Wave guides + cavities -

An enormous modern literature

metallic waveguides + cavities

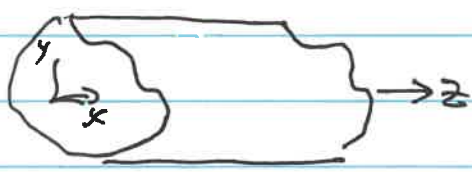
fiber optic cables

photonic band gap materials

Issue: solving coupled PDE's with boundary conditions is hard

but a lot of interesting (even ~~[illegible]~~) physics is accessible from a simple (!) subset -

Metallic waveguides + cylindrical cavities



- Fields confined in x, y
- have to discuss b.c., - can make this simple
- If walls are good conductors, an easy approximation gives power loss

$\rho = \vec{J} = 0$, $\vec{E}, \vec{B} \sim e^{-i\omega t} \Rightarrow$

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B}, \quad \vec{\nabla}\times\vec{B} = -i\mu\epsilon\omega\vec{E}, \quad \nabla\cdot E = 0, \quad \nabla\cdot B = 0$$

or

$$\left[\nabla^2 + \mu\epsilon\omega^2\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

Next, assume z dependence

$$\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{-i\omega t}e^{\pm ikz}$$

$\pm ikz$ for right or left moving wave - cavity needs superposition for standing wave

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \quad \text{so}$$

$$\Big[\nabla_t^2 + \underbrace{[\mu\epsilon\omega^2 - k^2]}_{\gamma^2} \Big] \begin{pmatrix} \vec{E}(x,y) \\ \vec{B}(x,y) \end{pmatrix} = 0 \qquad (*1)$$

so far, so good ... now boundary conditions

vacuum $B, H,$ D, E $\uparrow \hat{n}$

conductor B_c, H_c D_c, E_c

σ_F : free surface charge density
$\vec{K}$ = " " current

$$\hat{n}\cdot(\vec{D}-\vec{D}_c) = \sigma_F \qquad \hat{n}\cdot(\vec{B}-\vec{B}_c) = 0$$
$$\hat{n}\times(\vec{H}-\vec{H}_c) = \vec{K} \qquad \hat{n}\times(\vec{E}-\vec{E}_c) = 0$$

hmm ... For a static ~~conductor~~ ~~problem~~ conductor,
$\hat{n}\cdot\vec{D} = \sigma_F$, $\hat{n}\times\vec{E} = 0$.

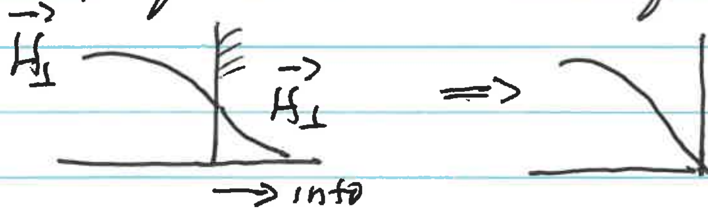
Static magnetic fields penetrate conductors – it's μ that counts, not σ.

Non-static: $\vec{J} = \sigma\vec{E} \Rightarrow \nabla\times H = J + i\frac{\omega}{\sigma}J \ldots$

Jump to the answer! Real time varying fields only penetrate a distance equal to the skin depth

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}} \longrightarrow 0 \text{ as } \sigma\to\infty,\ \omega\neq 0$$

"perfect conductor" defined as $\omega\sigma\to\infty$



$\vec{H}_\perp = 0 \quad (\vec{H}\cdot\hat{n} = 0)$

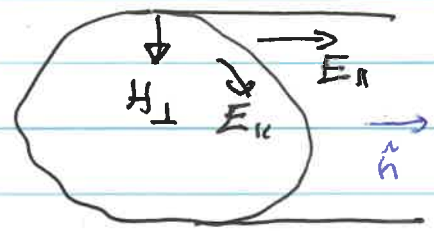
b.c. at a good conductor $\begin{cases} \vec{B}\cdot\hat{n} = 0 & \vec{E}_{tan} = 0 \\ \hat{n}\times\vec{H} = \vec{K} & \hat{n}\cdot\vec{D} = \sigma_F \end{cases}$ $*2$

So our problem is

$$(\nabla_t^2 + \gamma^2)\begin{pmatrix}\vec{E} \\ \vec{H}\end{pmatrix} = 0$$

$\hat{n} \times \vec{E} = 0$ or $\vec{E}_{\parallel} = 0$ or $E_{\tan} = 0$

$\hat{n} \cdot \vec{H} = 0$ or $\vec{H}_{\perp} = 0$



$\vec{S} = \frac{1}{2}(\vec{E} \times \vec{H}^*)$

gives energy flow

So far, no loss into walls.

