Set 2– due 15 September

"I don't see why you are talking about this problem when either of you is capable of sitting down and solving it." – H. Bethe, to his students

Several problems involving images.

1) Jackson 2.1 [15 points] This should just be a review... (a)-3, (b)-2 (c)-3, (d)-2, (e)-2, (f)-3

2) Jackson 2.7 [20 points] Not quite identical to the previous one. (a)-3, (b)-3, (c)-4, (d)-10

3) Jackson 2.11 [20 points] Do this one before trying the next one! (a+b)-12, (c)-5, (d) 3.

A useful hint: if the quantity

$$\frac{A + B\cos\theta}{C + D\cos\theta} \tag{1}$$

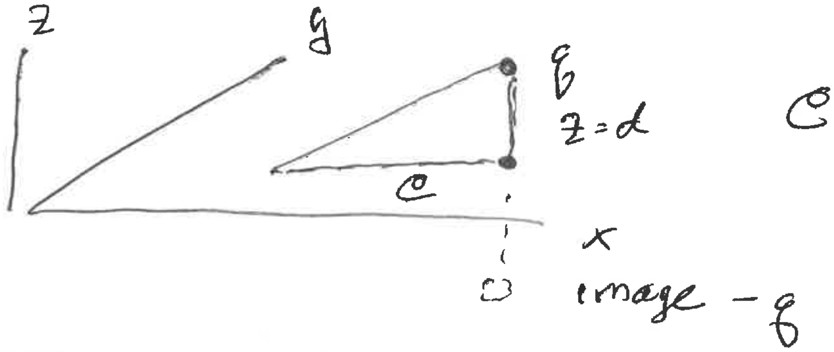
is supposed to be independent of θ, then $A/C = B/D$.

4) Jackson 2.8 parts a-b only [20 points]. I found it hard to get the answer quoted in the text. Here are two potentially useful hints:

Hint 1: If $x = \ln z$ and $\cosh^{-1} y = x$, then $y = (z + 1/z)/2$.

Hint 2: You probably have two equations giving d as a function of a, b, and the offsets of the image wires from the centers of the cylinders (call them d_1 and d_2). Try multiplying them together to give d^2. If you don't understand this, and can work the problem in some other way, ignore the hint! (The reason why Jackson's answer for the capacitance is desirable is that it only depends on simple paramters, the radii of the two cylinders and their separation.)

1) 2.1 – a classic image problem – point charge over a grounded plane

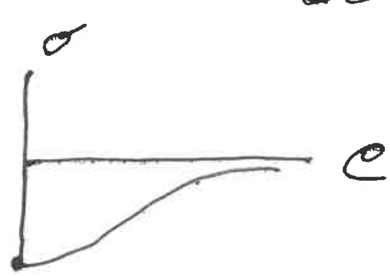


$$\rho^2 = x^2 + y^2$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[\rho^2 + (z-d)^2]^{1/2}} - \frac{1}{[\rho^2 + (z+d)^2]^{1/2}}\right]$$

a) $$\frac{\sigma}{\epsilon_0} = -\frac{\partial \Phi}{\partial z}\Big|_{z=0} = \frac{q}{4\pi\epsilon_0}\left[\frac{z-d}{[\rho^2 + (z-d)^2]^{3/2}} - \frac{z+d}{[\rho^2 + (z+d)^2]^{3/2}}\right]_{z=0}$$

$$\sigma = -\frac{qd}{2\pi}\frac{1}{[\rho^2 + d^2]^{3/2}} \Rightarrow$$



b) $$F = \frac{-q^2}{4\pi\epsilon_0 (2d)^2}$$ from the image – attractive, of course!

c) $$F = \int \frac{\sigma^2}{2\epsilon_0}\cdot 2\pi\rho\, d\rho = \frac{2\pi}{2\epsilon_0}\left(\frac{qd}{2\pi}\right)^2 \int_0^\infty \frac{\rho\, d\rho}{[\rho^2 + d^2]^3}$$

$$= \frac{(qd)^2}{4\pi\epsilon_0}\cdot\frac{1}{4d^4} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{(2d)^2}$$

This is the force on the plane, equal and opposite to the result in (b), as expected.

d) The force is attractive, positive work must be done to take the point charge to infinity

$$W = \frac{1}{4\pi\epsilon_0}\int_d^\infty \frac{q^2}{4x^2}\,dx = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

e) $P.E. = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{2d}$ between charge & image.

