Set 4 – due 29 September

"You have to think until it hurts, and then keep going" – Heisenberg

1) Jackson 3.7. [10 points] (a)–5, (b)–5.

Two multipole problems, both involving quadrupoles. When I saw them the first time, I didn't understand the question, so here is a translation:

In these problems, the Cartesian quadrupole moments are defined through

$$Q_{ij} = \int d^3x \rho(x)[3x_i x_j - \delta_{ij} r^2], \tag{1}$$

There could be five of them (Q_{ij} is real – 9 entries, symmetric – 6 entries, traceless – one constraint). But with azimuthal symmetry, there can be only two nonzero ones. You can find an axis where Q_{ij} is diagonal, and the phrase "Nucleus with a quadrupole moment Q" means $Q_{33} = eQ = -2Q_{11} = -2Q_{22}$.

In Prob. 4.6, the surface of the nucleus is an ellipsoid. Its semimajor axis (of radius a) is oriented along the z axis, and its semiminor axis (of radius b) lies in the $x - y$ plane. The surface is given by

$$1 = \frac{z^2}{a^2} + \frac{\rho^2}{b^2} \tag{2}$$

where $\rho^2 = x^2 + y^2$. Also, read carefully the remarks (and footnotes!) on pp. 150-151.

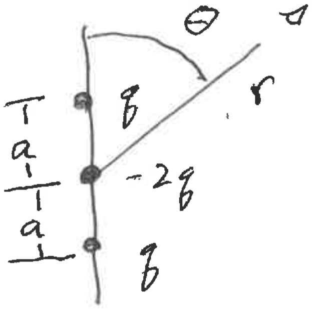
In Prob. 4.7, you will have to re-insert units to get a number.

2) Jackson 4.6 [20 points] (a)–5, (b)–5, (c)–10.

3) Jackson 4.7 [20 points]. (a)–7 (b)–7 (c)–6

4) [10 points] Show that the Green's function for the sphere, eq. 2.17, is equivalent to the "shell expression", eq. 3.125, in the limit that the radius of the inner shell is taken to zero. In the former expression, x and x' live inside a sphere of radius a, so in the latter expression, remember this, and relabel "b" as "a" for consistent notation to 2.17.

1) Jackson 3.7 – a quadrupole inside a grounded sphere.



a) $$\Phi = \frac{1}{4\pi\epsilon_0}\sum_j \frac{q_j}{|\vec{r}-\vec{r}_j|}$$

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r}+\hat{z}a|} + \frac{1}{|\vec{r}-\hat{z}a|}\right]$$

so expanding

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \sum_{\underline{l\ \text{even}}} \frac{a^l}{r^{l+1}} P_l(\cos\theta) + \frac{a^l}{r^{l+1}} P_l(\cos(\pi-\theta))\right]$$

The odd l's cancel because $P_l(\cos(\pi-\theta)) = (-)^l P_l(\cos\theta)$.

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{2}{r} + \frac{a}{r^2}(\cos\theta - \cos\theta) + \frac{2a^2}{r^3}P_2(\cos\theta) + \cdots\right]$$

so as $a \to 0$, $\Phi \to \frac{Q}{4\pi\epsilon_0 r^3} P_2(\cos\theta)$, $Q = 2qa^2$.

b) To deal with the grounded sphere, $\Phi(r=b)=0$, add a solution of $\nabla^2\Phi = 0$ which is regular at the origin, $\Phi_b = \sum_l A_l r^l P_l(\cos\theta)$

and find the A_l's from $\Phi + \Phi_b = 0$ at $r=b$.

