Set 10 – due 10 November

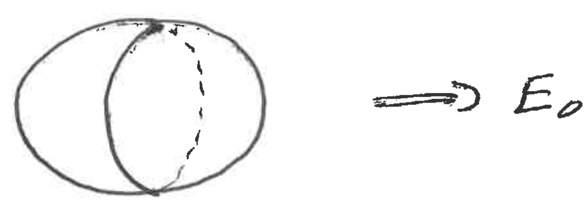
"Everything should be made as simple as possible, but not simpler."–A. Einstein

1) Jackson 6.14 [20 points] (a)–10; (b)–5; (c)–5; The purpose of this problem is to get you to walk through the Feynman lectures, vol. II, Ch 23. The business about Bessel functions at the end comes from thinking abut the wave equation for the interior of a cylindrical cavity, with no azimuthal dependence for the wave and the assumption that the wave vanishes at the surface. (This is actually not completely correct, either. But the resonant frequencies will be $\omega = c/a$ times order-unity constants.)

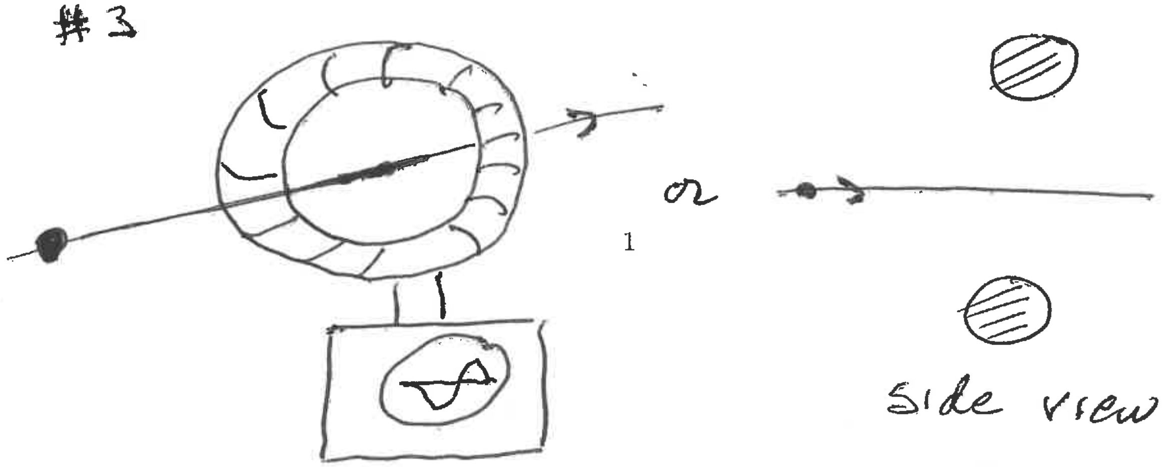
2) [20 points] A conducting spherical shell of radius a is placed in a uniform electric field $\vec{E}$. Find the force tending to separate the two halves of the sphere across a diametrical plane perpendicular to $\vec{E}$ (a) using the stress tensor [15 points] and (b) [5 points] integrating the appropriate projection of $\sigma^2/(2\epsilon_0)$ over a hemisphere.

3) [20 points] A toroid made of magnetic material is used to detect 10 MeV protons from a pulsed accelerator. Assume that the beam of protons is concentrated as a point and moves normal to the plane of the toroid. If there are 10^8 protons per pulse, the number of turns of wire on the toroid is 100, $\mu(\text{toroid})/\mu_0 = 100$; mean radius is 0.5 cm, and the cross sectional area is 0.1 cm^2, calculate the voltage output per burst.

