Set 8 – due 27 October

"As usual, mathematical calculations could not win over unexpected conditions." – D. Eisenhower

1) Jackson 5.3 [10 points] The classic problem of the solenoid...

2) (a) [7 points] Continuing this problem, show that near the axis and near the center of the solenoid (length L and radius R) the magnetic induction is mainly parallel to the axis, but has a small radial component

$$B_\rho = 24\mu_0 NI \frac{R^2 z\rho}{L^4} \tag{1}$$

correct to order R^2/L^2 and for $z << L$, $\rho << R$. The coordinate z is measured from the center point of the solenoid, with the ends of the solenoid at $z = \pm L/2$.
(b) [3 points] Show that at the end of a long solenoid the magnetic induction near the axis has components

$$B_z \simeq \frac{\mu_0 NI}{2}, B_\rho \simeq \frac{\mu_0 NI}{4}\frac{\rho}{R}. \tag{2}$$

Hint: there is a short way to do this, or a very long way.

3) Jackson 5.13 [20 points]

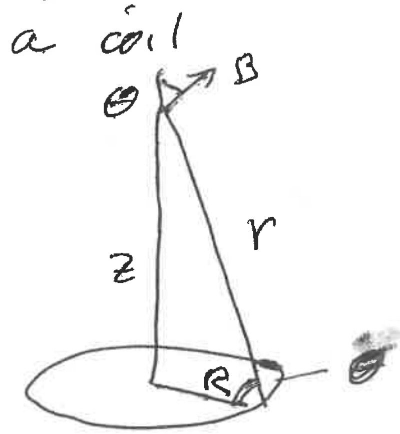
4) [20 points] The Helmholz coil is a pair of circular current loops of radius a with a common axis, separated by a distance $2b$ which is chosen to make the second derivative of $\vec{B}$ vanish at a distance at a point on the axis halfway between the two coils. Show that this distance is $2b = a$ and also find the coefficient C in the Taylor expansion

$$B_z(b+\delta) = B_z(b)(1 + C(\frac{\delta}{a})^4 + \ldots]. \tag{3}$$

You'll probably have to start by finding B for a single coil along its axis; this is simple Biot-Savart cranking.

This problem is a bit messy but if you are a table top experimentalist you might actually use Helmholz coils someday.

1) 5.3 - the Solenoid - use the Biot-Savart eqn for a loop, then stack loops to make a coil



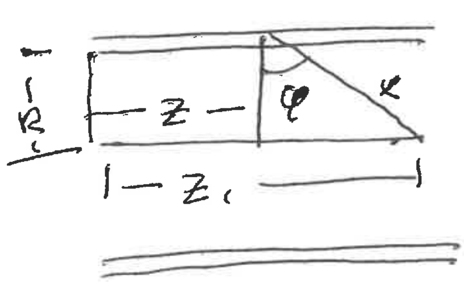
$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{r^3}$$

and on the z-axis, $\vec{B}$ averages to B_z

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{dl}{r^2} \cos\theta$$

$$B_z = \frac{\mu_0 I}{4\pi} \cdot \frac{2\pi R}{[z^2+R^2]} \cdot \frac{R}{[z^2+R^2]^{1/2}} = \frac{\mu_0 I}{2} \frac{R^2}{[z^2+R^2]^{3/2}}$$

Now integrate over coils with N turns per unit length. Do "left" & "right" separately



$$B_L(z_1) = \hat{z} \, \mu_0 I \frac{R^2 N}{2} \times \int_0^{z_1} \frac{dz}{[(z_1-z)^2+R^2]^{3/2}}$$

