Set 1– due 8 September

"You won't become a good pianist by listening to good concerts." – J. Wess

For this week, we have some short problems, while we build tools: First some (rather formal) Green's theorem manipulations...

1) [5 points] Jackson 1.10: note that this is only true if there is no charge inside the sphere!

2) [5 points] Jackson 1.12: this is needed for

3) [10 points] Jackson 1.13

4) [20 points] A sphere of radius R has a uniform charge density ρ and total charge Q. Find the electrostatic potential energy of the sphere, that is, the energy required from some non-electric interaction to hold the sphere together. On dimensional grounds, the answer will be

$$W = C\frac{1}{4\pi\epsilon_0}\frac{Q^2}{R} \tag{1}$$

and the interesting quantity is the constant C. Do this problem "like an undergraduate," that is, use Gauss's law to find $\vec{E}$, then integrate it to find $\Phi(r)$. Do the problem two ways: compute (a) [10 points]

$$W = \frac{1}{2}\int d^3x\rho(x)\Phi(x) \tag{2}$$

(b) [10 points]

$$W = \frac{\epsilon_0}{2}\int d^3x|\vec{E}(x)|^2. \tag{3}$$

We'll visit the third alternative

$$W = \frac{1}{2}\int \rho(r)\rho(r')\frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r'}|}d^3rd^3r' \tag{4}$$

later in the course–this looks like a lot but it is actually the easiest way to do the problem.

Jackson 1.10. From Jackson eq. 1.36 we have

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho(x)}{|x-x'|} + \frac{1}{4\pi}\oint_{S'} dA'\left[\frac{1}{R}\frac{d\Phi}{dn'} - \Phi\frac{\partial}{\partial n'}\frac{1}{R}\right]$$

If $\rho(x')=0$ the first term vanishes. Let S be a sphere of radius R, let x be a point on the interior — in fact let it be in the center and choose coordinates so $x=0$. Then $\hat{n}'$ and x' are parallel, $r' = |\vec{x}'| = R$

$$\frac{\partial\Phi}{\partial n'} = -\vec{E}\cdot\hat{n}', \quad \frac{\partial}{\partial r'}\frac{1}{r'}\Big|_{r'=R} = -\frac{1}{R^2},$$

using $\frac{\partial}{\partial n'} = n'\cdot\nabla' = \frac{\partial}{\partial r'}$. Put everything together

$$\Phi(0) = -\frac{1}{4\pi R}\int_S dA'\,\vec{E}\cdot\hat{n}' + \frac{1}{4\pi R^2}\int dA'\,\Phi(x')$$

Gauss' law says $\int dA'\, E\cdot\hat{n}' = \frac{Q_{enclosed}}{\epsilon_0} = 0$

so $$\Phi(0) = \frac{1}{4\pi R^2}\int dA'\,\Phi(x')$$

The potential at the center is the average value of Φ on the surface of the sphere.

2) 1.12 Call the potential due to ρ & σ Φ

Call the potential due to ρ' & σ' Φ'.

We are asked to show that

$$\int_V \rho \Phi' \, d^3x + \int_{S'} \sigma \Phi' \, dA = \int_{V'} \rho' \Phi \, d^3x + \int_{S'} \sigma' \Phi \, dA$$

Start with Jackson eq. 1.35, Green's 2nd identity,

set $\phi = \Phi \quad \psi = \Phi'$

$$\rho = -\epsilon_0 \nabla^2 \Phi \quad \rho' = -\epsilon_0 \nabla^2 \Phi'.$$

Also recall that outside a conductor

$\dfrac{\partial \Phi}{\partial n} = -E_n = -\dfrac{\sigma}{\epsilon_0}$ — But in this formula, $\hat{n}$ is an outward normal $\to n$ while in 1.35 $\hat{n}$ is a normal into the integration region.

$$1.35: \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x = \int_{S'} dA \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right]$$

$$\text{or} \quad -\frac{1}{\epsilon_0} \int_V (\Phi \rho' - \Phi' \rho) \, d^3x = -\frac{1}{\epsilon_0} \int dA \left[-\Phi \sigma' + \Phi' \sigma \right]$$

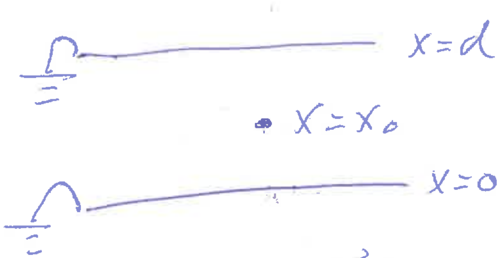
defn of normal

Just regroup

$$\int_V \Phi \rho' \, d^3x + \int_S \Phi \sigma' \, dA = \int_V \Phi' \rho \, d^3x + \int_S \Phi' \sigma \, dA$$

Now we apply it ---

unprimed system

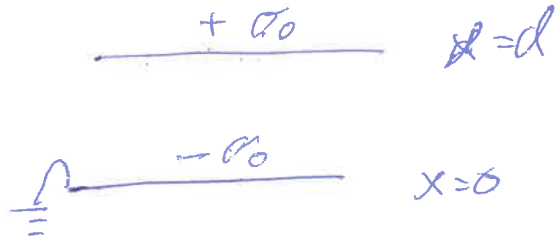


$\rho(x) = q\,\delta^3(x-x_0)$

$\vec{x}_0 = (x_0, 0, 0)$

$\Phi = 0$ on both surfaces

primed system



$E'_x = -\dfrac{\sigma_0}{\epsilon_0}$ ($\downarrow$!)

