Set 12 – due 1 December

"The task is not to see what no one else has seen, but to think what no one else has thought, about that which everyone else has seen." – Schrödinger

1) [20 points] Jackson 7.6 (a)–6, (b)–6, (c)–8.

2) [20 points] More about Jackson 7.2. You have already solved this problem to find the transmission and reflection coefficients. Now consider the case of an infinite planar dielectric medium of thickness d in the vacuum (i.e. $n_3 = 1$ in 7.2). Assume that there is an incident electromagnetic wave of energy density u_0 and frequency ω directed normal to the surface of the plane, and compute the radiation pressure of the wave on the plane in two ways: (a) [7 points] Calculate the field momentum of the incident wave, p_i, the transmitted wave, p_t, and the reflected wave p_r. Then use momentum conservation to write $\vec{p}_i = \vec{p}_t + \vec{p}_r +$ momentum of plate. (b) [7 points] Evaluate the stress tensor to the left and to the right of the slab. (c) [6 points] For what values of thickness d is the pressure a maximum?

3) [10 points] Suppose that there is a magnetic field $\vec{H}_0$ parallel to, and at the surface of, a good conductor. Beginning with the expressions for $\vec{E}$ and $\vec{H}$ in conductors (in Jackson Sec. 8.1), (a) [5 points] show that the time-averaged power loss into the conductor per unit area is

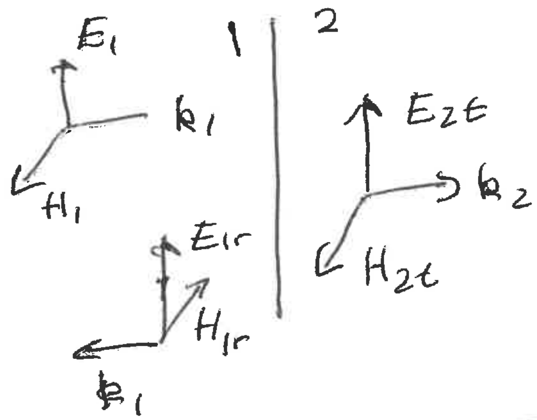
$$\frac{dP}{dA} = \frac{\mu_c \omega \delta}{4}|H_0|^2 = \frac{1}{2\sigma\delta}|H_0|^2. \tag{1}$$

(b) [5 points] What is the time-averaged magnetic energy density (energy per unit area) stored in the conductor, in terms of $\vec{H}_0$?

The point of this problem is that this is how loss occurs for electromagnetic waves confined in a region withmettalic boundaries: it is used to compute attenuation in a wavegiude or the Q of a cavity.

7.6 – Radiation incident on an absorbing slab – set $\mu = \mu_0$ so that the index of refraction is $n^2 = \epsilon/\epsilon_0$.

Choose coordinates so $\vec{E} = \hat{x} E$.



Then $\vec{H} = \hat{y} H = \sqrt{\frac{\epsilon}{\mu}}\, \hat{n} \times \vec{E}$

$$= \left(\frac{1}{\mu_0}\frac{n}{c}\right) \hat{n} \times \vec{E}.$$

$n_1 = 1,\ k_1 = \omega/c \equiv k$

$n_2 = n(\omega),\ k_2 = n(\omega) k_1 = n(\omega)\frac{\omega}{c}.$

In region 1

$$E_{1x} = E_1 e^{i(kz-\omega t)} + E_{1r} e^{i(-kz-\omega t)}$$

$$H_{1y} = \frac{1}{\mu_0 c}\left(E_1 e^{i(kz-\omega t)} - E_{1r} e^{i(-kz-\omega t)}\right)$$

In region 2

$$E_{2x} = E_{2t} \exp[i(n(\omega)kz - \omega t)]$$

$$H_{2y} = \frac{n(\omega)}{\mu_0 c} E_{2t} \exp[i(n(\omega)kz - \omega t)]$$

$E_{\parallel}$ and $H_{\parallel}$ continuous at $z = 0$: $E_1 + E_{1r} = E_{2t}$

$$E_1 - E_{1r} = n(\omega) E_{2t}$$

or $\dfrac{E_{1r}}{E_1} = \dfrac{1-n(\omega)}{1+n(\omega)}$ and $\dfrac{E_{2t}}{E_1} = \dfrac{2}{1+n(\omega)}$