Is the work done to remove the charge $\frac{q^2}{8\pi\epsilon_0 d}$?

No! this quantity includes the work to remove the image. (d) is correct because the image moves for free.

f) $W = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4d}$ so $\frac{q}{4\pi\epsilon_0}\frac{1}{4d}$ in volts

gives W in eV. With $\frac{1}{4\pi\epsilon_0} = 10^{-7}c^2 = 9\cdot 10^9$,

all in MKS.

$$1.6\times 10^{-19}\,C \times \frac{9\times 10^9}{4\times 10^{-10}\,m} = \frac{9}{4}\times 1.6\ V = 3.6\,V$$

$W = 3.6$ eV

In CGS $\frac{1}{4d}\frac{e^2}{\hbar c}\hbar c = \frac{1}{4\mathring{A}}\frac{1}{137}\times 1970\ eV\text{-}\mathring{A}$

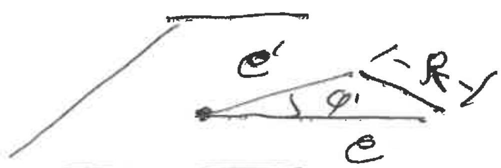
$= 3.6$ eV

2) Jackson 2.7 differs from 2.1 because $V \neq 0$ on the conducting surface. This is a Dirichlet problem – in 2.1, $\Phi = 0$ on the boundary, so the Φ of 2.1 is the Dirichlet G

a) $$G(x, x') = \frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}} - \frac{1}{[(x-x')^2 + (y-y')^2 + (z+z')^2]^{1/2}}$$

$$= \frac{1}{[R^2 + (z-z')^2]^{1/2}} - \frac{1}{[R^2 + (z+z')^2]^{1/2}}$$

$R^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\varphi'$, setting $\varphi = 0$



b)

$\hat{n}'$ is out of solution region $\quad \frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial z}$

$$\Phi(x) = -\frac{1}{4\pi} \int_S \Phi(x') \frac{\partial G}{\partial n'} dA'$$

$$= \frac{V}{4\pi} \int_0^{2\pi} d\varphi' \int_0^a \rho' d\rho' \cdot \frac{2z}{[R^2 + z^2]^{3/2}}$$

recycling prob 1, part b, $\frac{\partial G}{\partial n} \sim \sigma$

c) If $\rho = 0$ the integral is elementary

$$\Phi(z, \rho = 0) = \frac{V}{4\pi} \cdot 2z \cdot 2\pi \int_0^a \frac{\rho' d\rho'}{[\rho'^2 + z^2]^{3/2}}$$

$$= V \cdot \left[1 - \frac{z}{\sqrt{z^2 + a^2}} \right]$$

d) This is nasty. Expand in $(\rho^2 + z^2)$...

$$\frac{\rho'}{[\rho'^2 + \rho^2 - 2\rho\rho' \cos\varphi' + z^2]^{3/2}}$$

$$= \frac{\rho'}{[\rho^2 + z^2]^{3/2}} \left[1 + \frac{\rho'^2}{\rho^2 + z^2} \left(1 - 2\frac{\rho}{\rho'} \cos\varphi' \right) \right]^{-3/2}$$

$$= \frac{\rho'}{[\rho^2 + z^2]^{3/2}} \left[1 - \frac{3}{2} \frac{\rho'^2}{\rho^2 + z^2} \left(1 - \frac{2\rho}{\rho'} \cos\varphi' \right) + \frac{15}{8} \left(\frac{\rho'^2}{\rho^2 + z^2} \right)^2 \left(1 - \frac{2\rho}{\rho'} \cos\varphi' \right)^2 + \cdots \right]$$

This still looks awful, but note that the $\cos\varphi'$ terms integrate to zero, $\langle \cos^2\varphi' \rangle = 1/2$, so

$$\Phi(z, \rho) = \frac{2z \cdot V \cdot 2\pi}{4\pi} \left\{ \int_0^a \frac{\rho' d\rho'}{(\rho^2 + z^2)^{3/2}} - \frac{3}{2} \int_0^a \frac{\rho'^3 d\rho'}{(\rho^2 + z^2)^{5/2}} + \frac{15}{8} \frac{1}{(\rho^2 + z^2)^{7/2}} \int \rho'^5 d\rho' \left(1 + \frac{4\rho^2}{\rho'^2} \cdot \frac{1}{2} + \cdots \right) \right.$$

the $\rho^2 + z^2$'s all outside the $\int$, of course ...

