Set 5–due 6 October

"The climbing as a whole is not very esthetic or enjoyable; it is merely difficult."–Yvon Chouinard

1) Jackson 2.12 [15 points] Don't begin with (2.71) – take a solution with separate $\sin(n\phi)$ and $\cos(n\phi)$ coefficients.

2) Jackson 2.13 [20 points] (a)–15, (b)–5. For some psychological reason, I found it easier to do this problem measuring my angle from the intersection of the two potential values (i, e., $V = V_1$ for $0 < \phi < \pi$), then changing variables at the end to Jackson's convention.

3) Jackson 3.20 [20 points]. (a)–8 (b)–5 (c)–7. Omit all the discussion abuot 3.19. Just work this problem in cylindrical coordinates, and find the Dirichlet Green's function by beginning with

$$\delta(z - z') = \frac{2}{L}\sum_n \sin\frac{n\pi z}{L} \sin\frac{n\pi z'}{L} \tag{1}$$

and

$$\delta(\phi - \phi') = \frac{1}{2\pi}\sum_m \exp(im(\phi - \phi')) \tag{2}$$

(redoing the derivation on p. 125-126). In part (c) you will have to look up one integral over a Bessel function. Do you recall the reciprocity problem from Set 1?

2.12. Jackson 2.12 starts from the general solution in 2-d, $\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} \rho^n [A_n \sin n\phi + B_n \cos n\phi]$. (1)

We need $b_0 = 0$ if Φ is to be regular at the origin.

$$\left.\begin{aligned} 2\pi a_0 &= \int_0^{2\pi} d\phi' \, \Phi(b,\phi') \\ b^n \begin{bmatrix} A_n \\ B_n \end{bmatrix} &= \frac{1}{\pi}\int_0^{2\pi} d\phi' \, \Phi(b,\phi') \cdot \begin{bmatrix} \sin n\phi' \\ \cos n\phi' \end{bmatrix} \end{aligned}\right\} \quad (2)$$

Put (2) into (1) -

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \, \Phi(b,\phi') \left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \left\{\sin n\phi' \sin n\phi + \cos n\phi' \cos n\phi\right\}\right]$$

The expression in [ ] is

$$W = 1 + 2\sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos n(\phi-\phi') = 1 + \sum_{n=1}^{\infty} x^n \left[e^{in\chi} + e^{-in\chi}\right]$$

setting $x = \rho/b$ and $\chi = \phi - \phi'$. The sums are geometric series, so

$$W = 1 + \frac{x e^{i\chi}}{1 - x e^{i\chi}} + \frac{x e^{-i\chi}}{1 - x e^{-i\chi}} = 1 + \frac{x e^{i\chi} - x^2 + x e^{-i\chi} - x^2}{1 - 2x\cos\chi + x^2}$$

$$= 1 + \frac{2x\cos\chi - 2x^2}{1 - 2x\cos\chi + x^2} = \frac{1 - x^2}{1 - 2x\cos\chi + x^2}$$

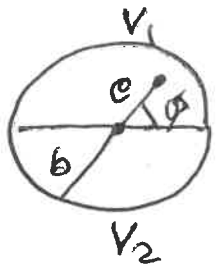
$$= \frac{1 - \left(\frac{\rho}{b}\right)^2}{1 + \left(\frac{\rho}{b}\right)^2 - 2\frac{\rho}{b}\cos(\phi-\phi')} \quad (!)$$

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \, \Phi(b,\phi') \left[\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')} \right]$$

This is Φ for $0<\rho<b$. If we want the exterior solution $\rho>b$, just replace $\frac{\rho}{b}$ by $\frac{b}{\rho}$. This is correct for all terms <u>except</u> the monopole. If we want Φ for $b<\rho<\infty$, we must specify $\Phi(\rho)$ as $\rho\to\infty$ because the monopole term is $a_0 + b_0 \ln\rho$. If Φ is finite as $\rho\to\infty$, it must be that $b_0=0$.

In books about complex variables our result is called the "Poisson integral formula for a circle" and it has a contour integral story.

