Set 7–due 20 October

1) Jackson 4.9. [20 points] (a)–10 points, (b)–4 points, (c)–6 points.

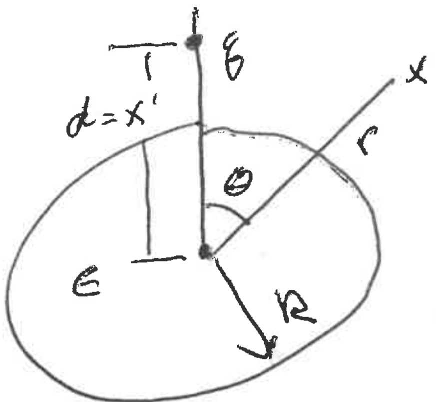
2) Jackson 4.13 [15 points]

3) Jackson 4.10 [10 points].

4) [15 points] A sphere of radius R is made of a uniaxial dielectric with dielectric constant ϵ along two principal axes and dielectric constant ϵ_3 along the third. The sphere is placed in a uniform electric field, with arbitrary orientation. Find the torque on the sphere.

4.9 – the point charge and the dielectric sphere.

Placing the charge on the z-axis gives



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

and $|\vec{x}| = r$, $|\vec{x}'| = r' = d$

Then add a homogeneous solution to account for the dielectric sphere.

Combining them, for $r<R$ $\Phi_{inside}(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_l A_l r^l P_l(\cos\theta)$

for $r<R$ $\Phi_{outside} = \frac{1}{4\pi\epsilon_0}\sum_l \left[q\frac{r_<^l}{r_>^{l+1}} + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta)$

$\Phi_{in} = \Phi_{out}$ at $r=R$ so $A_l R^l = q\frac{R^l}{d^{l+1}} + \frac{B_l}{R^{l+1}}$. (1)

Continuity of $\vec{D}\cdot\hat{n}$ is $\epsilon \frac{\partial \Phi_{in}}{\partial r}\Big)_{r=R} = \epsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big)_{r=R}$

$$\epsilon l A_l R^{l-1} = \epsilon_0 \left[q\frac{l R^{l-1}}{d^{l+1}} - (l+1)\frac{B_l}{R^{l+2}} \right]$$

algebra! $\frac{\epsilon}{\epsilon_0} l A_l = \frac{lq}{d^{l+1}} - (l+1)\frac{B_l}{R^{2l+1}}$ (2)

use (1): $\frac{\epsilon}{\epsilon_0} l \left[\frac{B_l}{d^{2l+1}} + \frac{q}{d^{l+1}} \right] = \frac{l}{d^{l+1}} q - (l+1)\frac{B_l}{R^{2l+1}}$

$$\frac{B_l}{R^{2l+1}}\left[\frac{\epsilon}{\epsilon_0} l + l + 1 \right] = q\frac{l}{d^{l+1}}\left(1 - \frac{\epsilon}{\epsilon_0}\right) \text{ - so ...}$$

$$B_\ell = -q\ell \frac{R^{2\ell+1}}{d^{\ell+1}} \frac{\left(\frac{\epsilon}{\epsilon_0}-1\right)}{\left[1+\ell\left(\frac{\epsilon}{\epsilon_0}+1\right)\right]}$$

$$A_\ell = \frac{q}{d^{\ell+1}}\left[1 - \frac{\ell\left(\frac{\epsilon}{\epsilon_0}-1\right)}{\left[1+\ell\left(\frac{\epsilon}{\epsilon_0}+1\right)\right]}\right] = \frac{q}{d^{\ell+1}} \frac{2\ell+1}{\left[1+\ell\left(\frac{\epsilon}{\epsilon_0}+1\right)\right]}$$

(sanity check - let $\epsilon \to \epsilon_0$ ...)

b) near $r=0$ $\quad \Phi = \frac{A_1 r\cos\theta}{4\pi\epsilon_0} = \frac{A_1 z}{4\pi\epsilon_0}$ so $\vec{E} = \hat{z} E_z$

$$E_z = \frac{-A_1}{4\pi\epsilon_0} = \frac{-3q}{4\pi\epsilon_0 d^2} \frac{1}{(2+\epsilon/\epsilon_0)}$$ from part (a)

c) As $\epsilon \to \infty$, $A_\ell \to 0$ and $B_\ell \to -q \frac{R^{2\ell+1}}{d^{\ell+1}}$ for $\ell \neq 0$

and if $\ell = 0$ $\quad A_0 = \frac{q}{d}$ and $B_0 = 0$. Then

$$\Phi_{in} = \frac{1}{4\pi\epsilon_0}\frac{q}{d}, \quad \Phi_{out} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{x}'|} - \sum_{\ell=1}^{\infty} \frac{q R^{2\ell+1} P_\ell(\cos\theta)}{r^{\ell+1} d^{\ell+1}}\right]$$