New! ~~Problem~~ $\vec{E}$ & $\vec{H}$ no longer transverse!

Follows from $\vec{E}(x,t) = \vec{E}_0(\vec{x}_T)\, e^{\pm i(k \cdot z - \omega t)}$

$= \vec{E}_t + \vec{E}_L$

$0 = \nabla \cdot E = \vec{\nabla}_t \vec{E}_t(x_T) + \frac{\partial}{\partial z} E_L(x_T) e^{\pm ikz}$

$0 = \vec{\nabla}_t \cdot \vec{E}_t(x_t) + ik\, E_L(x_T) e^{\pm ikz}$

only if E_t has no x_T dependence

is $E_z = 0$

step?

The ultimate separation of variables problem -

$\nabla \cdot E = 0 \implies$ i) $\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0$

ii) $\vec{\nabla}_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0$

iii) $\nabla \times E = \frac{\partial B}{\partial t} \implies \left(\vec{\nabla}_t + \hat{z}\frac{\partial}{\partial z}\right) \times \vec{E} = i\omega\left(\vec{B}_t + \vec{B}_z\right)$

$\nabla_t \times E_z = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & E_z \end{vmatrix}$ has no z component, pure t

$\nabla_t \times E_t = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ E_x & E_y & 0 \end{vmatrix}$ has no t component, pure z

$\nabla_z \times E = \begin{vmatrix} i & j & k \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \vec{\nabla}_z \times \vec{E}_t = \hat{z} \times \frac{\partial E_t}{\partial z}$

$= \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{\partial E_x}{\partial z} & \frac{\partial E_y}{\partial z} & 0 \end{vmatrix}$

so $\hat{z} \cdot (\vec{\nabla} \times \vec{E}) = \boxed{\hat{z} \cdot (\vec{\nabla}_t \times \vec{E}_t) = i\omega B_z}$ iiia

$\hat{z} \times (\vec{\nabla} \times \vec{E}) = \underbrace{\hat{z} \times (\vec{\nabla}_t \times \vec{E})}_{\nabla_t E_z - \cancel{(\hat{z} \cdot \nabla_t)} E} + z \times (\nabla_z \times E)$

$+ \hat{z} \times \left(z \times \frac{\partial \vec{E}_t}{\partial t}\right)$

$$\boxed{\vec{\nabla}_t E_z - \frac{\partial \vec{E}_t}{\partial z} = i\omega \hat{z} \times \vec{B}_t} \quad \text{iiib}$$

iv) $\vec{\nabla} \times \vec{B} = -i\omega\mu\epsilon\vec{E}$: iva $\hat{z} \cdot (\nabla_t \times B_t) = -i\omega\mu\epsilon E_z$

ivb $\nabla_t B_z - \frac{\partial B_t}{\partial z} = -i\omega\mu\epsilon\, \hat{z} \times \vec{E}_t$

The systematic derivation is very messy - just go to the result.

3 kinds of solutions in a metallic waveguide

1) Transverse Magnetic $B_z = 0$ (TM)

2) Transverse Electric $E_z = 0$ (TE)

These are the only solutions in hollow waveguides

3) ~~Transverse~~ TEM - transverse electromagnetic

(in coaxial cable, wires, a lot ...)

$$E_z = B_z = 0 \quad \text{as in free space}$$

$$\vec{E} = \vec{E}_t(x,y)\, e^{i(\pm kz - \omega t)}$$

$$k^2 = \mu\epsilon\omega^2 \quad \text{or } \gamma = 0$$

so $$\left(\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right) E_t = \nabla_t^2 \vec{E}_t = 0$$

$$\nabla \cdot E = 0 \Rightarrow \boxed{\vec{\nabla}_t \cdot \vec{E}_t = 0} \quad \text{because } E_z = 0$$

$$\hat{z} \cdot \left(\nabla \times E - \frac{\partial \vec{B}}{\partial t}\right) = 0 = \hat{z} \cdot (\nabla \times E_t) \quad \text{since } B_z = 0$$

$$\hat{z} \cdot (\vec{\nabla} \times \vec{E}_t) = \begin{vmatrix} i & j & \boxed{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = \vec{\nabla}_t \times \vec{E}_t$$

$$\boxed{\vec{\nabla}_t \times \vec{E}_t = 0} \Rightarrow \text{2-d electrostatics problem}$$

