Set 6–due 13 October

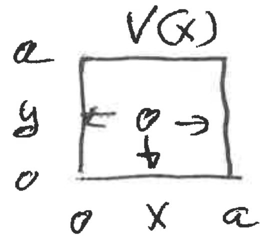
The midterm will be Tuesday, October 10, 7-830 PM, in G-131.

"If you're very good at calculus you could probably figure out a way to do it without thinking" – F. E. Low

1) [20 points] Write down the Dirichlet Green's function for electrostatics for a two-dimensional square of length a, $(0 < x < a,\ 0 < y < a)$, expanding in sine waves with a double sum, like in Eq. 3.167. Now suppose that the potential is specified to be $V = 0$ on all sides except the side at $y = a$ and $\Phi(x, a) = V(x)$. Work out down the appropriate formula for $\Phi(x, y)$ and look at it – does it not seem to show peculiar behavior as $y \to a$? To be definite, set $V(x) = \sin(\pi x/a)$, do the integral, and you'll find $\Phi(x, y) = \sin(\pi x/a)F(y)$. Plot partial sums of $F(y)$ (summing say the first n terms). What's going on? Are there alternate version of Green's functions which will not show this behavior?

1) $G(x, x') = 4\pi \left(\frac{2}{a}\right)^2 \sum_{\ell m} \dfrac{\sin\frac{\ell\pi x}{a} \sin\frac{\ell\pi x'}{a} \sin\frac{m\pi y}{a} \times \sin\frac{m\pi y'}{a}}{\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$

Suppose $\Phi =$



Green's theorem says

$$\Phi(x, y) = -\frac{1}{4\pi}\int_0^a dx' \, \frac{\partial}{\partial y} G(x, y; x', y')\Big|_{y' = a}$$

$$\frac{\partial G}{\partial y'} = -\frac{16\pi}{a^2} \sum_{\ell m} \frac{\sin\frac{\ell\pi x}{a} \sin\frac{\ell\pi x'}{a} \sin\frac{m\pi y}{a} \cdot \frac{m\pi}{a} \cos\left(\frac{m\pi y'}{a}\right)}{\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$$

and we evaluate this at $y' = a$, giving

$$\Phi = -\frac{16\pi}{4\pi a^2} \sum_{\ell m} \sin\frac{\ell\pi x}{a} \sin\frac{m\pi y}{a}$$

$$\times \left[\int_0^a V(x') \sin\frac{\ell\pi x'}{a}\, dx'\right] \cdot \frac{\frac{m\pi}{a} \cos m\pi}{\left(\frac{\ell\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$$

It looks like $\Phi(x, y)$ vanishes as y goes to a.
Hmm!

To continue, suppose $V(x') = V_0 \sin \frac{\pi x'}{a}$

so the integral is $\frac{V_0 a}{2} \delta_{\ell,1}$ and

$$\Phi(x,y) = -\frac{8V}{a} \sin\frac{\pi x}{a} \sum_{m=1}^{\infty} \frac{\sin\frac{m\pi y}{a} \cdot \left(\frac{m\pi}{a}\right)(-)^m}{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]}$$

The $y \to a$ behavior is an example of a "Gibbs phenomenon" - Φ vanishes at any finite order in m, the sum approaches a constant as $m \to \infty$. The enclosed pictures (generated by the enclosed c-code) show this. Any Green's function of the form

$$G(x,y;x',y') = \frac{2}{a} \sum_m \sin\frac{m\pi y}{a} \sin\frac{m\pi y'}{a}\, g_m(x,x')$$

will behave like this, as $y \to a$. There will be no issues as $x \to 0$ or $x \to a$.

In contrast, $\frac{2}{a}\sum_\ell \sin\frac{\ell\pi x}{a} \sin\frac{\ell\pi x'}{a}\, g_\ell(y,y')$ will not have a Gibbs phenomenon in y, but it will, in x. In this case,

$$g_\ell(a,y') \sim \sinh ky_< \sinh k(a-y_>)$$

and
$$\left.\frac{dg_\ell}{dy'}\right)_{y'=a} = -\sinh ky \cdot k\cosh k(a-y')\Big|_{y'=a}$$