$$\Phi = \frac{Vz}{(\rho^2+z^2)^{3/2}}\left\{\frac{a^2}{2} - \frac{3}{4\cdot 2}\frac{a^4}{(\rho^2+z^2)} + \frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\left(\frac{a^6}{6} + 2\,\frac{\rho^2 a^4}{4}\right) + \cdots\right\}$$

$$= \frac{Va^2 z}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3}{4}\frac{a^2}{(\rho^2+z^2)} + \frac{5}{8}\frac{(a^4+3a^2\rho^2)}{(\rho^2+z^2)^2} + \cdots\right]$$

which doesn't look useful. Jackson (and I) are setting you up for a revisit with better techniques.

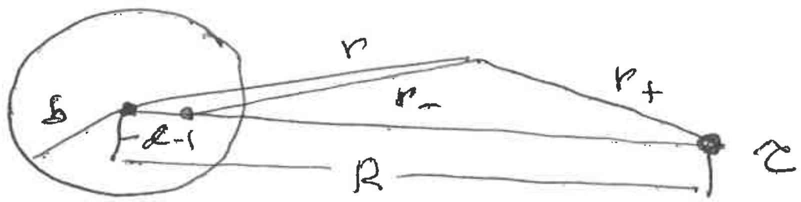
To compare with part (c), set $\rho = 0$

$$\Phi(z,0) = \frac{V}{2}\left(\frac{a}{z}\right)^2\left\{1 - \frac{3}{\cancel{2}4}\frac{a^2}{z^2} + \frac{5}{8}\left(\frac{a}{z}\right)^4 + \cdots\right\}$$

which is the Taylor expansion of

$$V\left[1 - \frac{1}{\sqrt{1+\left(\frac{a}{z}\right)^2}}\right] = V\left[\frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{3}{8}\left(\frac{a}{z}\right)^4 + \frac{5}{16}\left(\frac{a}{z}\right)^6 + \cdots\right]$$

3) 2.11 – Line charge, conducting cylinder, observer at (r, θ)



A line charge has a potential, a distance r away,

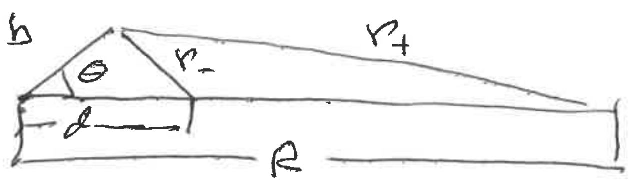
$$\Phi = -\frac{\tau}{2\pi\epsilon_0} \ln r + q \qquad q \text{ is a constant.}$$

To have $\Phi \to 0$ as $r \to \infty$, hide an image line charge $-\tau$ a distance d from the origin, and then

$$\Phi(r, \theta) = \frac{\tau}{2\pi\epsilon_0} \ln\left(\frac{r_-}{r_+}\right) \qquad (1)$$

combining true & image charges. Pick d so Φ is an equipotential on the $r = b$ cylinder. (This is a common motif of the last 2 problems.)

This means $\frac{r_-}{r_+} = \text{constant} \equiv k$ or $r_-^2 = r_+^2 k$



$$b^2 + d^2 - 2bd\cos\theta = k^2\left[b^2 + R^2 - 2bR\cos\theta\right]$$

or $$b^2 + d^2 - k^2(b^2 + R^2) - 2b(d - k^2 R)\cos\theta = 0$$

$k^2 = d/R$ to kill the cosine term

$$b^2 + d^2 - \frac{d}{R}(b^2 + R^2) = 0 \Rightarrow (d - R)\left(d - \frac{b^2}{R}\right) = 0$$

$\Rightarrow$ $\boxed{\text{part (a)} \quad d = b^2/R}$

b) eq (1) becomes $\Phi = \frac{\tau}{4\pi\epsilon_0} \ln \frac{r^2 + \left(\frac{b^2}{R}\right)^2 - 2r\frac{b^2}{R}\cos\theta}{r^2 + R^2 - 2rR\cos\theta}$