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 ;\ T = 1 - R = \frac{4\,\mathrm{Re}\, n(\omega)}{|1+n(\omega)|^2}$$

You can also get this directly from the Poynting vector: $\langle \vec{S}_1 \rangle = \frac{1}{2} \mathrm{Re}\,(E_1 \times H_1^*) = \hat{z} \frac{1}{2\mu_0 c} |E_1|^2$

$$\langle \vec{S}_R \rangle = -\hat{z} \frac{1}{2\mu_0 c} |E_{1r}|^2$$

$$\langle \vec{S}_T \rangle = \frac{\hat{z}}{2\mu_0 c} \mathrm{Re}\left[\frac{2}{1+n}\left(\frac{2n}{1+n}\right)^*\right] |E_1|^2$$

(using $H \propto n(\omega) E$

b) Eq. 6.134 gives the energy dissipated per unit time

$$W_{loss} = \mathrm{Re}\left[2i\omega\,(W_E - W_M)\right] = \frac{1}{2}\mathrm{Re}\left[i\omega\,(E\cdot D^* - B\cdot H^*)\right]$$

$$\vec{D} = \epsilon\vec{E} = \frac{n^2(\omega)}{\mu_0 c^2}\vec{E}, \quad \vec{B} = \mu_0\vec{H}, \quad \vec{H} = \frac{n(\omega)}{\mu_0 c}\hat{n}\times\vec{E}$$

so $$W_{loss} = \frac{\omega}{2\mu_0 c^2} \mathrm{Im}\left\{ (n^2(\omega))^* - |n(\omega)|^2 \right\} |E_{2t}(x,t)|^2$$

In the $\{\}$ we need $n^2 = (\mathrm{Re}\, n + i\,\mathrm{Im}\, n)^2$

$|n|^2 = (\mathrm{Re}\, n)^2 + (\mathrm{Im}\, n)^2$

so $$W_{loss} = \frac{\omega}{2\mu_0 c^2}\left[2\,\mathrm{Re}\, n(\omega)\cdot \mathrm{Im}\, n(\omega)\right]$$

$$\times \frac{4}{|1-n(\omega)|^2} \exp\left[-2(\mathrm{Im}\, n(\omega))kz\right].$$

The energy loss per unit area is found by integrating W_{loss} over z going into region 2

$$\int_0^\infty dz\, W_{loss}(z) = \frac{\omega}{2\mu_0 c^2}\cdot\frac{4}{|1+n(\omega)|^2}\cdot\frac{2\,\mathrm{Re}\, n\cdot \mathrm{Im}\, n}{2\cdot \mathrm{Im}\, n\cdot k} |E_1|^2$$

and $\omega = ck$ so this is

$$\frac{4 \,\mathrm{Re}\, n(\omega)}{|1+n(\omega)|^2} \frac{|E_i|^2}{2\mu_0 c^2} = T \cdot \langle S_i \rangle$$

the transmission coefficient times the incident flux.

c) Now we evaluate $n^2 = 1 + \frac{i\sigma}{\omega\epsilon_0}$ in terms of skin depth $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$. When $\frac{\sigma}{\omega\epsilon_0}$ is large,

$$n(\omega) \simeq \sqrt{\frac{i\sigma}{\omega\epsilon_0}} = \frac{\sqrt{i\sigma\omega}}{\omega} \frac{1}{\sqrt{\epsilon_0}} \sqrt{\frac{\mu_0}{\mu_0}} = \frac{\sqrt{2i}}{\omega} \sqrt{\frac{\mu_0 \sigma\omega}{2}} \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$= \sqrt{2i}\, \frac{c}{\omega\delta}.$$

$\frac{1}{k} = \frac{c}{\omega} = \frac{\lambda}{2\pi} \equiv \bar{\lambda}$ (barred notation by analogy to $\hbar = h/2\pi$)

$$n(\omega) = \sqrt{2i}\, \frac{\bar{\lambda}}{\delta}.$$

And $i = e^{i\pi/2}$ so $\sqrt{2i} = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right)$

or $\sqrt{2i} = 1 + i$ and
$\mathrm{Re}\, n(\omega) = \mathrm{Im}\, n(\omega) = \frac{\bar{\lambda}}{\delta}$. Plug this into

$$R = \left|\frac{1-n}{1+n}\right|^2 = \left|\frac{\sqrt{2i}\,\bar{\lambda} - \delta}{\sqrt{2i}\,\bar{\lambda} + \delta}\right|^2 = \left|\frac{1 - \frac{\delta}{\sqrt{2i}\bar{\lambda}}}{1 + \frac{\delta}{\sqrt{2i}\bar{\lambda}}}\right|^2.$$