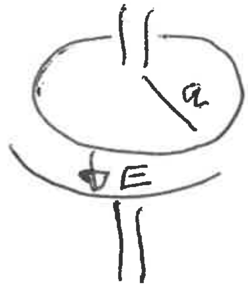
#2



#3



6.14



$I(t) = I_0 \cos\omega t$

a) Begin with the charge on the upper plate,

$Q(t) = \frac{I_0}{\omega} \sin\omega t = Q_0 \sin\omega t$ with $Q_0 = \frac{I_0}{\omega}$

For small ω, $\vec{E}^{(0)} = -\frac{\sigma}{\epsilon_0}\hat{z} = -\frac{Q_0}{\pi a^2}\frac{1}{\epsilon_0}\hat{z}$

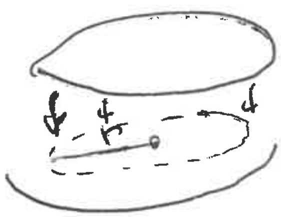
$$\vec{E}^{(0)}(t) = -\frac{Q_0}{\pi a^2 \epsilon_0} \sin\omega t \; \hat{z} \qquad (1)$$

Then $c^2 \vec{\nabla}\times\vec{B} = \frac{\partial \vec{E}}{\partial t}$ or $\oint \vec{B}^{(0)} \cdot d\ell = 2\pi r B^{(0)}(r)$

$$2\pi r B^{(0)}(r) = \frac{1}{c^2}\frac{d}{dt}\int \vec{E}\cdot\hat{n}\, dA$$

integrating around a circle of radius r

Fig 2



$\vec{B}^{(0)}(t) = -\frac{1}{c^2}\frac{\pi r^2}{2\pi r}\frac{I_0}{\pi a^2 \epsilon_0}\cos\omega t \cdot \hat{\varphi}$

or $\vec{B}^{(0)}(t) = -\hat{\varphi}\frac{r}{2\pi a^2}\frac{I_0}{c^2\epsilon_0}\cos\omega t$ for $r<a$.

($\hat{\varphi}$ = azimuthal direction)

Note $\mu_0\epsilon_0 = \frac{1}{c^2}$ so $\frac{1}{c^2\epsilon_0} = \mu_0$.

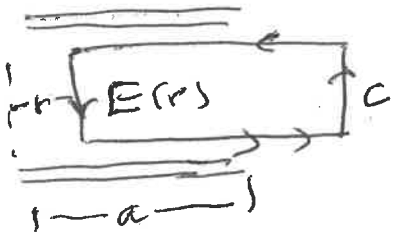
Next use $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/dt$ to find $\vec{E}^{(1)}$:

$$\oint \vec{E}^{(1)} \cdot d\ell = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\, dA$$

Take the contour as shown; note

Fig 3

$\vec{\nabla}\times\vec{E}^{(0)} = 0$ so $\int \vec{E}^{(0)} \cdot d\vec{\ell} = 0$



$$\oint E^{(1)}\cdot dl = \left[E^{(1)}(r) + C\right]\cdot d$$

$$= \frac{I_0\,\omega \sin\omega t}{2\pi a^2 \epsilon_0 c^2}\cdot d \cdot \int_r^a r'\,dr' \quad \text{integrating } B$$

$$E^{(1)}(r) + C = \frac{I_0\,\omega \sin\omega t}{2\pi a^2 \epsilon_0 c^2}\,\frac{1}{2}(a^2 - r^2)\,\hat{z} \qquad (2)$$

To determine C, think about integrating over the capacitor: $\int \vec{E}\cdot\hat{n}\,dA = Q/\epsilon_0$. This is saturated by $E^{(0)}$, hence $\int \vec{E}^{(1)}\cdot\hat{n}\,dA = 0$. Rewrite (2) as

$$\vec{E}^{(1)}(r) = \frac{I_0\,\omega \sin\omega t}{4\pi \epsilon_0 c^2}\left[\frac{a^2 - r^2}{a^2} + C'\right]\hat{z}$$

and fix C' so $\int E^{(1)}\cdot n\,dA = 0$. This is

$$\pi a^2 C' = -2\pi\int_0^a r\,dr\left[1 - r^2/a^2\right] = -\pi a^2 + \frac{\pi a^4}{2a^2} = -\frac{\pi a^2}{2}$$

so $C' = -\frac{1}{2}$ and

$$\vec{E}^{(1)}(r) = \hat{z}\, I_0\,\frac{\omega \sin\omega t}{4\pi \epsilon_0 c^2}\left[1 - \frac{r^2}{a^2} - \frac{1}{2}\right]$$

$$\vec{E}(r,t) = \vec{E}^{(0)} + \vec{E}^{(1)}$$

$$= \hat{z}\left\{-\frac{Q_0 \sin\omega t}{\pi \epsilon_0 a^2} - \frac{Q_0\,\omega^2 \sin\omega t}{8\pi\epsilon_0 c^2}\left(1 - \frac{2r^2}{a^2}\right)\right\}$$

$$= -\hat{z}\,\frac{Q_0 \sin\omega t}{\pi \epsilon_0 a^2}\left[1 + \frac{\omega^2 a^2}{8 c^2}\left(1 - \frac{2r^2}{a^2}\right) + \cdots\right]$$