($d = b^2/R$, ~~law~~ law of cosines, ~~...~~ $\ln r = \frac{1}{2}\ln r^2 \ldots$)

$$\Phi = \frac{\tau}{4\pi\epsilon_0} \ln \frac{1 - \frac{2}{r}\frac{b^2}{R}\cos\theta + \left(\frac{b^2}{rR}\right)^2}{1 - 2\frac{R}{r}\cos\theta + \frac{R^2}{r^2}}$$

As $r \to \infty$, this is

$$\Phi(r,\theta) \to \frac{\tau}{4\pi\epsilon_0} \ln\left[1 + \frac{2}{r}\cos\theta\left(R - \frac{b^2}{R}\right) + \cdots\right]$$

$$\simeq \frac{\tau}{2\pi\epsilon_0}\frac{1}{r}\left(R - \frac{b^2}{R}\right)\cos\theta + \cdots$$

There must be an easier way to get this!

c) $\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial r}\right)_{r=b}$

$$= -\frac{\tau}{4\pi}\left\{\frac{2r - 2\frac{b^2}{R}\cos\theta}{r^2 + \left(\frac{b^2}{R}\right)^2 - 2r\frac{b^2}{R}\cos\theta} - \frac{2r - 2R\cos\theta}{r^2 + R^2 - 2Rr\cos\theta}\right\}_{r=b}$$

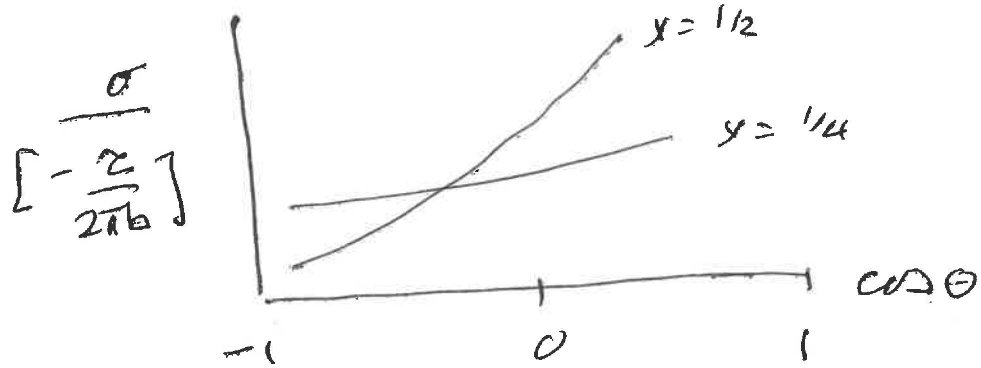
$$= \frac{-\tau}{4\pi}\left\{\frac{2b - 2\frac{b^2}{R}\cos\theta}{b^2 + \left(\frac{b^2}{R}\right)^2 - 2\frac{b}{R}b^2\cos\theta} - \frac{2b - 2R\cos\theta}{b^2 + R^2 - 2bR\cos\theta}\right\}$$

$$= -\frac{\tau}{4\pi}\frac{1}{b^2 + R^2 - 2bR\cos\theta}\left\{\frac{2bR^2 - 2b^2R\cos\theta}{b^2} - 2b + 2R\cos\theta\right\}$$

$$= \frac{-\tau}{2\pi b}\left[\frac{R^2 - b^2}{b^2 + R^2 - 2bR\cos\theta}\right]$$

To groom this, write $x = \frac{b}{R} = \frac{\text{radius of cylinder}}{\text{distance to line charge}}$

$$\sigma = -\frac{\tau}{2\pi b} \cdot \frac{1-x^2}{1-2x\cos\theta + x^2}$$



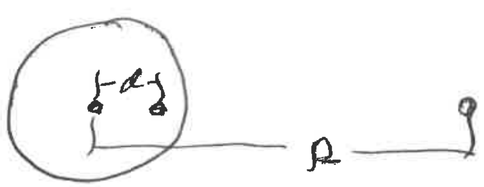
The smaller x is, the less charge piles up on one side

d) $\vec{E} = -\frac{\tau}{2\pi\epsilon_0} \frac{1}{r} \hat{r}$ for a line charge,

just from Gauss' law, so just look at the image

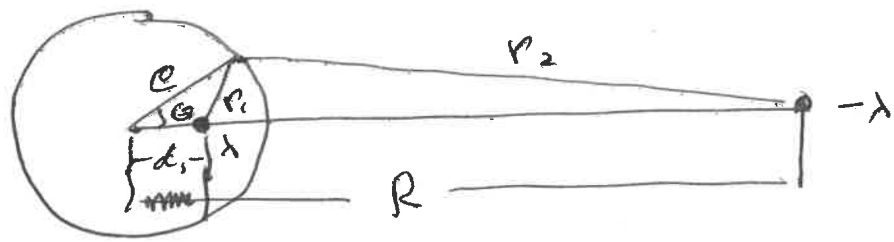
$$\frac{F}{l} = -\frac{\tau^2}{2\pi\epsilon_0} \frac{1}{R-d}$$

and $d = \frac{b^2}{R}$.



