Set 3 – due 22 September

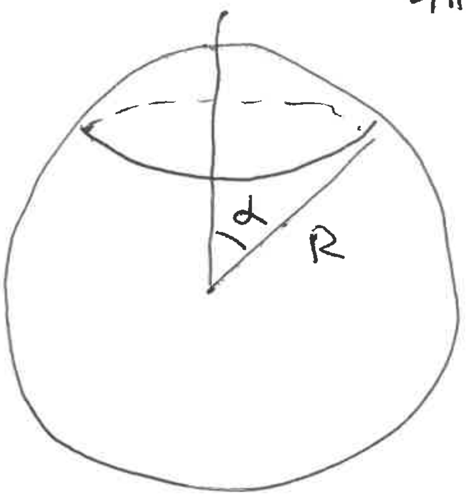
"The Fox knows many things, but the Hedgehog knows one great thing" – Archilochus

1) Jackson 3.2. [20 points] (a)–10, (b)–5, (c)–5. Expand Φ in Legendre polynomials so as not to duplicate the next problem.

2) [20 points] Find the Green's function appropriate to the geometry and boundary conditions of Problem 1, and use this Green's function to re-derive Φ in part (a).

3) Jackson 3.3 [25 points] Believe it or not, it is possible to get a closed form solution for $\Phi(z)$. I suggest a close examination of the inverse trig function section of your integral table. But physical reasoning can get you the leading term, too. (a)-10, (b)–10, (c)–5.

1) 3.2 $\sigma = \frac{Q}{4\pi R^2}$ even though the area depends on α!



$$\Phi_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \quad r<R$$

$$\Phi_{out} = \sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad r>R$$

Match boundary values at $r=R$:

$\Phi_{in}(R) = \Phi_{out}(R)$

1) $$\sum_\ell \left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}}\right] P_\ell(\cos\theta) = 0 \Rightarrow B_\ell = A_\ell R^{2\ell+1}$$

2) $$\sigma(\theta) = -\epsilon_0 \left[\frac{\partial \Phi_{out}}{\partial r} - \frac{\partial \Phi_{in}}{\partial r}\right]_{r=R}$$

$$= -\epsilon_0 \sum_\ell \left[\ell A_\ell R^{\ell-1} + \frac{(\ell+1) B_\ell}{R^{\ell+2}}\right] P_\ell(\cos\theta)$$

$$= -\epsilon_0 \sum_\ell (2\ell+1) A_\ell R^{\ell+1} P_\ell(\cos\theta).$$

Solve for the A_ℓ's using the orthogonality of the P_ℓ's

$$A_\ell = -\left[\frac{2\ell+1}{2}\right] \int_{-1}^{1} d\cos\theta \, P_\ell(\cos\theta) \frac{\sigma(\theta)}{\epsilon_0} \cdot \frac{1}{(2\ell+1) R^{\ell-1}}$$

$\sigma = \frac{Q}{4\pi R^2} \Theta(\theta - \alpha)$, so

$$A_\ell = -\frac{Q}{8\pi\epsilon_0} \frac{1}{R^{\ell+1}} \int_{-1}^{\cos^{-1}\alpha} d\cos\theta \, P_\ell(\cos\theta)$$

Jackson eq. 3.1.8 tells us that

$$P_\ell(\cos\theta) = \frac{1}{2\ell+1}\left[\frac{dP_{\ell+1}(\cos\theta)}{d\cos\theta} - \frac{dP_{\ell-1}(\cos\theta)}{d\cos\theta}\right]$$

so the integral is just

$$A_\ell = \frac{Q}{8\pi\epsilon_0 R^{\ell+1}}\frac{1}{(2\ell+1)}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]$$

and (this is part (a)) –

$$\left\{\begin{matrix}\Phi_{in}\\ \Phi_{out}\end{matrix}\right. = \frac{Q}{8\pi\epsilon_0}\sum_\ell \frac{1}{(2\ell+1)}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]$$

$$\times P_\ell(\cos\theta) \times \left\{\begin{matrix}\dfrac{r^\ell}{R^{\ell+1}} & \text{in}\\ \dfrac{R^\ell}{r^{\ell+1}} & \text{out}\end{matrix}\right.$$

b) $\Phi_{in}(r,\theta)$ gives $\vec{E}(r)$. At small r, symmetry says $\vec{E}$ has to point towards the spherical cap on the north pole. For small r, this means that only the monopole and dipole terms of Φ are important. The monopole is Φ = constant, so

$$E_r = -\frac{\partial\Phi_{in}}{\partial r}\Big)_{r=0} = -\frac{Q}{8\pi\epsilon_0}\sum_{\ell=1}\frac{1}{(2\ell+1)}\frac{\ell r^{\ell-1}}{R^{\ell+1}}\cdot P_\ell(\cos\theta)$$
$$\times\left[P_{\ell+1} - P_{\ell-1}\right]$$

$$\simeq -\frac{Q}{8\pi\epsilon_0}\frac{1}{3R^2}\left[P_2(\cos\alpha) - 1\right]\cos\theta$$