And $B^{(1)}$ from $\oint B^{(1)} \cdot dl = \frac{1}{c^2}\frac{d}{dt}\int \vec{E}^{(1)} \cdot \hat{n}\, dA$. 6-14.3

Use the contour of Fig 2 again –

$$2\pi r\, B^{(1)}(r) = -\frac{1}{c^2}\frac{\varphi_0\, \omega^3 \cos\omega t}{8\pi\epsilon_0 c^2}\int_0^r 2\pi r' dr' \left(1 - \frac{2r'^2}{a^2}\right)$$

$$= -\frac{2\pi\omega^3 \varphi_0 \cos\omega t}{8\pi\epsilon_0 c^4}\left[\frac{r^2}{2} - \frac{1}{2}\frac{r^4}{a^2}\right]$$

$$\vec{B}^{(1)} = -\hat{\varphi}\,\frac{\omega^3 \varphi_0 \cos\omega t}{16\pi\epsilon_0 c^4}\cdot r\cdot\left(1 - \frac{r^2}{a^2}\right)$$

$$\vec{B} = \vec{B}^{(0)} + \vec{B}^{(1)} = -\hat{\varphi}\,\mu_0 \frac{\varphi_0 \cos\omega t}{2\pi a^2}\left[1 + \frac{\omega^2 a^2}{8c^2}\left(1 - \frac{r^2}{a^2}\right) + \cdots\right]$$

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b) The energy densities give us the reactances.

$$w_e = \frac{\epsilon_0}{2}\left\langle |E|^2 \right\rangle_{\text{time average}} ; \quad U_e = \int w_e\, d^3x$$

$$w_e = \frac{\epsilon_0}{2}\cdot\frac{1}{2}\cdot\frac{I_0^2}{\omega^2\pi^2\epsilon_0^2 a^4}\left[1 + \frac{\omega^2 a^2}{8c^2}\left(1 - \frac{2r^2}{a^2}\right)\right]^2$$

Integrate over the volume between the plates

$$U_e = \frac{I_0^2}{4\pi^2\epsilon_0\omega^2}\cdot\frac{2\pi d}{a^4}\int_0^a r\,dr\left[1 + \frac{2\omega^2 a^2}{8c^2}\left(1 - \frac{2r^2}{a^2}\right) + \cdots\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{I_0^2}{\omega^2}\frac{2d}{a^4}\cdot\frac{1}{2}a^2 + \cdots = \frac{1}{4\pi\epsilon_0}\frac{I_0^2}{\omega^2}\frac{d}{a^2} + \cdots$$

$$U_m = \frac{1}{2\mu_0}\frac{1}{2}\frac{\mu_0^2}{4\pi^2 a^2} I_0^2 \cdot 2\pi d\int_0^a r\,dr \cdot r^2\left(1 + \frac{\omega^2 a^2}{4c^2}\left(1 - \frac{r^2}{a^2}\right) + \cdots\right)$$

$$= \mu_0\frac{I_0^2}{8\pi}\frac{d}{a^4}\left\{\frac{1}{4}a^4 + \frac{\omega^2 a^2}{4c^2}\left(\frac{1}{4}a^4 - \frac{1}{6}\frac{a^6}{a^2}\right) + \cdots\right\}$$

$$u_M = \frac{\mu_0}{4\pi} I_0^2 \frac{d}{8}\left(1 + \frac{1}{12}\frac{\omega^2 a^2}{c^2} + \cdots\right)$$

c) We can write the time-averaged electrical energy as $u_E = \frac{1}{2} \cdot \frac{1}{2} Q_0 V_0$ where one $\frac{1}{2}$ is for the time average and the other is $\frac{1}{2}\int \rho \Phi$.