To find $\vec{B}$, $\hat{z} \times (\vec{\nabla} \times \vec{E}) = i\omega\, \hat{z} \times \vec{B}_t$

$$\underset{0}{\hat{z} \times (\vec{\nabla}_t \times \vec{E}_t)} + \underset{\hat{z} \times (ik\,\hat{z} \times \vec{E}_t)}{\hat{z} \times (\vec{\nabla}_z \times \vec{E}_t)} = i\omega\, \hat{z} \times \vec{B}_t$$

$$\boxed{\vec{B}_t = (\hat{z} \times \vec{E}_t)\sqrt{\mu\epsilon}} \quad \text{as in free space}$$

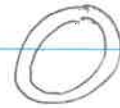
TEM $\quad \nabla \cdot E_t = 0, \; \nabla \times E_t = 0$

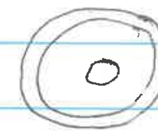
$B_t = \hat{z} \times \vec{E}_t \cdot \sqrt{\mu\epsilon} \qquad E_{\parallel} = 0, \; H_{\perp} = 0$

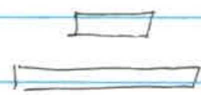
for TEM mode, transverse fields consistent w/ electrostatics in transverse dimensions

$\omega = ck$ dispersion rel.

Note - No TEM mode in hollow waveguide

by uniqueness theorem, Φ = constant on wall of conductor  hence $\vec{E} = 0$ inside.

Can have TEM mode in coaxial cable 

or slot  or transmission line 

(anything with holes)

Before going to other possibilities, pause to discuss energy flow & loss.

Flow is just $\hat{z} \cdot \vec{S} = \frac{1}{2}(\vec{E} \times \vec{H}^*) \cdot \hat{z}$

Power $= \int_{area} dA \; Re(\hat{z} \cdot \vec{S})$ -

Loss or Attenuation - Fields on surface $(H_{\parallel}, D_{\perp})$ penetrate into conductor, ohmic heating - or - just compute Poynting vector into the wall of the cavity -

To do this, we drop back & recall Maxwell's eqns in the conductor (basically Sec 8.1) - back away from perfection

i) $\nabla \times \vec{H}_c = J + \frac{\partial \vec{D}}{\partial t} \approx \sigma E_c$ if $J = \sigma E$ and $\frac{\sigma}{\omega\epsilon} \gg 1$

ii) $\vec{\nabla} \times \vec{E}_c = i\omega \vec{B}_c = i\omega\mu_c \vec{H}_c$

$\delta \approx \sqrt{\frac{2}{\sigma \mu_c \omega}}$

Attenuation – Fields on surface (~~magnitude $E_\perp$ & $H_\parallel$~~) penetrate into conductor: compute either ohmic heating or Poynting vector. Simple calculation after delicate approximation. uses $H_\parallel$ – it can be by In the conductor

i) $\nabla \times H_c = J + \dfrac{\partial D_c}{\partial t} \approx \sigma E$ if $J = \sigma E$, $D \approx 0$, $\dfrac{\sigma}{\omega \epsilon} \gg 1$

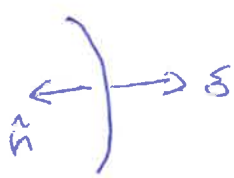
ii) $\vec{\nabla} \times \vec{E}_c = i\omega \vec{B}_c = i\omega \mu_c \vec{H}_c$

b.c. on surface: $\hat{n} \times (\vec{H}_{outside} - \vec{H}_c) = 0$

or $H_\parallel$ continuous $\Rightarrow$ input $H_c = H_\parallel$.

Next assume gradient dominated by component into conductor – rapid variation on scale δ, $\delta^2 = \dfrac{2}{\mu_c \omega \sigma}$

$$\vec{\nabla} \simeq -\hat{n} \frac{\partial}{\partial \xi}$$



$\hat{n}$ into cavity
ξ into material

then

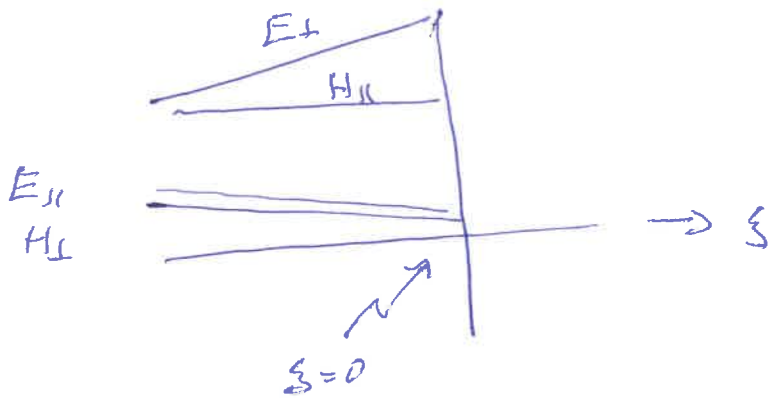
i) $-\hat{n} \times \dfrac{\partial \vec{H}_c}{\partial \xi} = \sigma \vec{E}_c$