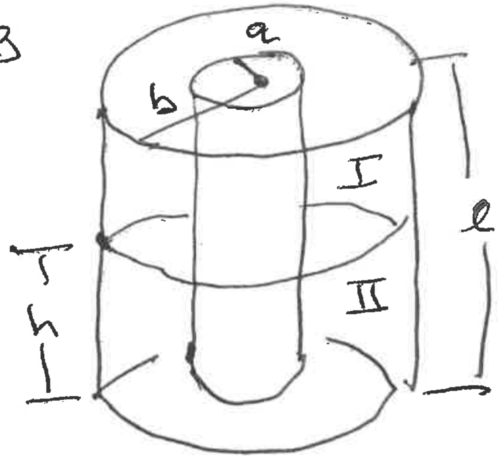
To make sense of this, rewrite it (add & subtract an $\ell=0$ term)

$$\Phi_{out} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{d}\sum_{\ell=0}^{\infty} \frac{\left(\frac{R^2}{d}\right)^\ell P_\ell(\cos\theta)}{r^{\ell+1}} + \frac{R}{d}\frac{1}{r}\right]$$

The middle term is an image charge $q' = -qR/d$, located at $y' = R^2/d$. The last term is an image charge at the origin which maintains the sphere's neutrality:

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|x-x'|} - q\frac{R}{d}\frac{1}{|\vec{x}-\vec{y}|} + \frac{qR}{d}\frac{1}{r}\right]$$

4.13



Compute the energy W stored in the capacitor when h is the height of the oil. The force of attraction will be $F_e = -\frac{\partial W}{\partial h}$

This is balanced by gravity. The gravitational potential energy associated with the height of the column is $\mathcal{U}$, and $F_g = -\frac{\partial \mathcal{U}}{\partial h}$. $F_g = F_e$ gives h.

Gravity first. To raise the column of oil from z to $z+dz$ costs $d\mathcal{U} = [\pi(b^2-a^2) z \rho] g\,dz$.

Here ρ = density of oil, $[\,]$ = mass of oil.

$$\mathcal{U} = \int_0^h d\mathcal{U} = \pi(b^2-a^2)\rho \frac{g h^2}{2}$$

$$-\frac{\partial \mathcal{U}}{\partial h} = -\pi(b^2-a^2)\rho g h.$$

Electrostatics: $W = \frac{1}{2}CV^2$, we need C vs h.

Gauss' law gives the radial $D_J = \lambda_J / 2\pi r$ for $J = \mathrm{I}$ or II, λ = charge / length. Above the oil $E_{\mathrm{I}} = \lambda_{\mathrm{I}}/(2\pi\epsilon_0 r)$, below the oil $E_{\mathrm{II}} = \lambda_{\mathrm{II}}/(2\pi\epsilon r)$

E_{tan} is continuous and E_{I} + E_{II} are tangential to the surface so

$$E_{\mathrm{I}} = E_{\mathrm{II}} \quad \text{or} \quad \frac{\lambda_{\mathrm{I}}}{\lambda_{\mathrm{II}}} = \frac{\epsilon_0}{\epsilon}.$$

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1+\chi) \vec{E} \text{ so}$$

$$\frac{\epsilon}{\epsilon_0} = 1+\chi \text{ and } \frac{\lambda_I}{\lambda_{II}} = \frac{1}{1+\chi}.$$

The total $Q = \lambda_I (l-h) + \lambda_{II} h$

$$Q = \lambda_I [(l-h) + (1+\chi)h] = \lambda_I (l+\chi h)$$

$$\therefore \lambda_I = \frac{Q}{l+\chi h}.$$

Integrate $E_I \rightarrow V_I = \frac{\lambda_I \ln \frac{b}{a}}{2\pi\epsilon_0} = \frac{Q}{2\pi\epsilon_0} \frac{1}{[l+\chi h]} \ln\frac{b}{a}$

and $V_I = V_{II}$. $CV = Q$ so $C = \frac{Q}{V}$

$$C = \frac{2\pi\epsilon_0 [l+\chi h]}{\ln b/a}$$

$$W = \frac{1}{2} C V^2 \text{ so } F_e = -\frac{1}{2} V^2 \frac{\partial C}{\partial h} = -\frac{\pi \epsilon_0 \chi V^2}{\ln b/a}$$

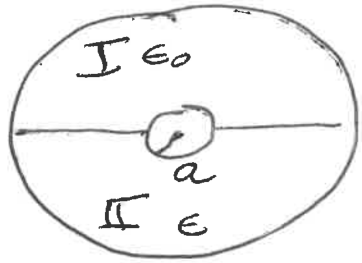
Putting the pieces together

$$\frac{\pi \epsilon_0 \chi V^2}{\ln b/a} = \pi (b^2 - a^2) \rho g h$$

$$\chi = \left[(b^2-a^2) \ln \frac{b}{a} \right] \frac{\rho g h}{\epsilon_0 V^2}$$

as we were asked to show.

4.10



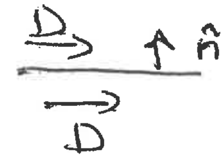
$\vec{E}$ is radial because E_{tan} is continuous.