$$= -k\sinh ky$$

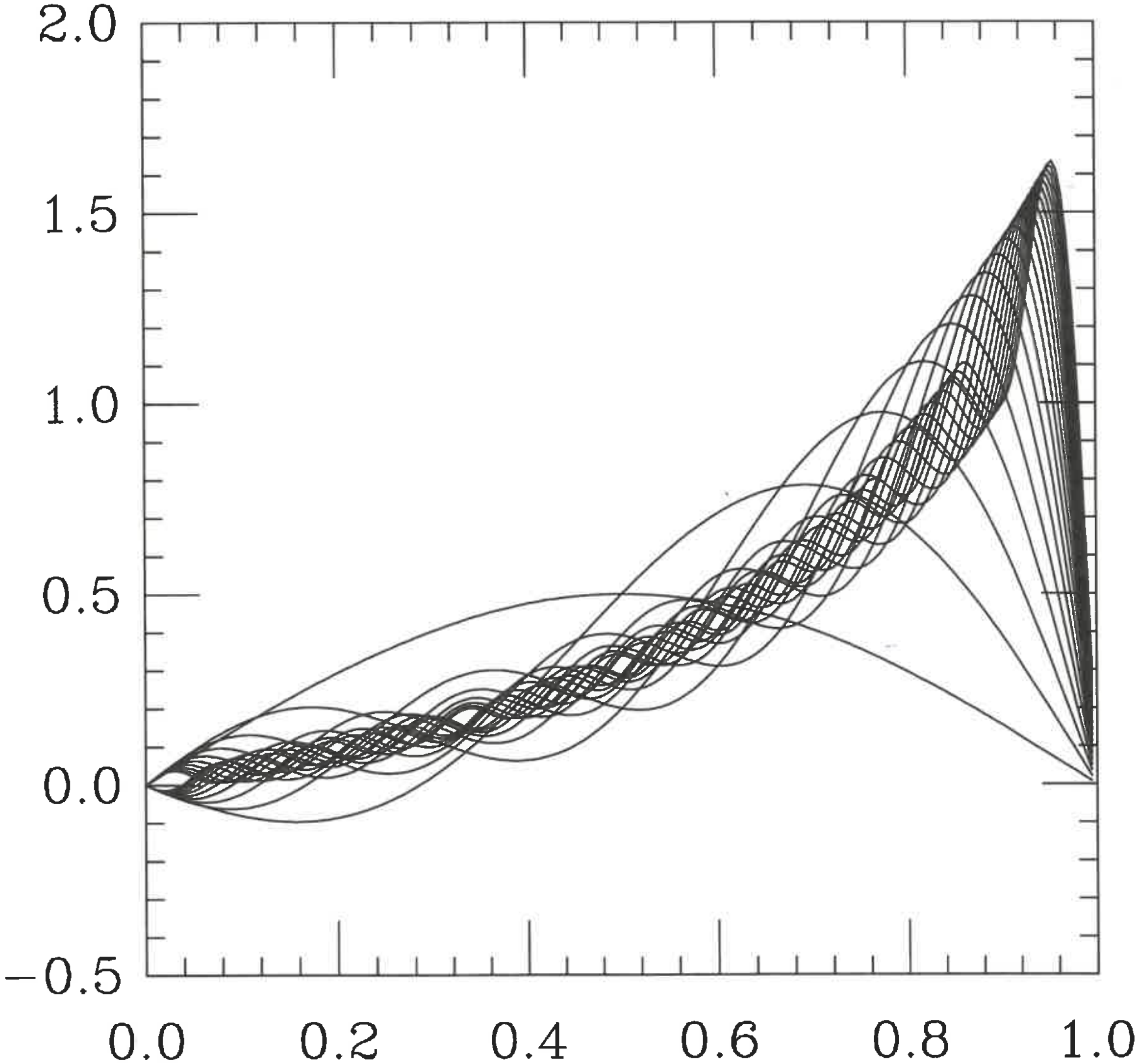
Φ will look like the "natural" separation of variables solution (for $\Phi = 0$ at $x = 0, a$ or $y = 0$)

$$\Phi(x,y) = \sum_\ell C_\ell \sin\frac{\ell\pi x}{a}\sinh\frac{\ell\pi y}{a}$$

which is smooth as $y \to a$.

```c
/* c-code plots sums for Phi, prob 2 set 6. Plotting uses
 ''axis'' -- c-package available on request */
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
FILE *output;
#define PI 3.1415926536

main()
{
char *test1 = "axis<test.ax|plot -T X";
int number,nt,dummy,i,j,k;
float value,sign;;
float siy,sum[200],y;

 output = fopen("test.ax","w");
 fprintf(output,"#r .1 h .8 w .8\n");
 number=20;

 /* 200 points in x */
 for(i=0;i<200;i++) sum[i]=0.0;
/* sum  up to 20 terms in expression */
  sign= 1.0;
 for(j=1;j<=number;j++){

   for(i=0;i<200;i++){
     y=0.005*((float)i);
     siy=sin((float)j*PI*y);
     sum[i] += sign*siy*((float)j)/(1.0+(float)(j*j));

     fprintf(output,"%e %e\n",y,sum[i]);
   }

   fprintf(output,"#\n");
   sign *= -1.0;
 }
fclose(output);
system(test1);
}
```

partial sums
from 1 to
20 terms



Sum 20 terms