Change variables to $\frac{z_1 - z}{R} = \tan\varphi$

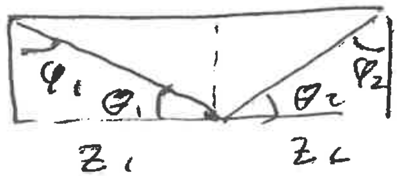
$$\frac{d(z-z_1)}{R} = \frac{d\varphi}{\cos^2\varphi} \quad , \quad x = (R^2 + (z_1-z)^2)^{1/2}$$

so $\cos\varphi = \frac{R}{x}$, $x^3 = \frac{R^3}{\cos^3\varphi}$

so $$B_L(z_1) = -\hat{z}\,\mu_0 \frac{IR^2}{2} N \int_{\varphi_1}^{0} \frac{R\,d\varphi}{\cos^2\varphi} \cdot \frac{\cos^3\varphi}{R^3}$$

$$= \hat{z}\,\mu_0 \frac{IN}{2} \sin\varphi_1$$

The "right" term is identical

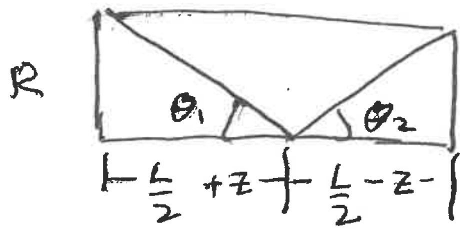


$$B = \hat{z}\,\mu_0 \frac{IN}{2}(\sin\varphi_1 + \sin\varphi_2)$$

$$= \hat{z}\,\mu_0 \frac{IN}{2}(\cos\theta_1 + \cos\theta_2)$$

$\longrightarrow \hat{z}\,\mu_0 IN$ for an infinitely long solenoid where $\theta_1, \theta_2 \to 0$.

2a) near the center



$$\cos\theta_{1,2} = \frac{\frac{L}{2} \pm z}{\sqrt{R^2 + \left(\frac{L}{2} \pm z\right)^2}}$$

$$= \frac{1}{\left[1 + \left(\frac{R}{\frac{L}{2} \pm z}\right)^2\right]^{1/2}}$$

expand the cosine: $\cos\theta_{1,2} = 1 - \frac{1}{2}\left(\frac{R}{\frac{L}{2} \pm z}\right)^2 + \dots$

$$\frac{\cos\theta_1 + \cos\theta_2}{2} = 1 - \frac{R^2}{4}\left(\frac{1}{\left(\frac{L}{2}+z\right)^2} + \frac{1}{\left(\frac{L}{2}-z\right)^2}\right) + \dots$$

$$= 1 - \frac{2R^2}{L^2}\left(\frac{1 + 4z^2/L^2}{\left(1 - \frac{4z^2}{L^2}\right)^2}\right) \simeq 1 - \frac{2R^2}{L^2}\left(1 + \frac{4z^2}{L^2}\right) \times \left(1 + \frac{8z^2}{L^2}\right)$$

$$= 1 - \frac{2R^2}{L^2} - \frac{24R^2z^2}{L^4} \quad \text{so}$$

$$B_z(z) = \mu_0 I N \left[1 - \frac{2R^2}{L^2} - \frac{24R^2z^2}{L^4} + \dots\right]$$

To find B_ρ, use $\vec{\nabla}\cdot\vec{B} = 0 = \frac{\partial B_z}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho$.

$-\frac{\partial B_z}{\partial z} = \frac{\partial}{\partial z} kz^2$ where $k = 24\mu_0 I N R^4/L^4$, so

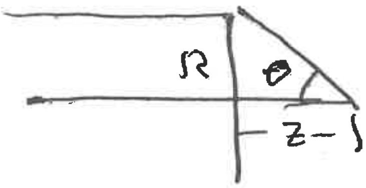
$2kz = \frac{1}{\rho}\frac{d}{d\rho}\rho B_\rho$. This says $2kz\rho = \frac{d}{d\rho}\rho B_\rho$

or $\rho B_\rho = kz\rho^2$, that is

$$\boxed{B_\rho = kz\rho = 24\mu_0 \frac{INR^2}{L^4} z\rho}$$

note as $L \to \infty$ $B_\rho \to 0$!