We are told that $\delta/\bar{\lambda}$ is small, so just Taylor expand

$$R = 1 - \frac{4\delta}{\bar{\lambda}} \,\mathrm{Re}\, \frac{1}{\sqrt{2i}},$$

$\sqrt{2i} = 1+i$ again so $\mathrm{Re}\,\frac{1}{\sqrt{2i}} = \mathrm{Re}\,\frac{1}{1+i} = \mathrm{Re}\,\frac{1-i}{2} = \frac{1}{2}$

$R = 1 - \frac{2\delta}{\bar{\lambda}}$, $T = 1 - R = \frac{2\delta}{\bar{\lambda}}$.

Alternatively $T = \dfrac{4\,\mathrm{Re}\,(\sqrt{2i}\;\bar{\lambda}/\delta)}{\left|1 + \sqrt{2i}\,\frac{\bar{\lambda}}{\delta}\right|^2} \simeq \dfrac{4\,\frac{\bar{\lambda}}{\delta}}{2\left(\frac{\bar{\lambda}}{\delta}\right)^2} = \frac{2\delta}{\bar{\lambda}}$.

Now return to p. 7.6.2,

$$W_{loss} = \frac{\omega}{2\mu_0 c^2}\,\frac{2\,\mathrm{Re}\,n\cdot\mathrm{Im}\,n}{|1+n(\omega)|^2}\cdot 4|E_1|^2 \exp\left[-2\,\mathrm{Im}\,n(\omega)\,kz\right]$$

$$= \frac{\omega}{2\mu_0 c^2}\cdot\frac{2\left(\frac{\bar{\lambda}}{\delta}\right)^2}{\left(1+\frac{\bar{\lambda}}{\delta}\right)^2 + \left(\frac{\bar{\lambda}}{\delta}\right)^2}\cdot 4|E_1|^2 \exp\left[-2\,\frac{\bar{\lambda}}{\delta}\,kz\right] \quad (k = 1/\bar{\lambda})$$

$$\simeq \frac{\omega}{2\mu_0 c^2}\cdot\frac{2(\bar{\lambda}/\delta)^2}{2\left(\frac{\bar{\lambda}}{\delta}\right)^2}\cdot 4|E_1|^2 \exp\left(-\frac{2z}{\delta}\right)$$

This again uses $\bar{\lambda}/\delta \gg 1$ and $k = 1/\bar{\lambda}$,

Again integrate over z: use $\omega = c/\bar{\lambda}$

$$W^{int}_{loss} = \int_0^\infty dz\, W_{loss}(z) = T\cdot\langle S_i\rangle = \frac{1}{2\mu_0 c^2}\cdot\frac{c}{\bar{\lambda}}\cdot 4\,\frac{\delta}{2} \times |E_1|^2$$

$$= \frac{\delta}{\mu_0 c\,\bar{\lambda}}\,|E_1|^2.$$

Ohmic loss is $\frac{1}{2}\,\mathrm{Re}\,\vec{J}_{2t}^{\,*}\cdot\vec{E}_{2t}$ and $\vec{J}=\sigma\vec{E}$ so

$$\frac{\sigma}{2}|E_{2t}|^2 = \frac{\sigma}{2}\,\frac{4}{|n+1|^2}\,|E_1|^2 \exp\left(-2\frac{z}{\delta}\right)$$

$$= \left\{\frac{\sigma}{2}\cdot\frac{4}{2(\bar{\lambda}/\delta)^2}\right\}|E_1|^2 \exp\left(-2\frac{z}{\delta}\right)$$

The prefactor in $\{\,\}$ = $\frac{\sigma\delta^2}{\bar{\lambda}^2} = \frac{1}{\bar{\lambda}^2}\,\frac{2}{\mu_0\omega}$.