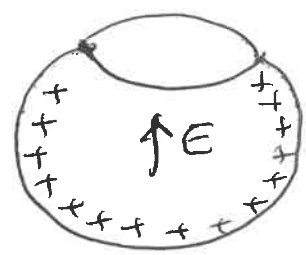
$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{Q}{8\pi\epsilon_0}\frac{1}{R^2}\sin\theta\left[P_2(\cos\alpha) - 1\right]$$

a- just look at $\Phi_{in} = \dfrac{Q}{8\pi\epsilon_0}\dfrac{[P_2 - 1]}{3R^2}\, r\cos\theta$ -

$r\cos\theta = z$ so $\Phi \propto z$ and

$$\vec{E} = -\frac{\hat{z}\,Q}{24\pi\epsilon_0}\,\frac{[P_2(\cos\alpha) - 1]}{R^2} \quad !$$

b) Note $[\;] < 0$



c) $$\frac{3\cos^2\alpha - 1}{2} - 1 = \frac{3\cos^2\alpha - 3}{2} = -\frac{3}{2}\sin^2\alpha \simeq -\frac{3}{2}\alpha^2$$

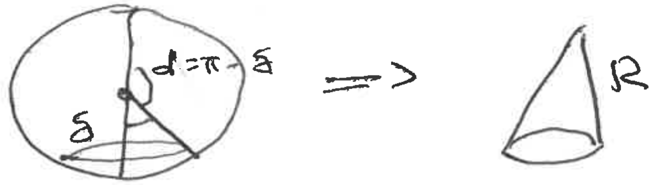
(if $\alpha \ll 1$)

so as $\alpha \to 0$ $$\vec{E} = \hat{z}\,\frac{Q}{R^2}\cdot\frac{3}{2}\,\frac{\alpha^2}{24\pi\epsilon_0} = \hat{z}\,\frac{Q\alpha^2}{16\pi\epsilon_0 R^2}$$

In the limit $\alpha \to 0$, $\vec{E}$ vanishes - not a surprise - think about it!

At large α $$\hat{E} = \frac{\hat{z}}{16\pi\epsilon_0}\,\frac{Q}{R^2}\sin^2(\pi - \delta) \simeq \frac{Q}{16\pi\epsilon_0}\,\frac{\delta^2}{R^2}\,\hat{z}$$

writing $\alpha = \pi - \delta$:



: note Φ & $\vec{E}$ are both $\propto \delta^2$.

Why the 16?

The 16 is because $\sigma = \frac{Q}{4\pi R^2}$, $Q_{TOT} = \int \sigma \, dA$

$$Q_{TOT} = \frac{Q}{4\pi R^2} \cdot 2\pi R^2 \int_\alpha^\pi \sin\theta \, d\theta = \frac{Q}{2}\left[1 + \cos\alpha\right]$$

If $\alpha = \pi - \delta$, $1 + \cos(\pi - \delta) = 1 - \cos\delta = \frac{1}{2}\delta^2$

so $Q_{TOT} = \frac{Q}{4}\delta^2$ and $\vec{E} = \hat{z}\,\frac{Q_{TOT}}{4\pi\epsilon_0 R^2}$.

This is what is expected! In the limit

$$\lim_{\alpha \to 0}\left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right]$$

$$= P_{\ell+1}(1) - P_{\ell-1}(1) = 1 - 1 = 0$$ for all

ℓ except $\ell = 0$. By definition, $P_{-1}(1) = -1$ (!)

so in Φ, only the monopole survives

$$\lim_{\alpha \to 0} \Phi = \frac{Q}{4\pi\epsilon_0}\frac{1}{R}.$$

As $\alpha \to \pi$ $P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) = (-)^{\ell+1} - (-)^{\ell-1}$