This can be achieved setting $A_l = 0$ for all $l \neq 2$,

$$A_2 b^2 + \frac{Q}{4\pi\epsilon_0 b^3} = 0 \qquad (l=2) \text{ so}$$

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r^3} - \frac{r^2}{b^5}\right] P_2(\cos\theta).$$

If we had not taken a to zero, the part (a) solution would be

$$\Phi_a = \frac{2q}{4\pi\epsilon_0} \sum_{l=2,4,6\ldots} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

and to make $\Phi(r=b)=0$, just write

$$\Phi = \frac{2q}{4\pi\epsilon_0} \sum_{l \text{ even}} \left[\frac{a^l}{r^{l+1}} - \frac{a^l}{b^{l+1}}\left(\frac{r}{b}\right)^l \right]$$

$$\times P_l(\cos\theta).$$

4.6) We begin with eq. 4.24

$$W = \dots - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_i}{\partial x_j}$$

$$Q_{ij} = \int d^3x\, \rho(x)\left[3x_i x_j - \delta_{ij} r^2\right]$$

$Q = \frac{1}{e} Q_{33}$ – see the eighth line on p. 151.

$\vec{E}$ is an external electric field so $\vec{\nabla}\cdot\vec{E} = 0$.

If $\vec{E}$ is axi-symmetric about the z-axis, $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$ and $\nabla\cdot E = 0$ says $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2}\frac{\partial E_z}{\partial z}$.

The hint in the footnote on p. 151 tells us the nucleus is azimuthally symmetric, and Q is traceless, so $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33} = -\frac{1}{2}eQ$.

Eq 4.24 becomes

$$W = -\frac{1}{6}\left[Q_{xx}\frac{\partial E_x}{\partial x} + Q_{yy}\frac{\partial E_y}{\partial y} + Q_{zz}\frac{\partial E_z}{\partial z}\right]$$

$$= -\frac{1}{6}\left[\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\times 2 + 1\right] eQ\frac{\partial E_z}{\partial z} = -\frac{eQ}{4}\frac{\partial E_z}{\partial z}$$

b) $$\frac{\partial E_z/\partial z}{e/a_0^3} = \frac{-4W/(eQ)}{e/a_0^3} = -\frac{4W}{Q}\frac{a_0^3}{e^2}$$

For a CGS number, multiply & divide by $\hbar c$

$$\left(-\frac{4W}{h}\right)\frac{1}{Q}\frac{2\pi\hbar}{e^2}\frac{a_0^3}{c} = \left(\frac{4\times 10^7}{sec}\right)\frac{2\pi\times 137\times(0.5\times 10^{-8}\text{ cm})^3}{2\times 10^{-24}\text{ cm}^2\times 3\times 10^{10}\text{ cm/sec}}$$

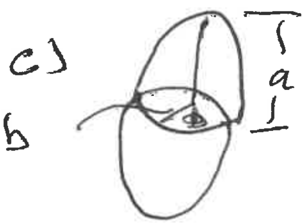
$$= 0.085$$

In MKS $$\frac{\partial E_z/\partial z}{\frac{e}{4\pi\epsilon_0 a^3}} = \frac{-\left(\frac{4W}{eQ}\right)}{\frac{e}{4\pi\epsilon_0 a^3}}$$

$$= -\frac{4W}{Q}\,\frac{4\pi\epsilon_0}{e^2}\,\frac{\hbar c}{\hbar c}\,a_0^3 = -\frac{4W}{hc}\,\frac{1}{Q}\,2\pi\left[\frac{4\pi\epsilon_0\hbar c}{e^2}\right]a_0^3$$

The same number as in cgs since $\frac{1}{137} = \frac{e^2}{\hbar c}\frac{1}{4\pi\epsilon_0}$.

c)



The surface of an ellipsoid of semi-major axis "a" along z and semi-minor axis "b" in the x-y plane is

$$1 = \frac{z^2}{a^2} + \frac{\rho^2}{b^2} \quad \text{with } \rho = \sqrt{x^2+y^2}.$$