Replace Q_0 by the capacitance $\mathcal{C}$, where $Q_0 = \mathcal{C} V_0$ -

$$u_E = \frac{1}{4}\frac{Q_0^2}{\mathcal{C}} = \frac{1}{4}\frac{I_0^2}{\omega^2 \mathcal{C}} = \frac{1}{4\pi\epsilon_0}\frac{I_0^2}{\omega^2}\frac{d}{a^2}$$ from

the last page. This is $\mathcal{C} = \frac{\epsilon_0 \pi a^2}{d} = \epsilon_0 \cdot \frac{\text{area}}{\text{separation}}$

(as expected)

$$U_M = \frac{1}{4} L I_0^2 = \frac{\mu_0}{4\pi} I_0^2 \frac{d}{8} \Rightarrow L = \frac{\mu_0}{4\pi}\frac{d}{2}.$$

Finally, the resonant frequency of an LC circuit is

$$\omega_{res}^2 = \frac{1}{LC} = 4\pi \cdot \frac{2}{d} \cdot \frac{1}{\mu_0} \cdot \frac{1}{\epsilon_0}\frac{d}{\pi a^2}$$

$$= \frac{8}{\mu_0 \epsilon_0}\frac{1}{a^2} = \frac{8c^2}{a^2}$$

$$\omega_{res} = \frac{c}{a}\sqrt{8} = 2.828\frac{c}{a}$$

The business about the Bessel function comes from thinking about a cylindrical resonant cavity with no fringing fields. In Lorentz gauge $\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$ and "no fringing fields" means $A = 0$ at $r = a$. Suppose we also ask for no φ or z dependence, and write

$$\vec{A}(\rho, \phi, z, t) = \vec{A}(\rho)\, e^{-i\omega t}$$

Then
$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial A}{\partial \rho} + \frac{\omega^2}{c^2} A = 0$$

or
$$\frac{\partial^2 A}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial A}{\partial \rho} + \frac{\omega^2}{c^2} A = 0$$

This equation is solved by $A(\rho) \propto J_0\left(\frac{\omega \rho}{c}\right)$.

If $A = 0$ at the edge of the cavity

$$J_0\left(\frac{\omega a}{c}\right) = 0$$

Tables give the quantization condition

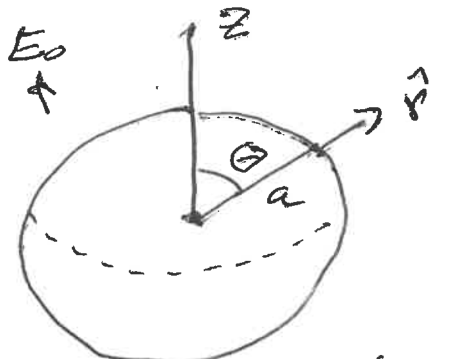
$$\frac{\omega a}{c} = 2.405,\ 5.52,\ \text{etc}$$

In part c) we guessed $\frac{\omega a}{c} = 2.82$.
Actually, neither answer is correct, due to the fringing fields!

What is the force needed to separate two conducting hemispheres in an external E field?

The force per unit area in the z direction at a point $\hat{r}$ on the surface of the sphere is

$$\frac{F_z}{A} = \hat{z}\, T_{zr}\, \hat{r} \quad \text{where } T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right]$$



Recall that if $r > a$

$$\Phi = \left(-E_0 r + E_0 \frac{a^3}{r^2} \right) \cos\theta .$$

On the surface of the sphere, $E_{tan} = 0$ because the sphere is a conductor.

Thus $\vec{E} = \hat{r}\, E_r$ is purely radial.

$$E_r = -\frac{\partial \Phi}{\partial r}\bigg)_{r=a} = -\cos\theta \left[-E_0 - 2E_0 \frac{a^3}{r^3} \right]_{r=a}$$

$$= 3E_0 \cos\theta$$

$E_z = E_r \cos\theta$ on the surface.

