Set 13 – due 8 December

"The laws of Nature are written in the language of mathematics," attributed to Galileo by E. Wigner

1) [10 points] Jackson 7.12. (a)–5, (b)–5.

2) [15 points] Jackson 7.13. (a)–10, (b)–5. Assume a flat earth. Amateur radio operators call this phenomenon "skip" (as in a stone skipping on the surface of a lake). You can hear the same effect on AM radio on winter nights.

3)[20 points] Jackson 7.22. (a)–10, (b)–10. In part (b), assume $\omega_0 > \gamma/2$).

7.12. This problem begins with Eq 7.58, the Drude formula, $\sigma = \frac{\sigma_0}{1-i\omega\tau} = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)}$. $(\equiv \sigma(\omega))$

The convention for the Fourier transform I'll take is

$$\underline{X}(x,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \underline{X}(x,\omega)\, e^{-i\omega t}.$$

And we need the continuity equation $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$,

Ohm's law, $\vec{J}(x,\omega) = \sigma(\omega)\, \vec{E}(x,\omega)$

Gauss' law, $\vec{\nabla} \cdot \vec{E}(x,\omega) = \frac{1}{\epsilon_0} \rho(x,\omega)$.

We Fourier transform the continuity equation

$$0 = \frac{1}{\sqrt{2\pi}} \int d\omega\, e^{-i\omega t} \left[\vec{\nabla} \cdot \vec{J}(x,\omega) - i\omega \rho(x,\omega) \right]$$

or (of course) $\vec{\nabla} \cdot \vec{J}(x,\omega) = i\omega \rho(x,\omega)$

$$= \sigma\, \nabla \cdot E \text{ (ohm)} = \frac{\sigma \rho}{\epsilon_0}$$

and $\frac{\sigma(\omega) \rho(x,\omega)}{\epsilon_0} = i\omega \rho(x,\omega)$, or

(part (a)) $\left[i\omega - \frac{\sigma(\omega)}{\epsilon_0} \right] \rho(x,\omega) = 0$

b) With $\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}$ and $\sigma_0 = \epsilon_0 \tau \omega_p^2$,

$$\left[i\omega - \frac{\sigma_0/\epsilon_0}{1-i\omega\tau} \right] \rho(x) = 0 = \left[i\omega - \frac{\tau\omega_p^2}{1-i\omega\tau} \right] \rho$$

so, if ρ is not zero, $i\omega = \frac{\omega_p^2 \tau}{1-i\omega\tau}$, and we solve for ω, $\omega_p^2 \tau - i\omega - \omega^2 \tau = 0$

$$\omega = \frac{-i \pm \sqrt{-1 + 4\omega_p^2 \tau^2}}{2\tau}$$

If $\omega_p \tau \gg 1$ this is $\omega = \frac{-i}{2\tau} \pm \omega_p$.

Now, everything in the plasma oscillates at this frequency – e, $\vec{E}$, --- so

$$\vec{E}(x,t) = \vec{E}(x,0)\, e^{-i\omega t}$$

$$= \vec{E}(x,0) \exp\left[-it\left(\frac{-i}{2\tau} \pm \omega_p\right)\right]$$

$$= \vec{E}(x,0) \exp\left[-\frac{t}{2\tau}\right] \exp(\pm i\omega_p t)$$

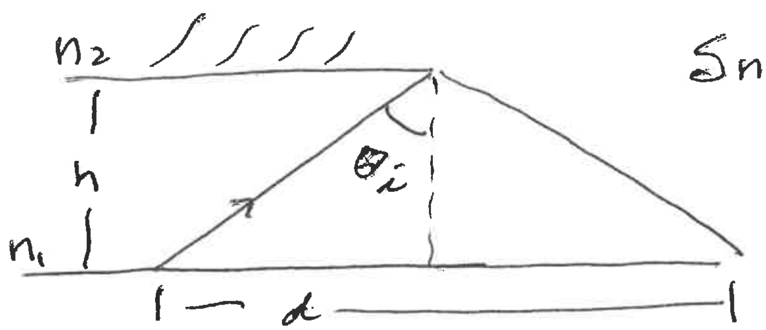
The amplitude of E decays with time as $\exp\left(-\frac{t}{2\tau}\right)$. Since energy is proportional to $|E|^2$, the energy of any disturbance decays as $\exp\left(-\frac{t}{\tau}\right)$.

7.13. $\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$ where $\omega_p = \frac{N_e e^2}{\epsilon_0 m_e}$,

The electron density $N_e = NZ \frac{\text{electrons}}{m^3}$.

Also $\frac{\epsilon(\omega)}{\epsilon_0} = n(\omega)^2$, the squared index of refraction.

The problem is about total internal reflection.