The volume is

$$V = 2\pi\int_{-a}^{a} dz \int_0^{b\sqrt{1-z^2/a^2}} \rho\, d\rho = \pi\int_{-a}^{a} dz\, b^2\left(1-\frac{z^2}{a^2}\right)$$

$$= \pi b^2\left(2a - \frac{2}{3}a\right) = \frac{4\pi}{3}ab^2.$$

The charge density is $\hat{\rho}(x) = \frac{eZ}{\left(\frac{4\pi}{3}a^2b\right)} = \frac{3}{4\pi}\frac{eZ}{a^2b}$.

$$Q_{33} = \int d^3x\, \hat{\rho}(x)\left[3z^2 - r^2\right]$$

$$= \frac{3}{4\pi}\frac{eZ}{ab^2}\cdot 2\pi\int_{-a}^{a} dz \int_0^{b\sqrt{1-z^2/a^2}} \rho\, d\rho\,\left[2z^2-\rho^2\right]$$

$$Q = \frac{Q_{33}}{e} = \frac{3}{4}\frac{Z}{ab^2}\frac{2\pi}{\pi}\int_{-a}^{a} dz\left[z^2\rho^2 - \frac{1}{4}\rho^4\right]_0^{b\sqrt{1-z^2/a^2}}$$

$$= \frac{3}{2}\frac{Z}{ab^2}\int_{-a}^{a} dz\left[b^2z^2 - \frac{b^2}{a^2}z^4 - \frac{b^4}{4} + \frac{1}{2}\frac{b^4z^2}{a^2} - \frac{1}{4}\frac{b^4z^4}{a^4}\right]$$

$$Q = \frac{3}{2}\frac{Z}{a}\left\{\frac{2}{3}a^3 - \frac{2}{5}a^3 - \frac{b^2 a}{2} + \frac{1}{3}b^2 a - \frac{1}{10}b^2 a\right\}$$

$$\{\ \} = \frac{4}{15}a^3 + a^2 b\left[-\frac{15}{30} + \frac{10}{30} - \frac{3}{30}\right]$$

$$= \frac{4}{15}a\,(a^2 - b^2)$$

so $$Q = \frac{4}{15}\cdot\frac{3}{2}Z(a^2-b^2) = \frac{2}{5}Z(a^2-b^2)$$

Sanity check: if the nucleus is spherical, $a = b$ and $Q = 0$.

Write Q as

$$Q = \frac{4}{5}Z\left(\frac{a+b}{2}\right)(a-b)$$

where $R = \dfrac{a+b}{2}$

$$Q = \frac{4}{5}ZR(a-b) = \frac{4}{5}ZR^2\left(\frac{a-b}{R}\right)$$

so $$\frac{a-b}{R} = \frac{5}{4}\frac{Q}{Z}\frac{1}{R^2} = \frac{5}{4}\,\frac{2.5\times10^{-24}\,cm^2}{63\times 49\times10^{-26}\,cm^2}$$

$$= 0.10$$

4.7 $\rho(r) = \frac{1}{64\pi} r^2 e^{-r} \sin\theta$

a) It's very useful to write

$$\sin^2\theta = 1-\cos^2\theta = -\frac{2}{3}\left[\frac{3\cos^2\theta - 1}{2}\right] + \frac{2}{3}$$

$$= -\frac{2}{3} P_2(\cos\theta) + \frac{2}{3} P_0(\cos\theta)$$

$$= -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\, Y_2^0(\theta,\varphi) + \frac{2}{3}\sqrt{4\pi}\, Y_0^0(\theta,\varphi)$$

Consequently $q_{\ell m} = \int Y_\ell^m(\theta,\varphi)^* r^\ell \rho(r)\, dr$

only has a q_{20} term and a q_{00} term.

$$q_{00} = \frac{2}{3}\sqrt{4\pi} \int_0^\infty \frac{r^2 dr}{64\pi} \cdot r^2 e^{-r} = \frac{1}{\sqrt{4\pi}} q$$

$$q_{20} \equiv \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}} \int_0^\infty r^2 dr \cdot r^2 \cdot \left[\frac{r^2 e^{-r}}{64\pi}\right]$$

which appear in Φ from eq. 4.10

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{1}{2} Q_{33} \frac{z^2}{r^3} + \cdots\right].$$