b) At the end, it's similar



$$\cos\theta = \frac{-z}{\sqrt{R^2+z^2}} = -\frac{z}{R} \quad \text{if } z \ll R$$

so $\vec{B} = \hat{z}\,\mu_0 \frac{I N}{2}\left(1 - \frac{z}{R}\right).$

The "1" is from the far end of the solenoid

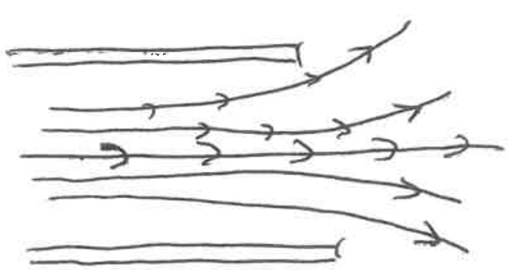
$$-\frac{\partial B_z}{\partial z} = \mu_0 \frac{I N}{2R} \equiv K$$

Again $\frac{\partial}{\partial \rho} \rho B_\rho = K\rho$

$$\rho B_\rho = \frac{1}{2} K \rho^2$$

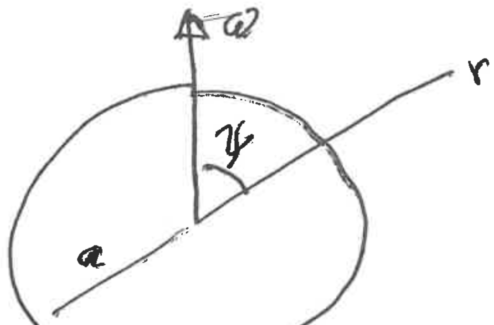
$$B_\rho = \frac{1}{2} K \rho = \mu_0 \frac{I N}{4} \frac{\rho}{R}$$

The radial term appears as we go slightly off axis. Here's the picture:



See the end of the solutions for the (very) long way.

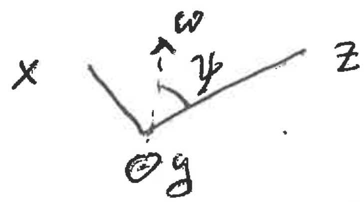
5.13 is most easily done using vector potential



$$\vec{A}(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')\, dA'}{|\vec{r} - \vec{r}'|}$$

$$\vec{J}(r') = \sigma \vec{V}(r') = \sigma\, \vec{\omega} \times \vec{r}'$$

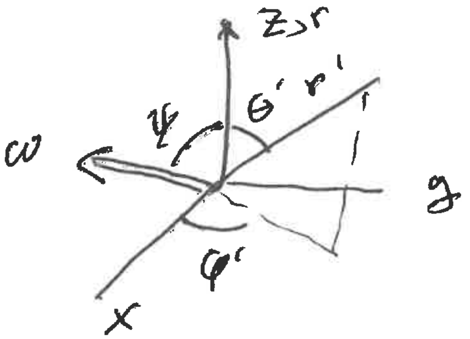
To simplify the problem, define the z axis along $\vec{r}$, put $\vec{\omega}$ in the x-z plane:



$$\vec{\omega} = \omega\left[\hat{x} \sin\psi + \hat{z}\cos\psi\right]$$

$$\vec{r}' = a\left[\hat{x}\sin\theta'\cos\varphi' + \hat{y}\sin\theta'\sin\varphi' + \hat{z}\cos\theta'\right]$$

OR



so $\vec{\omega} \times \vec{r}' = a\omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\psi & 0 & \cos\psi \\ \sin\theta'\cos\varphi' & \sin\theta'\sin\varphi' & \cos\theta' \end{vmatrix}$

$$\vec{\omega} \times \vec{r}' = \omega a \Big[-\hat{x}\cos\psi\sin\theta'\sin\varphi' + \hat{y}\left(\cos\psi\sin\theta'\cos\varphi' - \underline{\sin\psi\cos\theta'}\right) + \hat{z}\sin\psi\sin\theta'\cos\varphi' \Big]$$

Use $\int dA' = a^2 \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta'$ and

$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta')$ since $\vec{r}$ defines the z-axis

Now $\frac{1}{|r-r'|}$ has no ϕ dependence so the only term in $\vec{w}\times\vec{r}'$ which does not integrate to zero is the underlined one.