Snell's law is

$$n_2 \sin\theta_t = n_1 \sin\theta_i,$$

here $n_1 = 1$, $\theta_t = \pi/2$

$$\sin^2\theta_i = n_2^2 = 1 - \frac{\omega_p^2}{\omega^2}.$$

This is $\cos^2\theta_i = \frac{\omega_p^2}{\omega^2}$ and $\cos\theta_i = \frac{h}{\left[h^2 + \frac{d^2}{4}\right]^{1/2}}$

so $$\frac{\omega_p^2}{\omega^2} = \frac{h^2}{h^2 + (\frac{d}{2})^2} \Longrightarrow h^2 + \left(\frac{d}{2}\right)^2 = h^2 \frac{\omega^2}{\omega_p^2}$$

$$\left(\frac{d}{2}\right)^2 = h^2\left(\frac{\omega^2}{\omega_p^2} - 1\right)$$

or $$d = 2h\sqrt{\frac{\omega^2}{\omega_p^2} - 1}.$$

If $\theta < \theta_i$, or at smaller d, there is a transmitted wave. The signal passes through the ionosphere and is lost. For $\theta > \theta_i$ the signal is reflected back to Earth. Amateur radio operators call this phenomenon "skip" — like a stone skipping on the surface of a lake.

b) Numbers: $\lambda = \frac{c}{\nu} = \frac{2\pi c}{\omega} = 21\ m$

$$\omega = \frac{2\pi \times 3\times10^{8} m/s}{21\ m} = 90\times10^{6}\ sec^{-1}$$

(this is roughly $2\pi \times 14\ MHz$, short wave radio)

$d = 10^6\ m$, $h = 3\times10^5\ m$

$$\frac{\omega^2}{\omega_p^2} = \left(\frac{d}{2h}\right)^2 + 1 = \left(\frac{10^6}{6\times10^5}\right)^2 + 1 = 3.78$$

or $\frac{\omega_p^2}{\omega^2} = \frac{1}{3.78}$.

$\hbar c = 197\ MeV\text{-}fm$, $1\ fm = 10^{-13}\ cm = 10^{-15}\ m$

$\frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}$ in MKS, $\frac{e^2}{m_e \epsilon_0} = \frac{4\pi}{137} \times \frac{197\ MeV\cdot10^{-15} m\cdot c^2}{m_e c^2 = \frac{1}{2}\ MeV}$

$$\omega_p^2 = N_e\left[\frac{e^2}{4\pi\epsilon_0}\ \frac{4\pi}{\hbar c}\right]\frac{\hbar c}{m_e c^2}\cdot c^2$$

$$= N_e \cdot \frac{4\pi}{137} \times \frac{197\cdot10^{-15} MeV\text{-}m}{\frac{1}{2}\ MeV} \times 9\cdot10^{16}\ m^2/s^2$$

$$= N_e \times 3.2\times10^{3}\ sec^{-2}$$ ~~with~~

with N_e in units of $\frac{1}{m^3}$.

$$\omega_p^2 = \frac{\omega_p^2}{\omega^2}\cdot\omega^2 = \frac{1}{3.78}\times(9\cdot10^7)^2 = N_e \times 3.2\cdot10^3$$

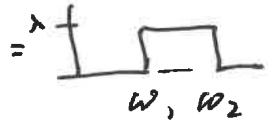
$\rightarrow N_e \sim 6.7\times10^{11}$ electrons$/m^3$.

This is in the range quoted by Jackson of

$N_e \sim 2\times10^{11}$ to 2×10^{12} electrons$/m^3$.

7.22 - rather mathy variations on the dispersion formula

$$\mathrm{Re}\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right] = \frac{2}{\pi}\,\mathcal{P}\int_0^\infty \frac{\omega'\,d\omega'}{\omega'^2-\omega^2}\,\mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0}$$

a) $\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = \lambda\left[\Theta(\omega-\omega_1)-\Theta(\omega-\omega_2)\right] =$ 

i) if $\omega<\omega_1$, $\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0}-1 = \frac{2\lambda}{\pi}\int_{\omega_1}^{\omega_2}\frac{\omega'\,d\omega'}{\omega'^2-\omega^2}$

$$= \frac{2\lambda}{2\pi}\ln\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}$$

ii) if $\omega>\omega_2$ the answer is identical

iii) If $\omega_1<\omega<\omega_2$ we have to use the careful definition for $\mathcal{P}$:

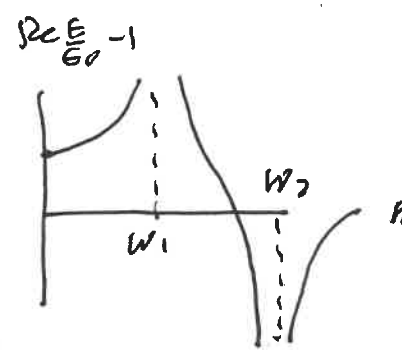
$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0}-1 = \lim_{\delta\to 0}\frac{2\lambda}{\pi}\left[\int_{\omega_1}^{\omega-\delta}\frac{\omega'\,d\omega'}{\omega'^2-\omega^2}+\int_{\omega+\delta}^{\omega_2}\frac{\omega'\,d\omega'}{\omega'^2-\omega^2}\right]$$

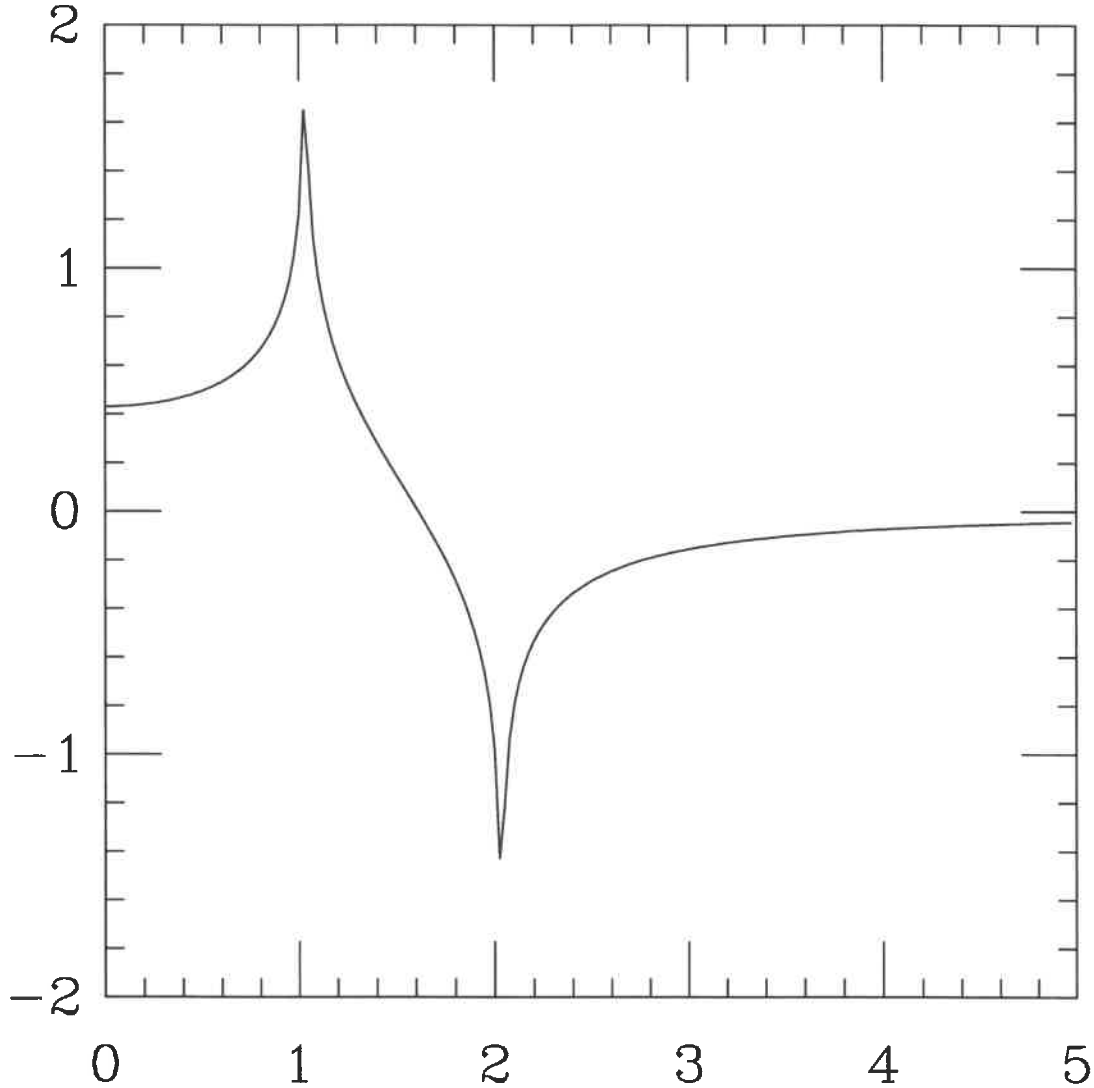
$$= \lim_{\delta\to 0}\frac{\lambda}{\pi}\left[\ln\frac{(\omega-\delta)^2-\omega^2}{\omega_1^2-\omega^2}+\ln\frac{\omega_2^2-\omega^2}{(\omega+\delta)^2-\omega^2}\right]$$

$$= \lim_{\delta\to 0}\frac{\lambda}{\pi}\ln\left[\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\cdot\frac{-2\omega\delta}{2\omega\delta}\right]$$

$$= \frac{\lambda}{\pi}\ln\frac{\omega_2^2-\omega^2}{\omega^2-\omega_1^2}$$

or - $\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0}-1 = \frac{\lambda}{\pi}\ln\left|\frac{\omega_2^2-\omega^2}{\omega_1^2-\omega^2}\right|$