4) Jackson 2.8

a) We basically already showed Φ = constant for an off set circle, but let's do it again (you don't have to – skip this page if bored –)



$$V = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r_2^2}{r_1^2} \;:\; r_1^2 = d_1^2 + e^2 - 2 d_1 e \cos\Theta$$

$$r_2^2 = \underbrace{(d_1+R)^2 + e^2}\;\underbrace{-2e(d_1+R)\cos\Theta}$$

For an equipotential $\dfrac{d_1^2+e^2}{(d_1+R)^2+e^2} = \dfrac{d_1}{d_1+R}$

which is solved by $d_1(d_1+R) = e^2$. Check this!

$$\frac{d_1 + e^2}{\left(\frac{e^2}{d_1}\right)^2 + e^2} = \frac{d_1^2\,[d_1^2+e^2]}{e^2\,[e^2+d_1^2]} = \frac{d_1^2}{e^2} = \frac{d_1^2}{d_1(d_1+R)} = \frac{d_1}{d_1+R}$$

e is (again) the geometric mean of d_1 and d_1+R.

Express in terms of V/λ

$$\exp\frac{4\pi\epsilon_0 V}{\lambda} = \frac{r_2^2}{r_1^2} = \frac{[(d_1+R)^2 + e^2] - 2e(d_1+R)\cos\Theta}{d_1^2 + e^2 - 2 e d_1 \cos\Theta}$$

$$= \frac{\frac{(d_1+R)}{d_1}(d_1^2+e^2) - 2e\,d_1\left(\frac{d_1+R}{d_1}\right)\cos\Theta}{d_1^2+e^2 - 2\cancel{de}\,e\,d_1\cos\Theta} = \frac{d_1+R}{d_1}$$

$\exp\dfrac{4\pi\epsilon_0 V}{\lambda} = \dfrac{d_1+R}{d_1}$ gives d_1 in terms of V, λ, R

note $(d_1+R)^2 + e^2 = (d_1+R)\dfrac{e^2}{d_1} + e^2$

$$= \left(\frac{d_1+R}{d_1}\right)e^2 + d_1(d_1+R) = (e^2+d_1^2)\left(\frac{d_1+R}{d_1}\right)$$

Now for the real problem

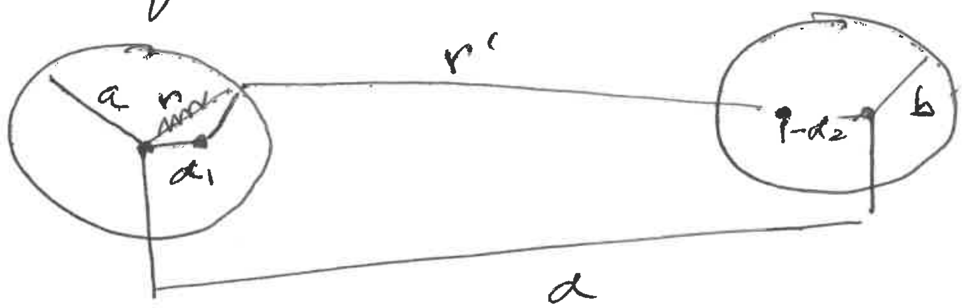
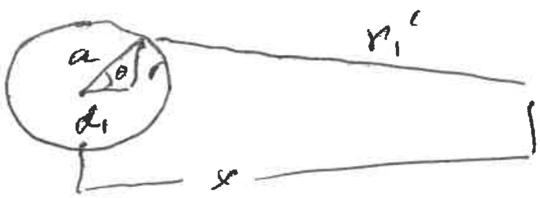