ii) $H_c = -\dfrac{i}{\omega \mu_c} \hat{n} \times \dfrac{\partial E_c}{\partial \xi} = \dfrac{i}{\sigma \omega \mu_c} \hat{n} \times \left(n \times \dfrac{\partial^2 H_c}{\partial \xi^2}\right)$

$$H_c - \frac{\delta^2}{2i}\left[\hat{n} \frac{\partial}{\partial \xi}(\hat{n} \cdot H_c) - \frac{\partial^2 H_c}{\partial \xi^2} \right] = 0$$

$$\hat{n} \times \vec{H}_c = \frac{\delta^2}{2i} \frac{\partial^2}{\partial \xi^2} (\hat{n} \times \vec{H}_c) \Rightarrow H_\parallel \,!$$

(set $\hat{n} \cdot \vec{H}_c = 0$ since $H_\perp$ related to $E_\parallel$, $E_\parallel = 0$ on surface) the approximation: WG-7



neglect the small components! energy flow involves $\vec{S}$ by $H_\parallel$ → ~~by $E_\perp$~~

small $E_\parallel$ → small $H_\perp$ - neglect

$$\cancel{\hat{n}\times\vec{H}_e} \quad \frac{\partial^2}{\partial \xi^2} H_\parallel = \frac{2i}{\delta^2} \vec{H}_\parallel$$

$$\Rightarrow H_\parallel = H_0 \exp\left(-\frac{\xi}{\delta}\right) \exp\left[\frac{i\xi}{\delta}\right]$$

$$\vec{E}_\parallel = -\frac{1}{\sigma} \hat{n}\times \frac{\partial \vec{H}}{\partial \xi} \sim \frac{1}{\sigma\delta} H_\parallel$$

small but need for S

$$\cancel{\frac{dP_{loss}}{dA} = -\frac{1}{2} \mathrm{Re}\, \hat{n}\cdot(\vec{E}\times\vec{H}^*)}$$

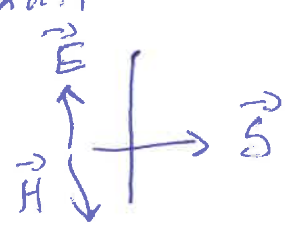
$$\frac{1}{\sigma\delta} = \sqrt{\frac{\mu_c \omega\sigma}{2\sigma^2}} = \sqrt{\frac{\mu_c\omega}{2\sigma}} \cdot = \sqrt{\frac{\mu_c^2\omega^2\cdot 2}{4\mu_c\omega\sigma}} = \frac{\mu_c\omega\delta}{2}$$

$$\cancel{\mu_c\omega\delta = \sqrt{\frac{\mu_c^2\omega^2\cdot 2}{\mu_c\omega\sigma}} = 2\frac{\mu_c\omega}{\sigma}}$$

$$\frac{dP_{loss}}{dA} = -\frac{1}{2} \mathrm{Re}\, \hat{n}\cdot(\vec{E}\times\vec{H}^*) = \frac{\mu_c\omega\delta}{4} |H_\parallel|^2$$

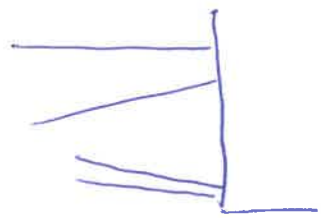
8.12

loss needs ∥!



$E_{\parallel}$ ~~in~~ $= \frac{\mu_c \omega \delta}{2} \hat{n} \times \vec{H}$ - in the conductor,

this is small compared to H. Look back at picture



A further approximation: Continuity says ~~$E_{\parallel}$~~

$E_{\parallel}$ inside $= E_{\parallel}$ outside

$\Rightarrow$ $E_{\parallel}$ outside is small: just replace by

$E_{\parallel}$ outside $= 0$ on surface.

Strictly speaking, this is not consistent - for ~~lossy~~ lossy materials it's a poor approximation.

