nabla^2 M_{12} = 0$. We just found $M_{12}(\vec{R} = \hat{z} R)$ so we can use the Great Legendre Trick to find M_{12} everywhere!

$$M_{12}(R,\theta) = \frac{\mu_0 \pi a}{2}\left[\left(\frac{a}{R}\right)^3 P_2(\cos\theta) - 3\left(\frac{a}{R}\right)^5 P_4(\cos\theta) + \frac{75}{8}\left(\frac{a}{R}\right)^7 P_6(\cos\theta) + \cdots\right]$$

The orientation  is $\theta = 90°$.

$P_2 = \frac{3\cos^2\theta - 1}{2} \to -\frac{1}{2}$ $\quad P_4(0) = \frac{3}{8}$, $P_6(0) = -\frac{5}{16}$ (see the table on p. 97 of Jackson) so

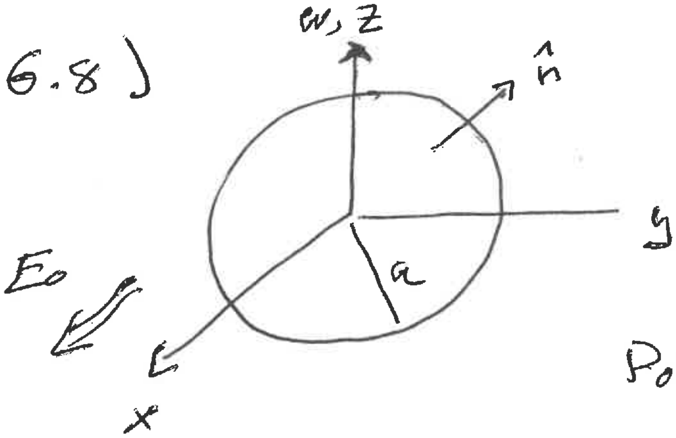
$$M_{12}\left(R, \frac{\pi}{2}\right) = -\frac{\mu_0 \pi a}{2}\left[\frac{1}{2}\left(\frac{a}{R}\right)^3 + \frac{9}{8}\left(\frac{a}{R}\right)^5 + \frac{375}{128}\left(\frac{a}{R}\right)^7 + \cdots\right] \quad (2)$$

d) $F = \left.\frac{\partial W}{\partial R}\right)_I = I_1 I_2 \frac{\partial M_{12}}{\partial R}$ so eq (1) says that F is attractive for loops on a common axis and (2) says F for coplanar loops is repulsive

$$F_1 = \mu_0 \frac{\pi}{2} I_1 I_2 \left[-3\left(\frac{a}{R}\right)^4 + 15\left(\frac{a}{R}\right)^6 - \frac{75}{8}\cdot 7\left(\frac{a}{R}\right)^8 + \cdots\right]$$

$$F_2 = \mu_0 \frac{\pi}{2} I_1 I_2 \left[\frac{3}{2}\left(\frac{a}{R}\right)^4 + \frac{45}{8}\left(\frac{a}{R}\right)^6 + \frac{375}{128}\cdot 7\left(\frac{a}{R}\right)^8 + \cdots\right]$$

6.8)



$\vec{E}$ is in the $\hat{x}$ (or $\hat{\imath}$) direction. $\vec{P}$ always follows $\vec{E}$ so $\vec{P} = \hat{\imath} P_0$ where

$$P_0 = 3\epsilon_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) E_0 .$$

For Φ_M we need $\vec{M}$. There are (at least) three approaches.

1) Use the surface current density $\vec{K}_M = \hat{M} \times \hat{n}$ (see 5.103). K_M comes from the surface polarization charge density $\sigma_P = \vec{P} \cdot \hat{n}$. A point on the surface has a velocity $\vec{v} = \vec{\omega} \times \vec{r}$ so $\vec{K}_M = \sigma_P \vec{\omega} \times \vec{r} = \vec{M} \times \hat{n}$. $\hat{n} = \vec{r}/a$ so $\vec{M} = \sigma_P \omega a \hat{z}$. This is a surface magnetization.

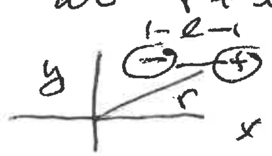
$$\sigma_P = \vec{P} \cdot \hat{n} = P_0 (\hat{\imath} \cdot \hat{n}) = P_0 \sin\theta \cos\varphi$$

$$\sigma_M = \vec{M} \cdot \hat{n} = \omega a (\hat{z} \cdot \hat{r}) P_0 \sin\theta \cos\varphi$$

$$= \omega a P_0 \cos\theta \sin\theta \cos\varphi$$

---

2) Use the volume current $\vec{J}_M = \vec{\nabla} \times \vec{M}$ to find $\vec{M}$. There is something counter intuitive, though. The picture shows a top view of a dipole located at $\vec{r}$ with its + head at $\vec{r} + \hat{\imath}\frac{l}{2}$, - tail at $\vec{r} - \hat{\imath} l/2$. If the dipole were being carried around the z axis, we would write



$$\vec{J} = Nq(\vec{v}_+ - \vec{v}_-) = Nq \vec{\omega} \times \left[\left(\vec{r} + \hat{\imath}\frac{l}{2}\right) - \left(\vec{r} - \hat{\imath}\frac{l}{2}\right) \right]$$

with $\vec{v} = \vec{\omega} \times \vec{r}$ and N the number of dipoles. This is $\vec{J} = \vec{\omega} \times \hat{\imath} N q l = \hat{\jmath} \omega P_0$ since $\hat{z} \times \hat{x} = \hat{y}$.