Integrate over z one last time

$$W^{int}_{ohmc} = \frac{2}{\mu_0\omega}\,\frac{1}{\bar{\lambda}^2}\,\frac{\delta}{2}\,|E_1|^2 = \frac{1}{2\mu_0 c}\cdot\frac{2\delta}{\bar{\lambda}}\,|E_1|^2$$

substituting $\omega = c/\bar{\lambda}$. This means that

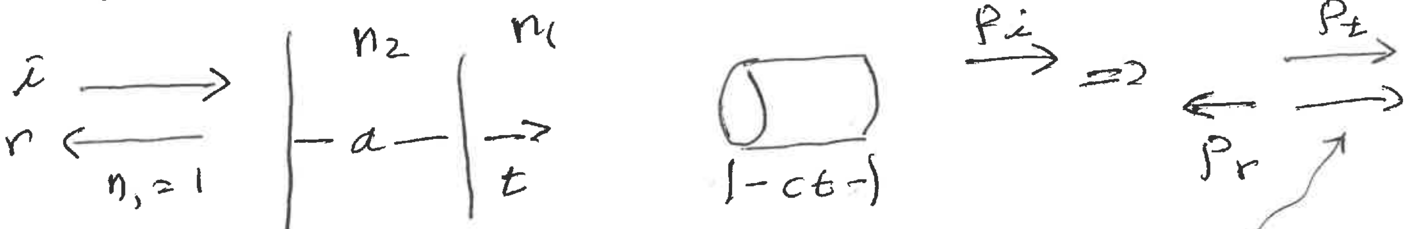
$$W^{int}_{ohmic} = W^{int}_{loss} = \langle S_i\rangle\cdot T = \frac{1}{2\mu_0 c}\,\frac{2\delta}{\bar{\lambda}}\,|E_1|^2.$$

The point of this is that you can compute dissipation either via ohmic heating, or by using complex index of refraction, but not by doing both together - that is double counting. See the discussion in Jackson below Eq. 7.57.

2) More about 7.2. The time-averaged field momentum density is

$$\vec{g} = \frac{1}{2c^2}(\vec{E} \times \vec{H}^*).$$

The time averaged momentum transferred to a unit area surface on the boundary can be found by computing $\vec{g}$ on both sides over a volume equal to V = unit area x ~~thickness~~ thickness, where thickness is equal to velocity ($v=c$) ~~time~~ x time. The pressure is the momentum absorbed by a unit area of the dielectric in a unit time



$$P_{plate} = p_i - p_t + p_r$$

$$\text{Pressure} = c\left[\frac{1}{2c^2}\vec{E}_i \times \vec{H}_i^*\right] - c\left[\frac{1}{2c^2}\vec{E}_r \times \vec{H}_r^*\right] - c\left[\frac{1}{2c^2}\vec{E}_t \times \vec{H}_t^*\right]$$

$$= \frac{1}{2\mu_0 c^2}\left\{|E_i|^2 + |E_r^2| - |E_t|^2\right\}.$$

Using $\frac{1}{\mu_0 \epsilon_0 c^2} = 1$, the initial energy density is u_0

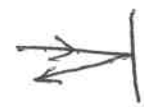
$$u_0 = \frac{1}{4}\left[\epsilon_0 |E_i|^2 + \frac{1}{\mu_0}|B_i|^2\right] = \frac{\epsilon_0}{2}|E_i|^2,$$

the pressure is $u_0\left[1 + \frac{|E_r|^2}{|E_i|^2} - \frac{|E_t|^2}{|E_i|^2}\right]$

$P = u_0[1 + R - T]$: R and T are reflection & transmission coeffs.

