Set 14 – due 14 December

The last day of classes is Thursday, 14 December. You can put this assignment in my mailbox, or hand it to me in person, any time up until 5 PM that day. The grader will try to turn it back to me by Monday (or Tuesday morning at the latest) and I'll post the solutions by Friday morning.

The final exam is Tuesday 19 December, 430-7 PM in our classroom.

"Modern physics is much too difficult for physicists." – D. Hilbert

1) [20 points] Jackson 8.2. (a)–5 points, (b)–5 points, (c)–3 points, (d)–7 points.

8.2 The classic problem of a TEM mode in a coaxial cable – the conductors have conductivity σ, skin depth δ, permeability μ_c. The dielectric (plastic) filler has index of refraction n and permeabilities μ and ϵ. ~~Eq 8.28 says~~

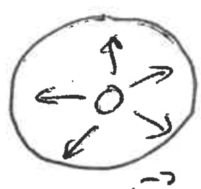
For a TEM mode $\vec{E}(x,t) = \vec{E}(\rho,\varphi)\, e^{i(kz-\omega t)}$

where $\vec{E}(\rho,\phi)$ is a solution from 2-d electrostatics, $\vec{\nabla}\cdot\vec{E}=0$, $\vec{\nabla}\times\vec{E}=0$. So Gauss' law says

$$\vec{E} = \hat{e}\,\frac{A}{\rho} \text{ for some constant } A. \quad (\equiv \hat{e}\, E_\rho)$$

$$\vec{B}(x,t) = \frac{n}{c}\,\hat{z}\times\vec{E}\; e^{i(k\cdot z-\omega t)} \qquad (8.28)$$

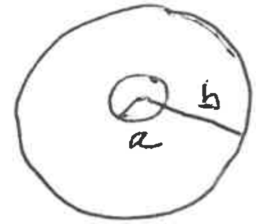
$$= \hat{\varphi}\,\frac{n}{c}\,\frac{A}{\rho}\, e^{i(k\cdot z-\omega t)} \equiv \hat{\varphi}\, B_\varphi\, e^{i(k\cdot z-\omega t)}$$



$\vec{E}$

$\vec{B}$

and



In terms of $H_0 = H(\rho=a)$, $H_\varphi(\rho) = H_0\,\frac{a}{\rho}$

and $E_\rho = \frac{cB}{n} = \sqrt{\frac{\mu}{\epsilon}}\, H = \sqrt{\frac{\mu}{\epsilon}}\,\frac{a}{\rho}\, H_0$

so the power along the line comes from the Poynting vector

$$\langle\vec{P}\rangle = \frac{1}{2}\left|\vec{E}\times\vec{H}^*\right| = \hat{z}\sqrt{\frac{\mu}{\epsilon}}\,\frac{|H_0|^2}{2}\,\frac{a^2}{\rho^2}$$

This must be integrated over the cross sectional area-

a) $$P = \int_a^b 2\pi\rho\, d\rho \, \langle \vec{P} \rangle = \sqrt{\frac{\mu}{\epsilon}}\, |H_0|^2 \, \pi a^2 \ln\frac{b}{a}.$$

b) The time averaged power absorbed by the conductors (per unit area) is

$$\frac{dP}{dA} = -\frac{1}{2\sigma\delta}|H_\parallel|^2 \quad \sim$$

$|H_\parallel| = H_0$ on the inner surface, $H_0 \frac{a}{b}$ on the outer one - so

$$\frac{dP}{d(\text{unit length})} = -\frac{1}{2\sigma\delta}|H_0|^2 \left\{ 2\pi a + 2\pi b\left(\frac{a^2}{b^2}\right)\right\}$$

$$= -\frac{1}{2\sigma\delta}\left[\sqrt{\frac{\epsilon}{\mu}}\,\frac{P}{(\pi a^2 \ln\frac{b}{a})}\right]\cdot 2\pi \cdot \left(a + \frac{a^2}{b}\right)$$

$$= -\sqrt{\frac{\epsilon}{\mu}}\,\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\sigma\delta \ln\frac{b}{a}}\,P \equiv -2\gamma P$$