(if $\ell \neq 0$) and this again is zero, so

in this limit there is again only a monopole

2) Jackson 3.2 by Green's function. We need a surface on which either $\Phi(x)$ or $\frac{\partial \Phi}{\partial n}$ is specified. The easiest surface to take is the one at spatial infinity, where $\Phi = 0$. This gives us a Dirichlet problem:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G_D(x, x') - \frac{1}{4\pi}\int dA'\, \Phi(x')\frac{\partial G}{\partial n'}$$

and the 2nd term is zero. Also

$$\rho(x') = \sigma(\theta)\,\delta(r' - R)$$

$$G_D(x, x') = \frac{1}{|\vec{x} - \vec{x}'|} \quad \text{or}$$

$$\frac{1}{|x - x'|} = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{(2\ell+1)} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$$

$r_<$ and $r_>$ are the lesser and greater of r and R, $\Omega = (\theta, \varphi)$, $\Omega' = (\theta', \varphi')$. This is still overkill, since the problem has azimuthal symmetry, only $m = 0$ terms contribute. And

$$Y_\ell^{m=0}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)\, P_\ell(\cos\theta')$$

Then $$\Phi_{in} = \frac{1}{4\pi\epsilon_0}\int r'^2 dr' d\Omega' \, \delta(r'-R) \times \sigma(\theta') \times \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta) \times P_\ell(\cos\theta')$$

$$= \frac{2\pi}{4\pi\epsilon_0} \sum_\ell \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta) \int_0^\pi \sin\theta' \, d\theta' \, \sigma(\theta') P_\ell(\cos\theta')$$

$$= \frac{Q}{8\pi\epsilon_0} \sum_\ell \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta) \int_{-1}^{\cos^{-1}\alpha} d\cos\theta \, P_\ell(\cos\theta)$$

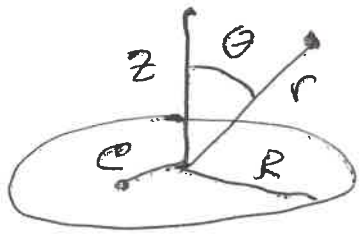
which is precisely what we had in problem 1.

No need to keep going!

$$\left\{ \begin{matrix} \Phi_{in} \\ \Phi_{out} \end{matrix} \right\} = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta)}{2\ell+1} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \right] \times \left\{ \begin{matrix} r^\ell / R^{\ell+1} \\ R^\ell / r^{\ell+1} \end{matrix} \right\}$$

Jackson 3.3 - more Legendres and a magic change of variables:

$$\sigma = \frac{k}{\sqrt{R^2-\rho^2}} \qquad \Phi(r,\theta) = \sum_{l} \frac{b_l}{r^{l+1}} P_l(\cos\theta)$$



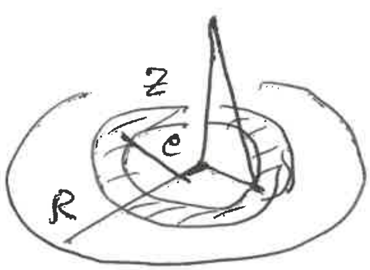
To find k, compute the potential at the center of the disc

$$V = \Phi(\rho=0) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma\, dA}{r} \Bigg)_{r=\rho}$$

$$= \frac{k}{4\pi\epsilon_0} \int_0^R \frac{2\pi\rho\, d\rho}{\rho} \frac{1}{\sqrt{R^2-\rho^2}} = \frac{2\pi k}{4\pi\epsilon_0} \sin^{-1}\frac{R}{R} \text{ (tables)}$$

$$= \frac{\pi^2 k}{4\pi\epsilon_0} \quad \text{so} \quad k = \frac{4\epsilon_0}{\pi} V$$

Next, find Φ along the z-axis



$$\Phi(z) = \frac{2\pi}{4\pi\epsilon_0} \int_0^R \frac{\rho\, d\rho\, \sigma(\rho)}{\sqrt{\rho^2+z^2}}$$

$$\Phi(z) = \frac{2\pi}{4\pi\epsilon_0} \cdot \frac{4\epsilon_0 V}{\pi} \int_0^R \frac{\rho\, d\rho}{\sqrt{\rho^2+z^2}} \frac{1}{\sqrt{R^2-\rho^2}}$$

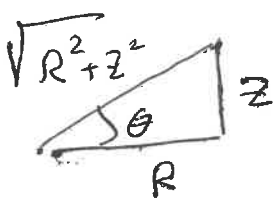
Try $u^2 = \rho^2 + z^2$ so $u\,du = \rho\,d\rho$

$$\Phi(z) = \frac{2V}{\pi}\int_{z}^{\sqrt{z^2+R^2}} \frac{u\,du}{u}\,\frac{1}{\sqrt{R^2+z^2-u^2}}$$