We can simplify this using $T+R=1$, so

$1-T+R = 1-(1-R)+R = 2R$. This makes

sense - maximum pressure when  is a maximum

We now look at last ~~week~~ week's solution, recall

$$r_{12} = \frac{n-1}{n+1} \equiv r \quad , \quad r_{23} = \frac{n_3-n_2}{n_3+n_2} \xrightarrow[\substack{n_3=1 \\ n_2=n}]{} \frac{1-n}{1+n} = -r$$

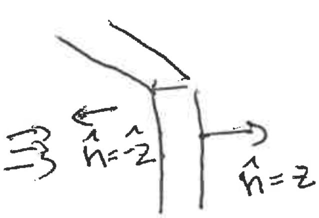
$$2R = \frac{2r^2[1-\cos\alpha]}{1-2r^2\cos\alpha + r^4} \quad \text{where } \alpha = 2k_2 d = 2\frac{\omega_0}{c} n d.$$

b) Repeat with the stress tensor

$$\overleftrightarrow{T} = \frac{\epsilon_0}{2}\left\{ \overleftarrow{E}\vec{E} + c^2 \overleftarrow{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\left(|E|^2 + c^2|B|^2\right)\right\}$$

(replace $\frac{1}{\mu_0}$ by $\epsilon_0 c^2$...). Integrate T over a surface of unit area on each side of the dielectric. At normal incidence, transversality says $\hat{n}\cdot\vec{E} = \hat{n}\cdot\vec{B} = 0$ (for pressure into/away from plate) so the first terms give nothing.

$$F_z = \underset{\text{on right}}{\int T_{zz} n_z \, dA} - \underset{\text{on left}}{\int T_{zz} n_z \, dA} \quad \text{(note - sign in T!)}$$



$$\frac{\text{Force}}{\text{area}} = \frac{1}{2\mu_0 c^2}\left[-|E_t|^2 + |E_i|^2 + |E_r|^2\right]$$

this is $P \propto -T+1+R$ as we had before!

7.2.3

c) The pressure is proportional to $1 - \cos 2k_2 d$ so we want $\cos 2k_2 d = -1$ for maximum pressure. This is

$$2k_2 d = (2n+1)\pi = \pi, 3\pi, 5\pi \ldots$$

$$k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{\lambda}\sqrt{\frac{\epsilon}{\epsilon_0}}$$ where λ_2 is the in-medium wavelength, λ the wavelenth in vacuum.

Thus $$d = \frac{(2n+1)\pi}{2k_2} = (2n+1)\frac{\pi}{2}\frac{\lambda_2}{2\pi} = \frac{2n+1}{4}\lambda_2 .$$

The minimum d is $\frac{\lambda_2}{4}$.

It's a bit like last week, except then we wanted $R=0$ for

$n=1$ | $\sqrt{n}$ | $n>1$

while here we have $n=1$ | n | $n=1$

$$R = \frac{4r^2}{(1+r^2)^2} \text{ at } \cos d = -1, \quad r = \frac{n-1}{n+1}.$$

3) Inside a good conductor,

$$H = \vec{H}_o e^{-\xi/\delta} e^{i\xi/\delta},$$

where ξ is the normal distance into the conductor and δ = skin depth. Jackson Eq. 8.10 says

$$\vec{E} = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(\hat{n}\times\vec{H}_o)\exp\left[-\frac{\xi}{\delta}+i\frac{\xi}{\delta}\right]$$

a) The power loss is

$$\frac{dP}{dA} = -\frac{1}{2}\,\mathrm{Re}\,\hat{n}\cdot(E\times H^*) = \frac{1}{2}\sqrt{\frac{\mu_c\omega}{2\sigma}}\,|H_o|^2$$

We introduce $\delta^2 = \frac{2}{\mu_c\omega\sigma}$ or $\frac{1}{\sigma} = \frac{\delta^2\mu_c\omega}{2}$ so

the $\sqrt{\ }$ is $\sqrt{\frac{\mu_c\omega}{2}\,\frac{\delta^2\mu_c\omega}{2}}$ and

$$\frac{dP}{dA} = \frac{\mu_c\omega\delta}{4}|H_o|^2 = \frac{\delta}{4}\,\frac{2}{\delta^2\sigma}|H_o|^2 = \frac{1}{2\delta\sigma}|H_o|^2$$

b) The energy stored a distance ξ in the metal is

$$u(\xi) = \frac{1}{4}\mu_c|H(\xi)|^2 = \frac{\mu_c}{4}|H_o|^2\exp\left[-\frac{2\xi}{\delta}\right].$$

The energy stored per unit area is $\int_0^\infty u(\xi)\,d\xi$ so we integrate inward, the integral is

$$\frac{\mu_c}{4}\cdot\frac{\delta}{2}|H_o|^2 = \frac{\mu_c\delta}{8}|H_o|^2.$$