We use $n! = \int_0^\infty x^n e^{-x} dx$ to evaluate

$$q = 4\pi \cdot \frac{2}{3} \cdot \frac{1}{64\pi} \times 4! = \frac{2}{3}\frac{1}{16} 24 = \underline{1}$$

$$Q_{33} = \frac{4\pi}{5}\left(-\frac{4}{3}\right)\frac{1}{64\pi} \cdot 6! = \frac{-6!}{3\cdot 4\cdot 5} = -2\cdot 6 = -12$$

all "in units of e and a_0 to the appropriate power."

Now we know $\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')\,d^3x'}{|\vec{x}-\vec{x}'|}$

and $\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell m} \frac{r_<^\ell}{r_>^{\ell+1}}\left(\frac{4\pi}{2\ell+1}\right) Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$.

The $\ell=0$ piece is

$$4\pi\epsilon_0 \Phi_{\ell=0}(r) = \frac{2}{3}\int\frac{d\Omega}{4\pi}\cdot 4\pi \cdot \left\{ \int_0^r r'^2 dr' \frac{1}{r}\left(\frac{r'^2 e^{-r'}}{64\pi}\right) + \int_r^\infty r'^2 dr' \frac{1}{r'}\left(\frac{r'^2 e^{-r'}}{64\pi}\right)\right\}$$

We want $r=0$. Let's do this in 2 ways. First, just set $r=0$ in the integral, naively dropping the first term $\int_0^0 dr'$. Then

$$\lim_{r\to 0} 4\pi\epsilon_0 \Phi_{\ell=0}(r) = \frac{2}{3}\cdot\frac{4\pi}{64\pi}\int_0^\infty r'^3 dr'\, e^{-r'}$$
$$= \frac{2}{3}\cdot\frac{1}{16}\cdot 3! = \frac{1}{4}.$$

That seemed too easy! As a check, write

$$\frac{1}{r}\int_0^r r'^4 e^{-r'} dr' + \int_r^\infty r'^3 e^{-r'} dr'$$
$$= \int_0^\infty r'^3 e^{-r'} dr' + \int_0^r dr' e^{-r'}\left[\frac{r'^4}{r} - r'^3\right]$$

adding and subtracting. The first term is the naive answer. In the new 2nd term, write $e^{-r'} = 1 - r' + \cdots$ since r' is small over the range of the $\int$,

$$\frac{1}{r}\int_0^r dr'\left[1 - r' + \cdots\right]\left[r'^4 - r r'^3\right]$$
$$= \frac{1}{r}\left[\frac{r^5}{5} - \frac{r r^4}{4} - \frac{r^6}{6} + \frac{r r^5}{5} + \cdots\right]$$ which goes as r^4 for small r - we can neglect it.

The $l=2$ term is

$$-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\int d\Omega'\, Y_2^0(\Omega')^* Y_2^0(\Omega)$$

$$\times \frac{4\pi}{(2\cdot 2+1)}\sqrt{\frac{5}{4\pi}}\, P_2(\cos\theta)\cdot\frac{r^2}{64\pi}\int_0^\infty \frac{(r')^4}{(r')^3}\,dr'\,e^{-r'}$$

$(r_> \text{ is } r' \in [0,\infty)]$

$$= -\frac{2}{3}\cdot\frac{1}{5}\cdot\frac{1}{16}\cdot r^2 P_2(\cos\theta)$$

so $\quad 4\pi\epsilon_0\, \Phi_{l=2} = -\frac{1}{120} r^2 P_2(\cos\theta)$

c) As in problem 4-6 "Nucleus with Q" means

$$Q_{33} = -2Q_{11} = -2Q_{22} = eQ$$

$$W = -\frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_i}{\partial x_j}$$