whence $P(z) = P_0 \exp(-2\gamma z)$

c) $V = Z_0 I$, $I = \oint H \cdot d\ell = 2\pi a H_0$

$$V = \int_a^b E(\rho)\, d\rho = \sqrt{\frac{\mu}{\epsilon}}\, H_0\, a \ln\frac{b}{a}$$

so $$Z_0 = \frac{V}{I} = \sqrt{\frac{\mu}{\epsilon}}\,\frac{\ln(b/a)}{2\pi}$$

8.2.3

The time averaged power loss per unit length is

$$\frac{dP}{dz} = \frac{1}{2} I^2 \left(\frac{R}{\text{length}}\right) = \frac{1}{2}(2\pi a H_0)^2 \frac{R}{\text{length}}$$

and part (b) said

$$\frac{dP}{dz} = \frac{1}{2\sigma\delta}|H_0|^2\left(\frac{1}{a}+\frac{1}{b}\right)2\pi a^2 = \frac{1}{2}(2\pi a H_0)^2\frac{R}{\text{length}}$$

so $$\frac{R}{\text{length}} = \frac{1}{\sigma\delta}\left(\frac{1}{2\pi a}+\frac{1}{2\pi b}\right) -$$

note that each surface has its own $\frac{R}{\text{length}}$ value

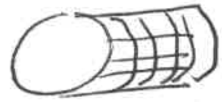
$$\left[\frac{R}{\text{length}}\right]^{-1} = \sigma \times \text{skin depth} \times \text{circumference}$$

Inductance is a measure of the magnetic energy stored in the field. So compute

$$\frac{dW_M}{dz} = \frac{1}{4} L I^2 = \frac{1}{4}\frac{d}{dz}\int dA\ B\cdot H^*$$

the $\frac{1}{4}$ has an extra $\frac{1}{2}$ from the time average.

A is the area of a part of the cylinder which is z units long -



|- z -|

There are 2 parts - first, from the fields between the 2 conductors

$$\frac{1}{4}\int B\cdot H^* dA = z\cdot\left(\frac{1}{4}\mu|H_0|^2\right)\int_a^b 2\pi\varrho d\varrho\cdot\left(\frac{a}{\varrho}\right)^2$$

This is $z \times \frac{dW}{dz}$. The integral is

$$\frac{\pi a^2}{2}\mu|H_0|^2 \ln\frac{b}{a} = \frac{\pi a^2}{2}\mu \ln\frac{b}{a}\left[\frac{I}{2\pi a}\right]^2$$

$$= \frac{I^2}{8\pi}\mu \ln\frac{b}{a}.$$

Then there is the contribution from the fields inside the conductors. Start with

$$\vec{H}_c = H_{\parallel s}\exp\left(-\frac{\xi}{\delta}\right)\exp\frac{i\xi}{\delta}.$$

Here $H_{\parallel s}$ is the surface value of $H_\parallel$, ξ is the distance into the conductor. For the inner conductor

$$\frac{dW_M^c}{dz} = \frac{1}{4}\mu_c\cdot 2\pi a\cdot|H_0|^2\int_0^\infty d\xi\, e^{-2\xi/\delta}$$

$$= \mu_c\frac{\pi a}{4}\delta\,|H_0|^2 \quad (\text{and } I = 2\pi H_0)$$

All that changes for the outer conductor is that H_0 scales down by a factor $\frac{a}{b}$, and $2\pi a \to 2\pi b$. Overall, this is $\frac{b}{a}\left(\frac{a^2}{b^2}\right) = \frac{a}{b}$

so

$$\frac{dW_M^c}{dz} = \mu_c \frac{\pi \delta}{4} \left[\frac{a^2}{a} + \frac{a^2}{b} \right] \frac{I^2}{(2\pi a)^2}$$

$$= \frac{\mu_c \delta}{16\pi} \left(\frac{1}{a} + \frac{1}{b} \right) I^2$$

The total L/length is

$$\frac{L}{\text{length}} = \frac{1}{4\pi} \left[2 \ln \frac{b}{a} + \mu_c \delta \left(\frac{1}{a} + \frac{1}{b} \right) \right]$$