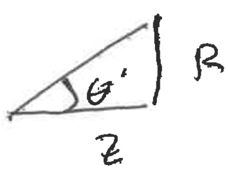
$$= \frac{2V}{\pi}\sin^{-1}\frac{u}{\sqrt{R^2+z^2}}\Bigg|_{u=|z|}^{u=\sqrt{R^2+z^2}}$$

$$\Phi = \frac{2V}{\pi}\left[\sin^{-1}(1) - \sin^{-1}\frac{|z|}{\sqrt{R^2+z^2}}\right]$$

Note the $|\;|$!



$$\Rightarrow [\;] = \frac{\pi}{2} - \sin^{-1}\frac{|z|}{\sqrt{R^2+z^2}} = \frac{\pi}{2} - \theta$$



$$= \tan^{-1}\frac{R}{z}$$ – look closely!

so, very improbably $\Phi(z) = \frac{2V}{\pi}\tan^{-1}\frac{R}{|z|}$

To go off axis, use the Great Trick – expand

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty}(-)^n\frac{x^{2n+1}}{2n+1}$$

$$\Phi(z) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}\left(\frac{R}{|z|}\right)^{2n+1}$$

Then $\left(\frac{R}{|z|}\right)^{2n+1} = \frac{R}{|z|}\left(\frac{R^2}{z^2}\right)^n$ and

a) $\Phi(r,\theta)$ for $r > R = \frac{2V}{\pi}\frac{R}{r}\sum_{\ell=0}^{\infty}\frac{(-)^\ell}{2\ell+1}\left(\frac{R^2}{r^2}\right)^\ell$

$\times P_{2\ell}(\cos\theta)$

(filling in $\frac{P_n}{r^{n+1}}$ ... note even powers only)

For $z \ll R$ $\quad \tan^{-1}\frac{R}{z} = \frac{\pi}{2} - \frac{z}{R} + \frac{1}{3}\left(\frac{z}{R}\right)^3 + \cdots$

$$\Phi(z) = \frac{2V}{\pi}\left[\frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}\left(\frac{|z|}{R}\right)^{2n+1}\right]$$

$$= V - \frac{2V}{\pi}\sum_{\ell=0}\frac{(-)^\ell}{2\ell+1}\left(\frac{r}{R}\right)^{2\ell+1}P_{2\ell+1}(|\cos\theta|)$$

Here there are odd powers (and absolute value) – $r^\ell P_\ell(\cos\theta)$-like

Note at $r=0$ $\Phi = V$!

To make sense of this, look at small z $(z>0)$

$$\Phi(z) = V - \frac{2}{\pi}V\frac{z}{R} + \cdots$$

For a flat plate, Gauss' law says $2EA = \frac{\sigma A}{\epsilon_0}$,

$E = \frac{\sigma}{2\epsilon_0}$, $\Phi = -\frac{\sigma z}{2\epsilon_0}$, and at $\rho = 0$ $\sigma = \frac{K}{R} = \frac{4\epsilon_0 V}{\pi R}$ so

$$\Phi = -\frac{2Vz}{\pi R} + \text{constant}. \quad -\underbrace{\left(\frac{4\epsilon_0 V}{\pi R}\right)}_{\sigma}\frac{z}{2\epsilon_0}$$

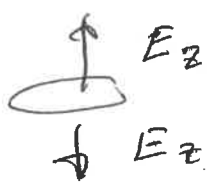
This checks. and E_z is + if $z > 0$.

But the $|z|$! at $z < 0$,

$$\Phi = - \frac{2V|z|}{\pi R} = \frac{2Vz}{\pi R}$$

since $z = -|z|$

$$-\frac{\partial \Phi}{\partial z} = -\frac{2V}{\pi R}$$ is negative;



is the (correct) picture

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c) $CV = Q$, $Q = \frac{4\epsilon_0 V}{\pi}\int_0^R \frac{2\pi\rho\, d\rho}{\sqrt{R^2-\rho^2}}$

$$Q = -8\epsilon_0 V \cdot \frac{2}{2}\sqrt{R^2-\rho^2}\,\Big|_0^R = 8\epsilon_0 V R$$

$$\underline{C = 8\epsilon_0 R}$$

This problem seems very contrived - how would you know $\sigma(\rho)$ in the first place? There is a cryptic discussion on p. 135 of Jackson...

$$\frac{2\rho\, d\rho}{\sqrt{R^2-\rho^2}} = \frac{-dx}{\sqrt{x}} = 2\sqrt{x} = -4\epsilon_0 V \cdot 2\sqrt{x}\,\Big|_R^0$$