Then

$$\frac{F_z}{A} = \epsilon_0 \left[E_z E_r - \frac{1}{2}\hat{z}\cdot\hat{r}\, E^2 \right]$$

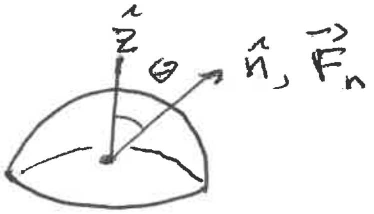
$$= \epsilon_0 \left[\cos\theta - \frac{1}{2}\cos\theta \right] (3E_0 \cos\theta)^2$$

$$= \frac{9}{2}\epsilon_0 E_0^2 \cos^3\theta .$$

To find the total force, integrate over a hemisphere

$$F = 2\pi a^2 \int_0^1 d\cos\theta \cdot \frac{9}{2}\epsilon_0 E_0^2 \cos^3\theta = \frac{9}{4}\pi a^2 \epsilon_0 E_0^2$$

The direct approach uses the local normal force per unit area, $\frac{\sigma^2(\theta)}{2\epsilon_0}$. We must project this onto the z-direction.



$$\frac{F_z}{\text{area}} = \frac{\sigma^2(\theta)}{2\epsilon_0} \cos\theta$$

From $\Phi = \left(-E_0 r + E_0 \frac{a^3}{r^2}\right)\cos\theta$,

$$\frac{\sigma(\theta)}{\epsilon_0} = E_r = 3E_0 \cos\theta$$

$$\frac{\sigma^2}{2\epsilon_0} = \frac{9}{2}\frac{\epsilon_0^2}{\epsilon_0} E_0^2 \cos^2\theta$$

$$F = 2\pi a^2 \cdot \frac{9}{2}\epsilon_0 E_0^2 \int_0^1 d\cos\theta \cos^3\theta = \frac{9}{4}\pi a^2 \epsilon_0 E_0^2$$

(as before)

Note #1: with $\vec{E} = \hat{r} E_r$

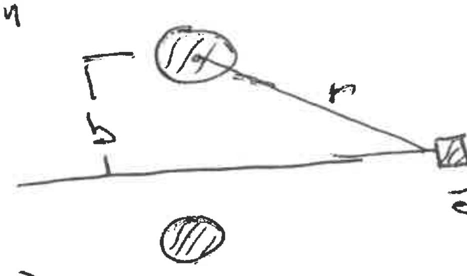
$$\overleftrightarrow{T} = \epsilon_0 E_r^2 \left[\overleftarrow{r}\vec{r} - \frac{1}{2}\left(\overleftarrow{r}\vec{r} + \overleftarrow{\theta}\vec{\theta} + \overleftarrow{\varphi}\vec{\varphi}\right)\right]$$

$$\hat{z}\cdot\overleftrightarrow{T}\cdot\hat{r} = \epsilon_0 E_r^2\Big[(\hat{z}\cdot\hat{r})(\hat{r}\cdot\hat{r}) - \frac{1}{2}\Big((\hat{z}\cdot\hat{r})(\hat{r}\cdot\hat{r}) + (\hat{z}\cdot\hat{\theta})(\hat{\theta}\cdot\hat{r}) + (\hat{z}\cdot\hat{\varphi})(\hat{\varphi}\cdot\hat{r})\Big)\Big]$$

$$= \frac{1}{2}\epsilon_0 E_r^2 (\hat{z}\cdot\hat{r}).$$

Note #2: $E=0$ inside the sphere so the total force from integrating over the total surface area just comes from the outer hemisphere - the flat base does not contribute.

3) The toroid: in cross-section

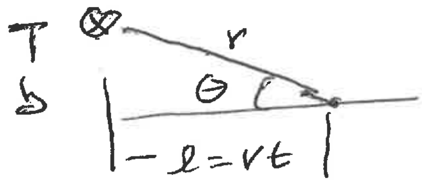


The pulse is a current density $\vec{J}(r',t) = q\vec{v}\,\delta^3(\vec{r}\,' - \vec{r}(t))$

and the induced magnetic field at ~~the~~ location $\vec{r}$ is

$$\vec{B}(r) = \frac{\mu_0}{4\pi}\int \vec{J}(r',t) \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r'$$

If the pulse moves to the left (as shown), the right hand rule sends B into the paper (above the center line



$\sin\theta = \frac{b}{r}$, $r^2 = b^2 + l^2$

Thus $B_\varphi = \frac{\mu_0}{4\pi}\, qv\, \frac{\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\, qv\, \frac{b}{(b^2+l^2)^{3/2}}$