$$\vec{A}(r) = -\hat{y}\cdot 2\pi\cdot\frac{\mu_0}{4\pi}\cdot\sigma w a^3 \sin\psi \int_{-1}^{1} d\cos\theta' \cos\theta'$$

$$\times \sum_{\ell} P_\ell(\cos\theta')\frac{r_<^\ell}{r_>^{\ell+1}}$$

Only $\ell=1$ survives so

$$\vec{A} = -\hat{y}\,\frac{\mu_0\sigma}{2}\,a^3\omega\sin\psi\cdot\frac{2}{3}\,\frac{r_<}{r_>^2}\,.$$

Noticing that $\hat{r}\times\vec{w} = \hat{y}\, r w \operatorname{sen}\psi$, we see that this is

$$\vec{A}(r) = \frac{\mu_0}{3}\,\frac{\sigma a^3}{r}\,\frac{r_<}{r_>^2}\,\vec{\omega}\times\vec{r}$$

or $\vec{A}(r) = \mu\,\frac{\sigma a}{3}\,\vec{w}\times\vec{r}$ if $r<a$

$$= \mu\sigma\frac{a^4}{r^3}(\vec{w}\times\vec{r}) \text{ if } r>a.$$

~~Ans~~ Notice that the $r>a$ solution is

$$\vec{A} = \frac{\vec{m}\times\vec{r}}{r^3} \quad \text{and} \quad \vec{m} = \frac{\mu\sigma a^4}{3}\vec{\omega}$$

which is a dipole, so we can immediately write

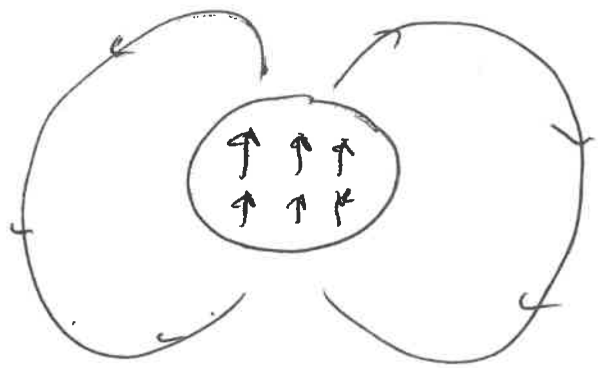
$$\vec{B} = \frac{3\vec{r}(\vec{m}\cdot\vec{r})-\vec{m}}{r^5} = \frac{\mu_0\sigma a^4}{3r^3}\left[3\hat{r}(\vec{w}\cdot\hat{r})-\vec{\omega}\right]$$

For $r<a$, use $\vec{B} = \vec{\nabla} \times \vec{A}$ and

$$\vec{\nabla} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{\nabla} \cdot \vec{r}) - (\vec{\omega} \cdot \vec{\nabla})\vec{r}$$

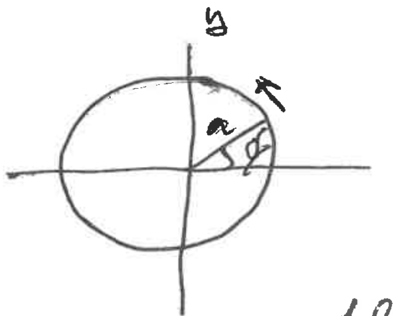
$$= 3\vec{\omega} - \vec{\omega} = 2\vec{\omega}$$

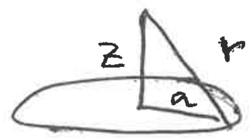
$$\vec{B} = \frac{2}{3}\mu_0 \sigma a \vec{\omega} \quad \text{if } r<a$$



The Helmholz coil: 2 concentric loops, radius a, separation ~~2B~~ 2b. Start with one loop on axis

$$B(x) = \frac{\mu_0 I}{4\pi}\int d\vec{l}' \times \frac{(\vec{x}-\vec{x}')}{|x-x'|^3}$$





$$dl' = a\,d\varphi\,[-\hat{i}\sin\varphi + \hat{j}\cos\varphi]$$

$$\vec{x}-\vec{x}' = z\hat{k} - a(\hat{i}\cos\varphi + \hat{j}\sin\varphi)$$

$$|x-x'| = (a^2+z^2)^{1/2}$$

The numerator in dB is

$$a\,d\varphi \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\varphi & a\cos\varphi & 0 \\ -a\cos\varphi & -a\sin\varphi & z \end{vmatrix}$$