Fig 1

Gauss' law says the charge on the wire is the charge on its cylinder. The surfaces of the two cylinders are equipotentials, so the ratio r'/r is a constant around the cylinder. The law of cosines for the left hand cylinder is



$$r_1'^2 = x^2 + a^2 - 2ax\cos\theta \quad (1)$$

$$r^2 = d_1^2 + a^2 - 2d_1 a\cos\theta \quad (2)$$

If the cylinder is an equipotential, the ratio of the cosine terms in (1) and (2) is equal to the ratio of the non cosine terms,

3) $\dfrac{x^2+a^2}{d_1^2+a^2} = \dfrac{x}{d_1}$ or 4) $x d_1 = a^2$ (geometric mean, again)

Check: $\dfrac{x^2+a^2}{\left(\frac{a^2}{x}\right)^2+a^2} = \dfrac{x^2}{a^2}\,\dfrac{(x^2+a^2)}{(x^2+a^2)} = \dfrac{x^2}{a^2} = \dfrac{x}{d_1}$ –

or, just recall problem 2.11. Either way, this is implicitly part (a). Apply this to both cylinders in the original picture (Fig 1)

$$d_1(d-d_2) = a^2 \quad \text{and} \quad d_2(d-d_1) = b^2 \quad (5)$$

The potentials on the two surfaces are

$$\phi_1 = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r(1)}{r'(1)} \quad \text{and} \quad \phi_2 = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r(2)}{r'(2)}.$$

$\phi_1 - \phi_2 = V$ is the voltage difference between the 2 cylinders, and the capacitance is $C = Q/V$. On each cylinder, r'/r = constant.

From Eqs (1), (2), (3), (4) on the LH cylinder

$$\frac{r^2}{r'^2} = \frac{d_1}{x} = \frac{a^2}{x^2}, \quad x = d - d_2, \text{ so}$$

$$\phi_1 = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r(1)}{r'(1)} = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{d-d_2} = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{d_1}{a}$$

$$(a^2 = d_1(d-d_2)\ \ldots)$$

$$\phi_2 = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r(2)}{r'(2)} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{d-d_1} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{d_2}{b}$$

$$V = \phi_2 - \phi_1 = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{d_1 d_2}{ab}$$

$$\frac{CV}{L} = \frac{Q}{L} = \lambda \text{ so } \quad C = \frac{2\pi\epsilon_0}{\ln\left(\frac{d_1 d_2}{ab}\right)}$$

Now we are ready for the hints. Call

$$x = \ln \frac{d_1 d_2}{ab} \equiv \cosh^{-1} y$$

$$y = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\frac{d_1 d_2}{ab} + \frac{ab}{d_1 d_2}\right]$$

Next, notice $d^2 = d \cdot d$ (!). Eq 5 says

$$d = \frac{a^2}{d_1} + d_2 \quad \text{and} \quad d = \frac{b^2}{d_2} + d_1, \text{ so}$$

$$d^2 = \left(\frac{a^2 + d_1 d_2}{d_1}\right)\left(\frac{b^2 + d_1 d_2}{d_2}\right)$$

$$= \frac{a^2 b^2}{d_1 d_2} + a^2 + b^2 + d_1 d_2$$

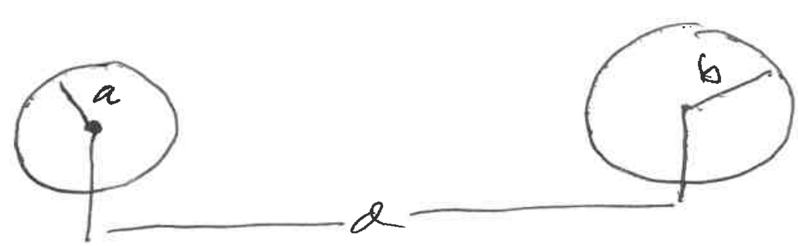
$$d^2 - a^2 - b^2 = \frac{a^2 b^2}{d_1 d_2} + d_1 d_2 = ab\left[\frac{ab}{d_1 d_2} + \frac{d_1 d_2}{ab}\right]$$

or $$\frac{1}{2}\left[\frac{d_1 d_2}{ab} + \frac{ab}{d_1 d_2}\right] = \frac{d^2 - a^2 - b^2}{2ab}$$

which gives the beautiful though highly improbable result

$$C = \frac{2\pi\epsilon_0}{\cosh^{-1}\left[\frac{d^2 - a^2 - b^2}{2ab}\right]} \quad (!)$$

This is more useful than the first formula for C because d, a, and b are easy to read off, as opposed to d_1 & d_2



That was hard!