2) 2.13 is similar. My definition of ϕ (see picture) says



$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \rho^n \left[A_n \cos n\phi + B_n \sin n\phi\right]$$

As in 2.12, orthogonality gives

$$\int_0^{2\pi} d\phi\, \Phi(b,\phi) = 2\pi a_0 = \pi(V_1+V_2) \Rightarrow a_0 = \frac{V_1+V_2}{2}$$

$$\pi b^n A_n = \int_0^{2\pi} d\phi\, \Phi(b,\phi) \cos n\phi = 0 \text{ by parity}$$

$$\pi b^n B_n = \int_0^{2\pi} d\phi\, \Phi(b,\phi) \sin n\phi$$

$$= V_1 \int_0^{\pi} d\phi \sin n\phi + V_2 \int_{\pi}^{2\pi} d\phi \sin n\phi$$

$$= -\frac{V_1}{n}\left[\cos n\pi - 1\right] - \frac{V_2}{n}\left[1 - \cos n\pi\right]$$

$$= 0 \text{ if } n \text{ is even}$$

$$= 2\left(\frac{V_1 - V_2}{n}\right) \text{ if } n \text{ is odd} \Rightarrow B_n = \frac{2}{\pi}\left(\frac{V_1-V_2}{b^n}\right)\frac{1}{n}$$

$$\Phi = \frac{V_1+V_2}{2} + \frac{2}{\pi}(V_1-V_2) \sum_{n \text{ odd}} \left(\frac{\rho}{b}\right)^n \frac{\sin n\phi}{n}$$

This series can be summed. Define $z = \frac{\rho}{b} e^{i\phi}$

so $\operatorname{Im} z^n = \frac{\rho}{b} \sin n\phi$. Call the sum S,

$$S = \operatorname{Im} \sum_{n \text{ odd}} \frac{z^n}{n}$$

You can find the sum on p. 74 of Jackson (if you don't know it already!) -

$$\sum_{n\ \text{odd}} \frac{z^n}{n} = \frac{1}{2}\ln\frac{1+z}{1-z}$$

$$= \frac{1}{2}\ln\frac{1+z}{1-z}\left(\frac{1-z^*}{1-z^*}\right) = \frac{1}{2}\ln\frac{1-|z|^2+(z-z^*)}{1+|z|^2-(z+z^*)} \qquad (*)$$

Call this $\frac{1}{2}\ln W$. Now if $W = Ae^{i\theta}$, $\ln W = A + i\theta$.

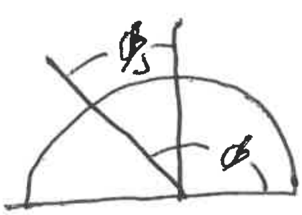
$$\tan\theta = \frac{\operatorname{Im} W}{\operatorname{Re} W} = \frac{2\operatorname{Im} z}{1-|z|^2}$$ (the denominator of $(*)$ is real!)

and $$S = \frac{1}{2}\tan^{-1}\left(\frac{2\operatorname{Im} z}{1-|z|^2}\right)$$

This is a typical improbable result in 2-d.

Recall $z = \frac{\rho}{b}e^{i\phi}$ so $\operatorname{Im} z = \frac{\rho}{b}\sin\phi$

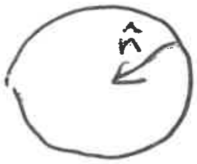
$$S = \frac{1}{2}\tan^{-1}\left\{\frac{2\frac{\rho}{b}\sin\phi}{1-\rho^2/b^2}\right\}$$



Jackson's angle is related to mine by $\phi = \phi_J + \pi/2$ so $\sin\phi = \cos\phi_J$

$$\Phi = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho\sin\phi}{b^2-\rho^2}\right)$$

b) $\frac{\sigma}{\epsilon_0} = \frac{\partial \Phi}{\partial \rho}\Big)_{\rho=b}$: plus sign since outward normal from surface is <u>into</u> ρ



$$\frac{\sigma}{\epsilon_0} = \left(\frac{V_1-V_2}{\pi}\right)\left[\frac{1}{1+\left(\frac{2b\rho\sin\phi}{b^2-\rho^2}\right)^2}\right]$$

$$\times \left[\frac{2b\sin\phi}{b^2-\rho^2} + \frac{2b\rho\sin\phi\cdot 2\rho}{(b^2-\rho^2)^2}\right]\Bigg|_{b=\rho}$$

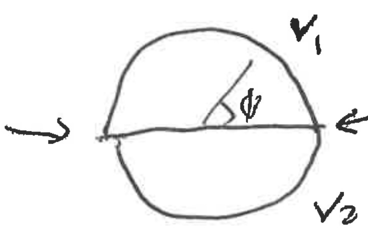
simplify first!