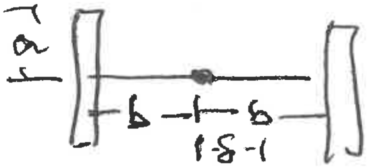
$$d\vec{B}_{\text{numerator}} = a\,d\varphi\,[\hat{i}\,az\cos\varphi - \hat{j}\,az\sin\varphi + \hat{k}a^2]$$

The first 2 terms integrate to zero, leaving

$$\vec{B} = \frac{\mu_0 I}{4\pi}\cdot\frac{2\pi a^2}{(a^2+z^2)^{3/2}}\,\hat{k}$$ for the field

on the z-axis. This is just a repeat of problem 11, a tune-up for the real coil,

$$B_z(\delta) = \frac{\mu_0 I a^2}{2}\left[\frac{1}{(a^2+(b+\delta)^2)^{3/2}} + \frac{1}{(a^2+(b-\delta)^2)^{3/2}}\right]$$



It's just a messy Taylor expansion

$$\frac{1}{[a^2+(b\pm\delta)^2]^{3/2}} = \frac{1}{[a^2+b^2+(\delta^2\pm 2b\delta)]^{3/2}}$$

$$B_z(\delta) = \frac{\mu_0 I a^2}{2[b^2+a^2]^{3/2}}\Bigg\{1+1-\frac{3}{2}\delta\left(\frac{(2b+\delta)+(-2b+\delta)}{b^2+a^2}\right)$$

$$+\frac{3}{2}\cdot\frac{5}{4}\delta^2\left(\frac{(2b+\delta)^2+(-2b+\delta)^2}{(b^2+a^2)^2}\right)$$

$$-\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\delta^3\left(\frac{(2b+\delta)^3+(-2b+\delta)^3}{(b^2+a^2)^3}\right)$$

$$+\frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8}\delta^4\left[\frac{(2b+\delta)^4+(-2b+\delta)^4}{(b^2+a^2)^4}\right]+\cdots\Bigg\}$$

We should then continue expanding ... but by symmetry, odd powers of δ should vanish - easy to see that's so, for δ + δ^3.

The δ^2 term is

$$-\frac{3}{2}\cdot\frac{2\delta^2}{(b^2+a^2)}\left[1-\frac{5}{4}\frac{(2b)^2}{b^2+a^2}\right].$$

We can make this vanish if $b^2+a^2 = 5b^2$ or

$a^2 = 4b^2$ or $\boxed{a = 2b}$

Then, the ratio of the δ^4 term to the δ^0 term is

$$R = \frac{3\cdot 5}{2\cdot 4}\,\frac{\delta^4}{(b^2+a^2)^2}\left[1 - \frac{7}{6}\left(\frac{3\cdot(2b)^2}{b^2+a^2} - \frac{9}{8}\,\frac{(2b)^4}{(b^2+a^2)^2}\right)\right]$$

Using $\dfrac{(2b)^2}{b^2+a^2} = \dfrac{4}{1+4} = \dfrac{4}{5}$ and $\dfrac{1}{b^2+a^2} = \dfrac{1}{a^2\left(\frac{1}{4}+1\right)} = \dfrac{1}{5}$

$$R = \frac{3\cdot 5}{2\cdot 4}\left(\frac{\delta}{a}\right)^4\left(\frac{4}{5}\right)^2\left[1 - \frac{7}{6}\left(3\cdot\frac{4}{5} - \frac{9}{8}\cdot\frac{16}{25}\right)\right]$$

$$= \left(\frac{\delta}{a}\right)^4 \frac{3}{2}\cdot\frac{4}{5}\left[1 - \frac{7}{6}\left(\frac{60-18}{25}\right)\right]$$

$$1 - \frac{7}{6}\cdot\frac{42}{25}$$

$$\frac{150-294}{6\cdot 25} = \frac{-144}{6\cdot 25}$$

$$R = -\left(\frac{\delta}{a}\right)^4 \cdot \frac{6}{5}\cdot\frac{144}{6\cdot 25}$$ so

$$B_z(\delta) = B_z(0)\left[1 - \frac{144}{125}\left(\frac{\delta}{a}\right)^4 + \cdots\right]$$