However, the answer is wrong – the dipole remains oriented in the $\hat{x}$-direction due to E, so the charge moves in the $-\hat{j}$ direction, hopping from molecule to molecule as the molecules rotate away. Only the sign of $\vec{J}$ is wrong, fortunately.

Now we use $\vec{J} = \vec{\nabla} \times \vec{M} = \hat{i}\dfrac{\partial M_z}{\partial y} - \hat{j}\dfrac{\partial M_z}{\partial x} = -\hat{j}\omega P_0$

so $\dfrac{\partial M_z}{\partial x} = \omega P_0$ and we integrate to find the (volume) magnetization on the interior,

$$M_z = \omega P_0 x.$$

Note that $\vec{M} = \hat{z}\,\omega P_0 x$ so $\nabla \cdot M = \dfrac{\partial M_z}{\partial z} = 0$.

There is a surface pole density $\sigma_M = \vec{M}\cdot\hat{n}\Big|_{r=a}$

$$\sigma_M = (\hat{z}\cdot\hat{r})\,\omega P_0 x \quad \text{at } r=a$$

$$= \omega a P_0 \cos\theta \sin\theta \cos\varphi \quad \text{as in method (a)}$$

---

3) Eq 6.100 of Jackson says that a material in bulk motion acquires an effective magnetization density

$$\vec{M}_{eff} = \vec{P} \times \vec{v}$$

The derivation is in the part of the discussion of the magnetic low energy effective field theory for E&M, around Eqs 6.93–6.96. With $\vec{v} = \vec{\omega} \times \vec{r}$

$$\vec{M}_{eff} = \vec{P} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{P}\cdot\vec{r}) - \vec{r}\,(\vec{P}\cdot\vec{\omega}).$$

$\vec{\omega}$ is along $\hat{z}$, $\vec{P}$ is along $\vec{x}$, so the 2nd term is zero

The first term, $\vec{M} = \vec{\omega}(\underline{P}\cdot\vec{r}) = \vec{\omega}\,\sigma_{P}$.

$\sigma_M = \vec{M}\cdot\hat{n} = (\hat{z}\cdot\hat{r})\,\omega\, a\, P_0 \frac{x}{r}$ at $x = r$

$\sigma_M = \omega a P_0 \cos\theta \sin\theta \cos\varphi$ for the 3rd time.

After all that, σ_M is anticlimatic. Look up $Y_2^1(\theta,\varphi)$, and you discover $\sigma_M = -\omega P_0 a\sqrt{\frac{8\pi}{15}}\,\mathrm{Re}\,Y_2^1(\theta,\varphi)$.

Plug this into $\Phi_M = \frac{1}{4\pi}\int dA' \frac{\sigma_M}{|\vec{x}-\vec{x}'|}$

$$\Phi_M = \mathrm{Re}\left\{ a^2 \int d\Omega' \left(-\omega P_0 a\sqrt{\frac{8\pi}{15}}\, Y_2^1(\theta',\varphi')\right) \times \frac{4\pi}{4\pi}\sum_{\ell m}\frac{Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\right\}$$

$$= \mathrm{Re}\left\{-\omega P_0 a^3 Y_2^1(\Omega)\sqrt{\frac{8\pi}{15}}\,\frac{1}{5}\,\frac{r_<^2}{r_>^3}\right\}$$

$$= \frac{1}{5}a^3\omega P_0 \sin\theta\cos\theta\cos\varphi\,\frac{r_<^2}{r_>^3}$$

$r_<$ and $r_>$ are the lesser and greater of r and a.

To clean up, $\cos\theta\sin\theta\cos\varphi = \frac{xz}{r^2}$ so

if $r < a$ $\Phi_M = \frac{1}{5}\omega P_0\, a^3 \frac{xz}{r^2}\frac{r^2}{a^3} = \frac{1}{5}\omega P_0\, xz$

if $r > a$ $\Phi_M = \frac{1}{5}\omega P_0 a^3 \frac{xz}{r^2}\left(\frac{a^2}{r^3}\right) = \frac{1}{5}\omega P_0\, xz\left(\frac{a}{r}\right)^5$

and $P_0 = 3\epsilon_0\left[\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right]E_0$.