$$\frac{\sigma}{\epsilon_0} = \left(\frac{V_1-V_2}{\pi}\right)\frac{(2b\sin\phi)(b^2-\rho^2+2\rho^2)}{(b^2-\rho^2)^2\left[1+\frac{4b^2\rho^2\sin^2\phi}{(b^2-\rho^2)^2}\right]}\Bigg|_{b=\rho}$$

$$= \left(\frac{V_1-V_2}{\pi}\right)\frac{(2b\sin\phi)(b^2+\rho^2)}{(b^2-\rho^2)^2+4b^2\rho^2\sin^2\phi}\Bigg)_{b=\rho}$$

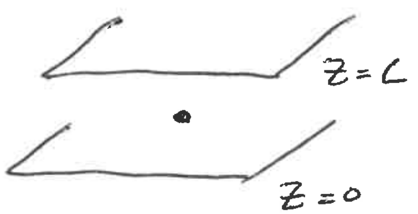
<u>Now</u> set $\rho=b$!

$$\frac{\sigma}{\epsilon_0} = \frac{V_1-V_2}{\pi}\cdot\frac{4b^3\sin\phi}{4b^4\sin^2\phi} = \frac{V_1-V_2}{\pi}\frac{1}{b\sin\phi}$$



$\leftarrow$ σ diverges at the interfaces at $\phi = 0, \pi$

3) 3.20 - a point charge at $x' = (0, 0, z_0)$, we want $\Phi(x = \rho, \phi, z)$ in cylindrical coordinates. This is a Green's function in cylindrical coords - see pp. 125-126. Use



$$\delta(z - z_0) = \frac{2}{L} \sum_n \sin\frac{n\pi z_0}{L} \sin\frac{n\pi z}{L}$$

$$\delta^2(x - x_0) = \frac{4\pi}{\rho} \delta(\rho - \rho_0)\, \delta(\phi - \phi_0)$$

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_m e^{im(\phi - \phi')}$$

in

$$G(\rho, \phi, z; \rho', \phi', z') = \sum_{mn} g_{nm}(\rho, \rho') \frac{e^{im(\phi - \phi')}}{2\pi} \times \frac{2}{L} \sin\frac{n\pi z'}{L} \sin\frac{n\pi z}{L}.$$

Writing $k = n\pi/L$, "standard manipulations" give

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d g_{km}}{d\rho} - \left[k^2 + \frac{m^2}{\rho^2} \right] g_{km}(\rho) = -\frac{4\pi}{\rho} \delta(\rho - \rho')$$

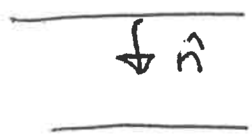
(3.141) - this is the Bessel eqn of imaginary argument, the solution is

$$g_{km}(\rho) = 4\pi I_m(k\rho_<) K_m(k\rho_>)$$

(see the end for the 4π). For us, $\rho' = 0$. Then we look up that all $I_m(k\rho = 0) = 0$ if $m \neq 0$, $I_0(0) = 1$, $\Phi = G \times (q/4\pi\epsilon_0)$ so (on-point charge - $m = 0$)

$$\Phi(z, \phi, \rho) = \left[\frac{q}{4\pi\epsilon_0} \cdot \frac{2}{L} \cdot \frac{4\pi}{2\pi} \right] \sum_{n=1}^{\infty} \sin\frac{n\pi z_0}{L} \sin\frac{n\pi z}{L} K_0\!\left(\frac{n\pi\rho}{L}\right)$$

note no ϕ dependence of course. The prefactor is $q/(\pi\epsilon_0 L)$.



b) $$\frac{\sigma}{\epsilon_0} = \frac{\partial\Phi}{\partial z}\Big)_{z=L}$$

$$= \frac{q}{\pi\epsilon_0 L}\sum_{n=1}^{\infty} K_0\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\frac{\partial}{\partial z}\sin\frac{n\pi z}{L}\Big)$$

The derivative gives $\frac{n\pi}{L}\cos n\pi = \frac{n\pi}{L}(-)^n$, so $\quad z=L$

$$\sigma = \frac{q}{L^2}\sum_{n=1}^{\infty} K_0\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\cdot n(-)^n.$$

c) Can we recover the Set 1 result that on the $z=L$ surface, $Q(L) = \int\sigma\, dA = -q\frac{z_0}{L}$?