— More about $\nabla\cdot B$ in Problem 2 — L1

Warmup: dipole $B = \dfrac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}$

Let $r = (\rho, 0, z)$ $\quad \vec{m} = (0, 0, M)$

$\hat{r} = \dfrac{1}{\sqrt{\rho^2+z^2}}(\rho, 0, z) \qquad m\cdot\hat{r} = \dfrac{Mz}{\sqrt{\rho^2+z^2}}$

$3\rho M/z^4$ at small ρ

$$B_\rho = B_x = \frac{3\rho}{\sqrt{\rho^2+z^2}}\frac{Mz}{\sqrt{\rho^2+z^2}}\frac{1}{(\rho^2+z^2)^{3/2}} = \frac{3\rho z M}{(\rho^2+z^2)^{5/2}}$$

$$B_z = \left(\frac{3z}{\sqrt{\rho^2+z^2}}\frac{Mz}{\sqrt{\rho^2+z^2}} - M\right)\frac{1}{(\rho^2+z^2)^{3/2}} = \frac{3Mz^2 - M(\rho^2+z^2)}{(\rho^2+z^2)^{5/2}}$$

$$= \frac{M(2z^2-\rho^2)}{(\rho^2+z^2)^{5/2}}$$

① Note $B_\rho = 0$ at $\rho = 0$

$$\frac{\partial B_z}{\partial z} = M\left\{\frac{4z}{(\rho^2+z^2)^{5/2}} - \frac{5}{2}\cdot\frac{2z(2z^2-\rho^2)}{(\rho^2+z^2)^{7/2}}\right\} = M\frac{4z(\rho^2+z^2) - 10z^3 + 5z\rho^2}{(\rho^2+z^2)^{7/2}}$$

$$\frac{\partial B_z}{\partial z} = zM\frac{(9\rho^2 - 6z^2)}{[\rho^2+z^2]^{7/2}} \xrightarrow[\rho=0]{} -\frac{6z^3M}{z^7} \quad -\frac{6M}{z^4}$$

$$\rho B_\rho = \frac{3\rho^2 zM}{(\rho^2+z^2)^{5/2}}, \quad \frac{\partial}{\partial\rho}\rho B_\rho = zM\left\{\frac{6\rho}{(\rho^2+z^2)^{5/2}} - \frac{3\cdot5\,\rho^2\cdot\rho}{(\rho^2+z^2)^{7/2}}\right.$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho = \frac{zM}{[\rho^2+z^2]^{7/2}}[6\rho^2+6z^2-15\rho^2] = \frac{zM(-9\rho^2+6z^2)}{(\rho^2+z^2)^{7/2}}$$

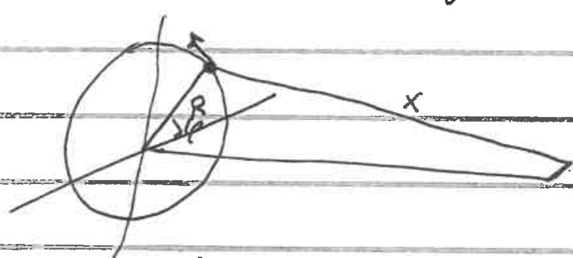
② Note $\nabla\cdot B = 0$.

③ At $\rho\to0$ $\quad \rho B_\rho = 3\rho^2\frac{M}{z^4}, \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho = \frac{6M}{z^4}$

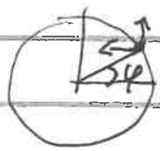
or $\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho = +\frac{6M}{z^4}$ from $\frac{\partial B}{\partial z}$ $\qquad \frac{d}{d\rho}\rho B_\rho = \frac{6M}{z^4}\rho$

and $\rho B_\rho = \frac{3M}{z^4}\rho^2 \qquad B_\rho = \frac{3M}{z^4}\rho^2$. So everything works.