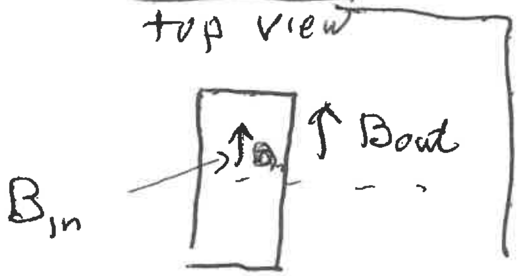
Also $\mu_0 \epsilon_0 = \frac{1}{c^2}$ so $\mu_0 = \frac{1}{c^2\epsilon_0}$.

Set $l = vt$ so the pulse passes through the toroid at $t=0$.

$$B_\varphi = \frac{1}{4\pi\epsilon_0}\,\frac{qvb}{c^2}\,\frac{1}{[b^2+v^2t^2]^{3/2}}$$

Now to go inside the ~~solenoid~~ toroid. $\vec{B}$ and $\vec{H}$ are tangential to the toroid. H_{tan} is continuous so

top view



$$B_{in} = \frac{\mu}{\mu_0} B_{out}.$$

The magnetic flux is multiplied times N for N terms of wire, so

$$\Phi = N \cdot A \cdot \frac{\mu}{\mu_0} B_{out}(t)$$

where A is the cross sectional area of the toroid. Putting everything together,

$$\Phi(t) = \frac{1}{4\pi\epsilon_0} N A \left(\frac{\mu}{\mu_0}\right) \frac{q v b}{c^2 [b^2 + v^2 t^2]^{3/2}} .$$

Faraday's law is $\mathcal{E} = -\dot{\Phi}$.

$$-\frac{d}{dt} \frac{1}{[b^2+v^2t^2]^{3/2}} = \frac{3}{2} \frac{-2v^2 t}{[b^2+v^2t^2]^{5/2}} .$$

The maximum $\mathcal{E}$ is at $\ddot{\Phi} = 0$ or

$$\frac{1}{[b^2+v^2t^2]^{5/2}} = \frac{5 v^2 t^2}{[b^2+v^2t^2]^{7/2}}$$

or at $b^2 + v^2 t^2 = 5 v^2 t^2$ or $\boxed{vt = \pm \frac{b}{2}}$

$$\mathcal{E}_{max} = 3 \frac{q}{4\pi\epsilon_0} \left(\frac{\mu}{\mu_0}\right) N A \left(\frac{v^2}{c^2}\right) \frac{bv \cdot \left(\frac{b}{2v}\right)}{\left[b^2 + \frac{b^2}{4}\right]^{5/2}}$$

$$= \frac{q}{4\pi\epsilon_0} \left(\frac{\mu}{\mu_0}\right) NA \left(\frac{v^2}{c^2}\right) \frac{1}{b^3} \cdot \frac{3}{2} \cdot \frac{1}{(5/4)^{5/2}}$$

The constant is $\frac{3}{2} \cdot \frac{32}{5^{5/2}} = 48/5^{5/2}$

$$\frac{v^2}{c^2} = \frac{2E}{mc^2} \quad \text{since } E = \frac{1}{2}mv^2 \quad mc^2 = 10^3 \text{ MeV}$$

(940 MeV) really

$$\text{so } \frac{v^2}{c^2} = \frac{2 \times 10 \text{ MeV}}{10^3} = 0.02$$

$$\frac{1}{4\pi\epsilon_0} = 9 \cdot 10^9, \quad q = 10^8 \times 1.6 \times 10^{-19} \text{ C}$$

$$\frac{\mu}{\mu_0} = 100, \quad N = 100, \quad A = 10^{-5} \text{ m}^2, \quad b = 0.5 \cdot 10^{-3} \text{ m}$$

$$\mathcal{E}_{max} = (9 \times 10^9)(1.6 \times 10^{-11})(100)(100)(10^{-5} \text{ m})^2 \times \frac{0.02}{(0.5 \times 10^{-3} \text{ m})^2} \times \frac{48}{5^{5/2}} \text{ Volts}$$

$$= 1978 \text{ V} \approx 2 \text{ kV}$$