$E(r) \propto 1/r^2$

$$D_I = \epsilon_0 E_I = \epsilon_0 E$$

$$D_{II} = \epsilon E_{II} = \epsilon E$$

Note $|D|$ is discontinuous on the boundary, but $\vec{D} \cdot \hat{n}_{boundary} = 0$



Call $E(r=a) \equiv E$. The free ~~charge is density~~ surface charge density on the inner sphere is $\sigma_F = \vec{D} \cdot \hat{n} = D$; $\sigma_F^{I} = D_I = \epsilon_0 E$

$$\sigma_F^{II} = D_{II} = \epsilon E.$$

In region II there is a polarization charge –

$D_{II} = \epsilon_0 E_{II} + P_{II}$ so $P_{II} = (\epsilon - \epsilon_0) E$.

The bound or polarization charge is $\sigma_b = \vec{P} \cdot \hat{n}$. $\hat{n}$ points away from the dielectric, so this is $\hat{n} = -\hat{r}$ on the inner sphere. There

$$\sigma_b = -(\epsilon - \epsilon_0) E(a).$$

Because E is radial, σ is uniform over the sphere. So $\sigma_F^{I} = \sigma_F^{II} + \sigma_B^{II}$ at $r = a$.

The total free charge at $r = a$ is

$$Q = 2\pi a^2 (\sigma_F^{I} + \sigma_F^{II}) = 2\pi a^2 [\epsilon_0 + \epsilon] E(a)$$

so $$E(a) = \frac{Q}{2\pi a^2} \frac{1}{(\epsilon_0 + \epsilon)}$$

and $$\sigma_b^{II} = -\left[\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right] \frac{Q}{2\pi a^2}$$

$$\sigma_F^{I} = \frac{\epsilon_0}{\epsilon_0 + \epsilon} \frac{Q}{2\pi a^2}$$

$$\sigma_F^{II} = \frac{\epsilon}{\epsilon_0 + \epsilon} \frac{Q}{2\pi a^2}$$

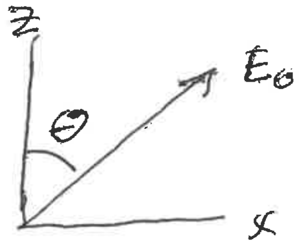
and check $$\sigma_F^{II} + \sigma_B^{II} = \left(\frac{\epsilon - \epsilon + \epsilon_0}{\epsilon_0 + \epsilon}\right) \frac{Q}{2\pi a^2}$$

$$= \sigma_F^{I}$$ as expected.

4) A dielectric sphere in an external E-field. For an isotropic dielectric

$$\vec{E}_{in} = \frac{3}{\kappa+2}\vec{E}_{out} \quad \vec{D}_{in} = \frac{3\epsilon}{\kappa+2}\vec{E}_{out}, \quad \kappa = \frac{\epsilon}{\epsilon_0}$$

Choose the coordinates so the unique axis is aligned along $\hat{z}$ (so $\epsilon_3 = \epsilon_z$) and let the external E field lie in the x-z plane. Then use superposition - consider each component separately. With $\kappa = \frac{\epsilon}{\epsilon_0}$, $\kappa_z = \frac{\epsilon_z}{\epsilon_0}$,



$$\vec{E}_{in} = (E_x, E_y, E_z)$$

$$= \left(\frac{3}{\kappa+2}E_0\sin\theta,\ 0,\ \frac{3}{\kappa_z+2}E_0\cos\theta\right)$$

$$\vec{D}_{in} = (D_x, D_y, D_z) = \left(\frac{3\epsilon_0}{\kappa+2}E_0\sin\theta,\ 0,\ \frac{3\epsilon_z}{\kappa_z+2}E_0\cos\theta\right).$$

The polarization $\vec{\mathbb{P}}$ is the dipole moment $\vec{p}$ per unit volume: $\vec{p} = \left(\frac{4\pi R^3}{3}\vec{\mathbb{P}}\right)$

With $\vec{\mathbb{P}} = \vec{D} - \epsilon_0\vec{E}$

$$\vec{p} = \frac{4\pi R^3}{3}\epsilon_0\left[3\left(\frac{\kappa-1}{\kappa+2}\right)\sin\theta,\ 0,\ 3\left(\frac{\kappa_z-1}{\kappa_z+2}\right)\cos\theta\right]E_0$$

The torque is $\vec{\tau} = \vec{p}\times\vec{E} = \begin{vmatrix} i & j & k \\ p_x & 0 & p_z \\ E_x & 0 & E_z \end{vmatrix}$

$$\vec{\tau} = \hat{j}\ 12\pi R^2\epsilon_0\sin\theta\cos\theta E_0^2\left[\frac{\kappa_z-\kappa}{(\kappa+2)(\kappa_z+2)}\right]$$

vanishes if $\kappa = \kappa_z$ or $\theta = 0°$ or $\theta = 90°$