2
1
0
−1
−2
0
1
2
3
4
5

b) $$\frac{\operatorname{Im}\epsilon(\omega)}{\epsilon_0} = \frac{\lambda\gamma\omega}{[\omega_0^2-\omega^2]^2+\gamma^2\omega^2}$$

$$= \frac{-\lambda}{2i}\left[\frac{1}{\omega^2-\omega_0^2+i\omega\gamma} - \frac{1}{\omega^2-\omega_0^2-i\omega\gamma}\right]$$

The easiest way to proceed is to go backwards - unfold the integral

$$\operatorname{Re}\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\operatorname{Im}\frac{\epsilon(\omega')}{\epsilon_0}$$

and then to use

$$\mathcal{P}\frac{1}{\omega'-\omega} = \frac{1}{\omega'-\omega-i\eta} - i\pi\,\delta(\omega-\omega').$$

Then $$\operatorname{Re}\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{-i\lambda\omega\gamma}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2} + \frac{\gamma\lambda}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega-i\eta}\operatorname{Im}\frac{\epsilon(\omega)}{\epsilon_0}.$$

The second term is

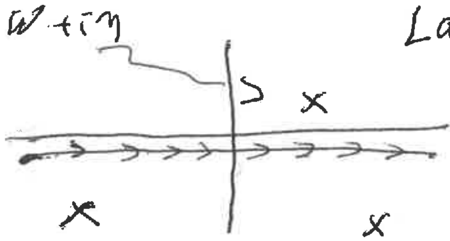
$$I_2 = \frac{\lambda}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega-i\eta}\left[\frac{-1}{\omega'^2-\omega_0^2+i\omega'\gamma} + \frac{1}{\omega'^2-\omega_0^2-i\omega'\gamma}\right]$$

As $\omega'\to\infty$ the integrands fall off as $\frac{1}{\omega^3}$, so we can convert the integral into a contour integral, choosing the return path for convenience. We need to find the ~~[illegible]~~ locations of poles. They are at

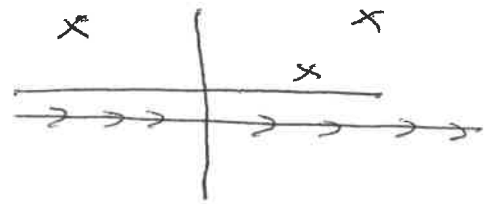
$$\omega'^2 \pm i\omega'\gamma - \omega_0^2 = 0 \quad \text{or}$$

$$\omega' = \mp\frac{i\gamma}{2} \pm\sqrt{-\frac{\gamma^2}{4}+\omega_0^2}$$

Taking $\omega_0 > \frac{\gamma}{2}$, the picture of the complex plane is



and



For the second term, close the contour in the lower half plane – this gives zero. Close the contour in the upper half plane, for the first term. The residue of a simple pole is $2\pi i$, so

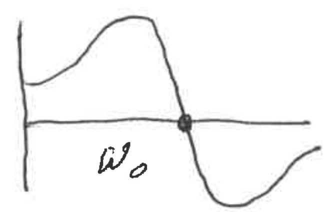
$$I_2 = \frac{-\lambda}{2\pi i} \cdot 2\pi i \cdot \frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}$$

and

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{-i\lambda\omega\gamma}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma^2} - \frac{\lambda}{\omega^2-\omega_0^2+i\omega\gamma} \cdot \left(\frac{\omega^2-\omega_0^2-i\omega\gamma}{\omega^2-\omega_0^2-i\omega\gamma}\right)$$

$$= \frac{-i\lambda\omega\gamma}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma^2} - \frac{\lambda(\omega^2-\omega_0^2-i\omega\gamma)}{(\omega^2-\omega_0^2) + \omega^2\gamma^2}$$

$$= \frac{\lambda(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma^2} \implies$$



$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} \sim$$

It's not a Lorentzian, though it is similar. Near the peak, if $\omega = \omega_0 + \delta$, $\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} \sim \frac{\lambda\gamma\omega_0}{(2\omega_0\delta)^2 + \gamma^2\omega_0^2}$ which is $\frac{\lambda\gamma}{4\omega_0^2}\left(\frac{1}{\delta^2 + \frac{\gamma^2}{4}}\right)$, which is Lorentzian