It is $$\frac{Q(L)}{q} = \frac{2\pi}{L^2}\sum_n \int_0^\infty \rho\, d\rho\, K_0\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right) \times (-)^n \cdot n$$

To separate sum and integral, change variables to $x = n\pi\rho/L$

$$\frac{Q(L)}{q} = \frac{2\pi}{L^2}\left[\int_0^\infty x\,dx\,K_0(x)\right]\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)^2\cdot n\cdot(-)^n\cdot\sin\frac{n\pi z_0}{L}$$

$$= \frac{2}{\pi}\left[\int_0^\infty x\,dx\,K_0(x)\right]\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi z_0}{L}.$$

By now, the sum is familiar. The integral is a library exercise. dlmf.nist.gov 10.22.1 says

$$\int_0^\infty x\,dx\, K_0(x) = x K_1(x)\Big|_0^\infty$$

$$K_1(x) = -\frac{\pi}{2}\left[J_1(ix) + i N_1(ix)\right]$$

Both $J_1(x)$ and $N_1(x)$ vanish as $x \to \infty$. As $x \to 0$ $J_1(x) \to 0$ and $N_1(ix) \sim \frac{1}{\pi}\frac{2}{ix}$ so

$$x K_1(x)\Big)_{x=0} = -\frac{i\pi}{2} \cdot \frac{1}{\pi}\frac{2}{i} = -1$$

The integral is -1 (!)

The sum is $$\frac{\varphi(L)}{q} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-)^n}{n}\left[\frac{e^{in\psi} - e^{-in\psi}}{2i}\right]$$

with $\psi = \pi z_0/L$.

$$\frac{\varphi(L)}{q} = \frac{2}{\pi}\,\mathrm{Im}\sum_{n=1}^{\infty}(-)^n\frac{e^{i\psi n}}{n} = \frac{2}{\pi}\,\mathrm{Im}\sum_{n=1}^{\infty}\frac{(-w)^n}{n}$$

$$= \frac{2}{\pi}\,\mathrm{Im}\ln\frac{1}{1+w}$$ (check: $\ln(1+w) = w - \frac{1}{2}w^2 + \dots$

Again, for $z = Re^{i\phi}$, $\ln z = \ln R + i\phi$, $\mathrm{Im}\ln z = \phi$

$$\frac{1}{1+e^{i\psi}} = \frac{1+e^{-i\psi}}{1+e^{-i\psi}}\,\frac{1}{1+e^{i\psi}} = \frac{1+e^{-i\psi}}{\text{something real}} = \frac{1+\cos\psi - i\sin\psi}{}$$

$$\tan\phi = \frac{\mathrm{Im}\frac{1}{1+w}}{\mathrm{Re}\frac{1}{1+w}} = -\frac{\sin\psi}{1+\cos\psi}.$$

Do you know your half angle formulas? $\frac{\sin\psi}{1+\cos\psi} = \tan\frac{\psi}{2}$

so $\phi = -\frac{\psi}{2} = -\frac{\pi}{2}\frac{z_0}{L}$

$$\frac{\varphi(L)}{q} = \frac{2}{\pi}\left(-\frac{\pi}{2}\frac{z_0}{L}\right) = -\frac{z_0}{L}$$!!!

In problem set 1, the reciprocity theorem gave us this result. To get it took less work, but more cleverness. Problem 3.20 is technically much more demanding, but it is also more straight forward. It also gives a series formula for σ (end of part b).

About the 4π on p. 3.20.1:

$$\int_{\varrho'-\epsilon}^{\varrho'+\epsilon} d\varrho \frac{d}{d\varrho}\varrho \frac{dg}{d\varrho} = -4\pi \int \delta(\varrho-\varrho')d\varrho' = -4\pi$$

$$\varrho \frac{dg}{d\varrho}\Big)_{\varrho'-\epsilon}^{\varrho'+\epsilon} = -4\pi$$

With $g_m(\varrho,\varrho') = A\, I_m(k\varrho_<)\, K_m(k\varrho_>)$, this is

$$-4\pi = A k\varrho' \left[\frac{dK_m(z)}{dz} I_m(z) - K_m(z)\frac{dI_m(z)}{dz}\right]$$

evaluated at $z = k\varrho'$.

To avoid a mess, notice that $k\varrho'[\;]$ must be independent of $k\varrho'$. Evaluate it where I_m and K_m are simple - at huge z. There

$$I_m(z) \sim \frac{e^z}{\sqrt{2\pi z}} \Rightarrow \frac{dI_m}{dz} = \frac{e^z}{\sqrt{2\pi z}} + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)$$

$$K_m(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \Rightarrow \frac{dK_m}{dz} = -e^{-z}\sqrt{\frac{\pi}{2z}}$$

$$-4\pi = A z \frac{1}{\sqrt{2\pi z}}\sqrt{\frac{\pi}{2z}}\left[-e^{-z}e^{z} - e^{-z}e^{z}\right]$$

$$= -\frac{A}{2}\cdot 2$$

$$A = 4\pi \;\; \text{!}$$