$P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2}$ so $-\frac{1}{120} r^2 P_2 = -\frac{[2z^2 - x^2 - y^2]}{240}$

$E_z = \frac{1}{4\pi\epsilon_0}\cdot\frac{4z}{240}$, $E_x = -\frac{1}{4\pi\epsilon_0}\cdot\frac{2x}{240}$, $E_y = -\frac{1}{4\pi\epsilon_0}\frac{2y}{240}$

$$W = -\frac{1}{4\pi\epsilon_0}\frac{1}{6}\left[Q_{33}\cdot\frac{1}{60} + (Q_{11}+Q_{22})\left(-\frac{1}{120}\right)\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{eQ}{360}\left[1 - 2\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$= \frac{-eQ}{4\pi\epsilon_0}\frac{1}{360}\cdot\frac{3}{2} = \boxed{\frac{-eQ}{4\pi\epsilon_0}\frac{1}{240}}$$

Now to clean up the units

$$W = \frac{q^2}{\text{distance}} \quad \text{in CGS}$$

$[Q] = [\text{length}]^2$ so we need an overall factor of $\frac{e}{a_0^3}$

$$\left[\frac{W}{h}\right] = \frac{1}{4\pi\epsilon_0}\,\frac{e^2}{2\pi\hbar c}\cdot c\cdot\frac{Q}{a_0^3}\times\frac{1}{240}$$

In CGS drop the $\frac{1}{4\pi\epsilon_0}$, use $\frac{e^2}{\hbar c} = \frac{1}{137}$

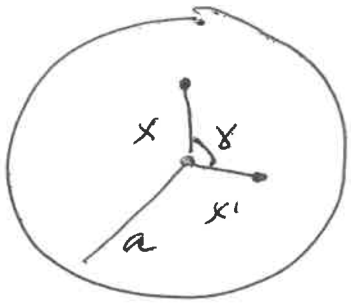
$$\frac{W}{h} = \frac{1}{2\pi}\,\frac{1}{137}\,\frac{1}{240}\,\frac{Q = 10^{-24}\ \text{cm}^2}{(0.53\times 10^{-8}\ \text{cm} = a_0)^3}$$

$$\times\ 3\times 10^{10}\ \text{cm/sec}$$

$$\sim 10^{6}\ \text{sec}^{-1}$$

4) Show that the "interior" Green's function, eq. 2.17, is equal to 3.125. (Exchange "b" for "a" appropriately to make the notation consistent.)

For the sphere $G = \dfrac{1}{|\vec{x}-\vec{x}'|} - \dfrac{a}{x'} \dfrac{1}{\left|\vec{x} - \dfrac{a^2}{x'^2}\vec{x}'\right|}$



Inside the sphere $\dfrac{x^2}{a^2}$ and $\dfrac{x'^2}{a^2}$ are <1 so $\dfrac{x^2 x'^2}{a^2} < a^2$

or $x^2 < \dfrac{a^4}{x'^2}$ or $|\vec{x}| < \left|\dfrac{a^2}{x'^2}\vec{x}'\right|$.

We can expand

$$G(x,x') = 4\pi \sum_{\ell m} \frac{Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)}{2\ell+1} \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{a}{x'} \frac{x^\ell}{\left(\frac{a^2}{x'}\right)^{\ell+1}} \right\}$$

$$\{\ \} = \frac{r_<^\ell}{r_>^{\ell+1}} - \left(\frac{x x'}{a^2}\right)^\ell \frac{1}{a}, \text{ so}$$

(relabel $a \to b$)

$$G(x,x') = 4\pi \sum_{\ell m} \frac{Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)}{2\ell+1} \cdot \left\{ \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r_<^\ell r_>^\ell}{b^{2\ell+1}} \right\}$$

which is 3.125.