Next we consider a single current loop at the origin



$\vec{r} = (\rho, 0, z)$



$\vec{x}' = (R\cos\varphi, R\sin\varphi, 0) \Rightarrow \vec{r} - x' = (\rho - R\cos\phi, -R\sin\phi, z)$

$I\, d\vec{l} = IR(-\sin\phi, \cos\varphi, 0)\, d\varphi$

$$dB(r) = \frac{\mu_0}{4\pi} \int I\, d\vec{l} \times \frac{(\vec{r} - \vec{x}')}{|r - x'|^3}$$

$$dl \times (r - x') = IR\,d\theta \begin{vmatrix} i & j & k \\ -\sin\varphi & \cos\varphi & 0 \\ \rho - R\cos\varphi & -R\sin\varphi & z \end{vmatrix}$$

$$= IR\,d\theta \left[\hat{i}\, z\cos\varphi + \hat{j}\, z\sin\varphi + \hat{k}\,[R - \rho\cos\varphi] \right]$$

$$\vec{B} = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} d\varphi \frac{\{ z(\hat{i}\cos\varphi + \hat{j}\sin\varphi) + \hat{k}[R - \rho\cos\varphi]}{[z^2 + \rho^2 + R^2 - 2\rho R\cos\varphi]^{3/2}}$$

1) At $\rho = 0$ can we show $B = \hat{z}$ only? Yes, trivial, denominator has no φ dependence $C = z^2 + R^2$

2) $$\frac{1}{(C + \rho^2 - 2\rho R\cos\phi)^{3/2}} = \frac{1}{C^{3/2}} \left[1 - \frac{3}{2}\left(\frac{\rho^2 - 2\rho R\cos\phi}{C}\right) + \frac{15}{8}\left(\frac{\rho^2 - 2\rho R\cos\phi}{C}\right)^2 \right.$$

or perhaps easier, $\rho \gg R$, $D = z^2 + \rho^2$ (check the dipole)

$$\frac{1}{(\quad)} = \frac{1}{D^{3/2}} \left[1 - \frac{3}{2}\left(\frac{R^2 - 2\rho R\cos\varphi}{D}\right) + \frac{15}{8}\left(\frac{R^2 - 2\rho R\cos\varphi}{D}\right)^2 + \cdots \right.$$

$= \frac{1}{D^{3/2}}$ Want to compute B_x & B_y

$$B_y = \frac{\mu_0 I R z}{4\pi}\int_0^{2\pi} d\varphi \sin\varphi \left\{ \frac{1}{D^{3/2}}\left[1 - \frac{3}{2}\frac{R^2}{D} + \cdots \right.\right.$$
$$\left.\left. + \frac{3}{2}\cdot 2\frac{\rho R\cos\varphi}{D} + \cdots\right\} = 0$$

$$\int_0^{2\pi} d\varphi \sin\varphi\cos\varphi = \frac{1}{2}\int_0^{2\pi}\sin 2\varphi\, d\varphi = 0$$

$$B_x = \frac{\mu_0 I R z}{4\pi}\int_0^{2\pi} d\varphi \cos\varphi\left\{\frac{1}{D^{3/2}}\cdot\frac{3\rho R\cos\varphi}{D} + \cdots\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{I R^2 \rho z}{D^{5/2}}\cdot\frac{1}{2}\cdot 2\pi = \frac{\mu_0}{4\pi}\,\frac{I R^2 \rho z}{(\rho^2+z^2)^{5/2}}$$