$\rho'(x) = 0$ $\Phi'(x) = \dfrac{\sigma_0 x}{\epsilon_0}$

$\Phi' = 0$ at $x=0$

$\sigma = \sigma(d)$ on the top plate
also σ on lower plate

$\sigma = \pm\sigma_0$

Both systems are infinite in the transverse directions - we need this to be true to solve the primed system.

Reciprocity formula says

$$\left.\begin{aligned} & q\int_V \delta(x-x_0)\,\delta(y)\,\delta(z)\left[\frac{\sigma_0 x}{\epsilon_0}\right] d^3x \quad \Phi'\rho \\ & + \int_{S'_-} \sigma_0 \cdot 0 \cdot dA + \int_{S'_+} \sigma(d)\cdot\frac{\sigma_0 d}{\epsilon_0}\, dA \quad \Phi'\sigma \end{aligned}\right.$$

$$= \underbrace{\int_V 0\cdot\Phi\, d^3x}_{\rho'\Phi} + \underbrace{\int_S \sigma'\cdot 0\, dA}_{\sigma'\Phi} = 0$$

or $$q\frac{\sigma_0 x_0}{\epsilon_0} + \frac{\sigma_0 d}{\epsilon_0}\int_{S'_+} \sigma(d)\, dA = 0$$

$$-\frac{x_0}{d}q = \int_{S'_+}\sigma(d)\, dA = \text{induced charge on } S'_+.$$ We'll derive this less mechanically later in the term.

4). $W = \frac{1}{2}\int \rho \Phi \, d^3x$ or $\frac{\epsilon_0}{2}\int E^2 d^3x$

Gauss: for $r<R$ $\quad 4\pi r^2 E = \frac{1}{\epsilon_0}\left(\frac{r^3}{R^3} Q\right)$

$\left(\oint E \cdot 4\pi r^2 = \frac{Q}{[\frac{4\pi}{3}R^3]} 4\pi \int_0^r r'^2 dr' \right)$

so $E_r = \frac{1}{4\pi\epsilon_0} Q \frac{r}{R^3}$ inside

$= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$ outside

$\Phi = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$ outside so $\Phi \to 0$ as $r \to \infty$.

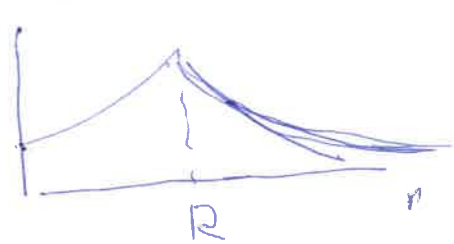
Then inside $\Phi(r) - \Phi(0) = -\int_0^r E \, dl = -\frac{Q}{4\pi\epsilon_0}\frac{1}{R^3}\int_0^r r\,dr$

$= -\frac{Q}{4\pi\epsilon_0}\frac{1}{R^3}\cdot\frac{1}{2} r^2$

$\Phi(R) = \Phi(0) - \frac{Q}{4\pi\epsilon_0}\frac{1}{2}\frac{1}{R} \quad = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$

so $\Phi(0) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{R} + \frac{Q}{2R}\right] = \frac{3}{2}\frac{Q}{4\pi\epsilon_0}\frac{1}{R}$

or $\Phi(r) = \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{r^2}{R^3} + \frac{3}{2}\frac{1}{R}\right]$ if $r<R$



Part (a)

$$W = \frac{1}{2}\int_0^R 4\pi r^2 dr\, \rho(r)\, \Phi(r)$$

$$= \frac{1}{2}\left[\frac{Q}{\frac{4}{3}\pi R^3}\right] 4\pi \int_0^R r^2 dr \cdot \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{2}\frac{r^2}{R^3} + \frac{3}{2}\frac{1}{R}\right)$$

$$= \frac{Q^2}{4\pi\epsilon_0}\cdot\frac{3}{2}\cdot\frac{1}{R^3}\cdot\int_0^R dr\left(-\frac{1}{2}\frac{r^4}{R^3} + \frac{3}{2}\frac{r^2}{R}\right)$$

$$= \frac{Q^2}{4\pi\epsilon_0}\;\frac{3}{2}\;\frac{1}{R^3}\left\{-\frac{1}{2}\cdot\frac{1}{5}\frac{R^5}{R^3} + \frac{3}{2}\cdot\frac{1}{3}\frac{R^3}{R}\right\}$$

$$= \frac{Q^2}{4\pi\epsilon_0}\,\frac{1}{R}\cdot q$$

$$q = \frac{3}{2}\left\{-\frac{1}{10} + \frac{1}{2}\right\} = \frac{3}{2}\cdot\frac{4}{10} = \underline{\underline{\frac{3}{5}}}$$

b) $W = \frac{\epsilon_0}{2} \int d^3x \, E^2(x)$

$$= \frac{\epsilon_0}{2} \int_{r<R} 4\pi r^2 dr \, E_{in}^2 + \frac{\epsilon_0}{2} \int_{r>R} 4\pi r^2 dr \, E_{out}^2$$

$$= \frac{\epsilon_0}{2} \cdot \left(\frac{1}{4\pi\epsilon_0}\right)^2 4\pi \cdot Q^2 \times$$

$$\left\{ \int_0^R r^2 dr \left(\frac{r}{R^3}\right)^2 + \int_R^\infty r^2 dr \frac{1}{r^4} \right\}$$

$$= \frac{Q^2}{4\pi\epsilon_0} \cdot \frac{1}{2} \cdot \left[\frac{1}{5} \frac{R^5}{R^6} + \frac{1}{R} \right]$$

$$= \frac{Q^2}{4\pi\epsilon_0} \frac{1}{R} \cdot q$$

$q = \frac{1}{2}\left(\frac{1}{5} + 1\right) = \frac{1}{2} \cdot \frac{6}{5} = \frac{3}{5}$ again!