This checks perfectly with the dipole ✓ Note on-axis, $B_x = B_y = 0$

Now go inside the loop

$$\vec{B} = \frac{\mu_0 I R}{4\pi}\int_0^{2\pi} d\varphi \left\{ z(\hat{i}\cos\varphi + \hat{j}\sin\varphi) + \hat{k}(R - \rho\cos\varphi)\right\}$$
$$\times \frac{1}{C^{3/2}}\left\{1 - \frac{3}{2}\frac{\rho^2}{C} + \frac{3}{C}\rho R\cos\varphi + \cdots\right\}$$

$$= \frac{\mu_0 I R}{4\pi}\left[\frac{3z\rho R}{C^{5/2}}\cdot\frac{2\pi}{2}\hat{i} + \hat{j}\cdot 0 + \hat{k}\cdot\frac{2\pi(R\ \cdots)}{C^{3/2}}\right]$$

Again on-axis $B_i = 0$ $\quad \vec{B} \propto \hat{k}$.

Integrate z' from $-\frac{L}{2}$ to $\frac{L}{2}$

Now we go to the solenoid. ~~With~~

$$B = \frac{\mu_0 I N R}{4\pi}\int_{-L/2}^{L/2} dz' \int_0^{2\pi} d\varphi\, \frac{\{(z-z')(\hat{i}\cos\varphi + \hat{j}\sin\varphi) + \hat{k}[R - \rho\cos\varphi]\}}{[(z-z')^2 + R^2 + \rho^2 - 2R\rho\cos\varphi]^{3/2}}$$

We assume $D = (z-z')^2 + R^2$ is large and expand a la "D"

$$B = \frac{\mu_0 I N R}{4\pi} \int_{-L/2}^{L/2} dz' \int_0^{2\pi} d\varphi \left\{ (z-z')[\hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi] + \hat{k}[R - e\cos\varphi] \right\}$$

$$\cdot \frac{1}{D^{3/2}} \left[1 - \frac{3}{2}\frac{e^2}{D} + \frac{3eR\cos\varphi}{D} + \cdots \right]$$

$$= \frac{\mu_0 I N R}{4\pi} \int_{-L/2}^{L/2} dz' \left\{ \hat{\imath}\, 3\frac{eR(z-z')}{D^{5/2}} \cdot \frac{2\pi}{2} + \hat{\jmath}\cdot 0 + \hat{k} \cdot 2\pi \frac{R}{D^{3/2}} \left(1 - \frac{3}{2}\frac{e^2}{D} + \cdots \right) \right.$$

or in leading order

$$B_x = \frac{\mu_0 I N e R^2}{4} \int_{-L/2}^{L/2} dz' \frac{(z-z')}{[(z-z')^2 + R^2]^{5/2}}$$

$$B_z = \frac{\mu_0 I N R^2}{2} \int_{-L/2}^{L/2} \frac{dz'}{[(z-z')^2 + R^2]^{3/2}}$$

The B_x integral is $B_x = \mu_0 \frac{I N R^2 e}{4} \frac{3}{3} \left. \frac{1}{[(z-z')^2 + R^2]^{3/2}} \right|_{z'=-L/2}^{z'=L/2}$

$$B_x = \mu_0 \frac{I N R^2 e}{4} \left\{ \frac{1}{[(\frac{L}{2}-z)^2 + R^2]^{3/2}} - \frac{1}{[(\frac{L}{2}+z)^2 + R^2]^{3/2}} \right\}$$

$$\{\} = \frac{1}{[\frac{L^2}{4} - Lz + R^2 + z^2]^{3/2}} - \frac{1}{[\frac{L^2}{4} + Lz + R^2 + z^2]} \sim \frac{\frac{3}{2}\cdot 2 \cdot Lz}{[\frac{L^2}{4}]^{5/2}}$$

$$\{\,\} = \frac{3Lz}{L^5} \times \frac{32}{4}$$

$$B_x = 24 \mu_0 I N R^2 \frac{c z}{L^4}$$ by direct integration.

At $z = \frac{L}{2} + \delta$, $\frac{1}{\left[\left(\frac{L}{2} - z\right)^2 + R^2\right]^{3/2}} \sim \frac{1}{R^3}$, drop other term

$$B_x = \mu_0 \frac{I N}{4} \frac{c}{R}$$

Agrees with use of $\nabla \cdot B$ ...