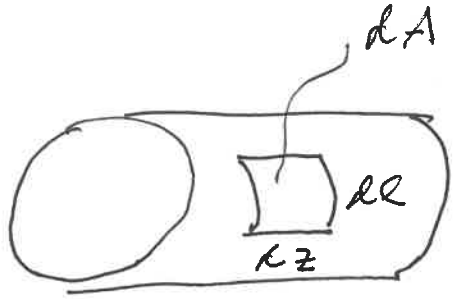
Most likely place to encounter this - resonant frequency of cavity shifted by leakage of fields into walls.

Also note

$$\frac{dP}{d\,\text{area}} = \frac{\mu_c \omega \delta}{4} |H_{\parallel}|^2 \rightarrow 0 \text{ as } \delta \rightarrow 0$$

WG 7.1

Use



$dA = dz$ (along waveguide) $\times\ dl$ (circumferential)

$$\frac{dP_{loss}}{dz} = \oint dl \, \frac{dP_{loss}}{dA}$$

Energy loss ≡ "attenuation"

Usually the RHS is proportional to P the power ~~down~~ along the waveguide

so

$$\frac{dP}{dz} = -\beta P$$

$$\Rightarrow P(z) = e^{-\beta z} P_0$$

$$\frac{1}{\beta} \equiv \text{"attenuation length"}$$

Ohm's law analogy

In medium $\frac{dP}{dA} = \mu_c \frac{\omega\delta}{4} |H_{\parallel}|^2$

$$\vec{H}_c = \vec{H}_{\parallel} e^{-\xi/\delta} e^{i\xi/\delta}$$

$$\vec{E}_c = \sqrt{\frac{\mu_c\omega}{2\sigma}} (1-i) (\hat{n}\times\vec{H}_{\parallel}) e^{-\xi/\delta} e^{i\xi/\delta}$$

Replace E_c by an effective current density

$$\vec{K}_{eff} = \int_0^\infty J(\xi) d\xi = \sigma \int_0^\infty E_c(\xi) d\xi$$

integrating into the conductor.

$$\vec{K}_{eff} = \sigma \sqrt{\frac{\mu_c\omega}{2\sigma}} (1-i)(\hat{n}\times\vec{H}_{\parallel}) \cdot \frac{\delta}{1-i}$$

$$= n \times H_{\parallel} \quad \text{since } \delta = \sqrt{\frac{2}{\mu_c\sigma\omega}}$$

$$\frac{dP_{loss}}{dA} = \frac{\mu_c\omega\delta}{4} |H_{\parallel}|^2$$

$$\frac{\mu_c\omega\delta}{4} = \sqrt{\frac{\mu_c^2\omega^2}{16}\frac{2}{\mu_c\sigma\omega}} = \sqrt{\frac{\mu_c\omega}{8\sigma}}$$

$$= \frac{1}{2}\sqrt{\frac{\mu_c\sigma\omega}{2\sigma^2}} = \frac{1}{2\sigma\delta}$$

$$\frac{dP}{dA} = \frac{1}{2\sigma\delta} |K|^2 \sim I^2 R$$

$R = \frac{1}{2\sigma\delta}$ = surface resistance per unit area

$$(\nabla^2 + \mu\epsilon\omega^2)\,\psi(x,y)\,e^{ik_g z} = 0$$
$$[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)]\,\psi(x,y) = 0$$

2 kinds of modes for hollow ~~conductor~~ waveguides

2) $E_z = 0$ everywhere, $\frac{\partial B_z}{\partial n} = 0$ on surface

Transverse Electric or TE mode
$(\vec{E}, \vec{B}) = (\vec{E}_t, \vec{B}_t, \vec{B}_z)$

3) $B_z = 0$ everywhere, $E_z = 0$ on surface
Transverse Magnetic or TM mode
$(\vec{E}, \vec{B}) = (\vec{E}_t, E_z, \vec{B}_t)$

Method of solution: use the z-field with the b.c.!

Recall, $\gamma^2 = \mu\epsilon\omega^2 - k^2$

TE $\quad (\nabla_t^2 + \gamma^2)\,B_z = 0, \quad \frac{\partial B_z}{\partial n} = 0$ on surface

TM $\quad (\nabla_t^2 + \gamma^2)\,E_z = 0 \quad E_z = 0$ on surface
$n \times E = 0$

$\Rightarrow$ An Eigenvalue Problem: only get a solution for special γ's, γ_λ

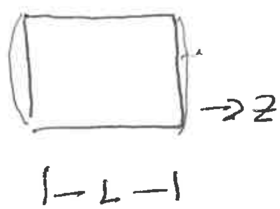
note different BC's for TE, TM, different γ_λ's

2 interesting physics effects

- dispersion and cutoff freq
- longitudinal field in waveguide (E_z, B_z)

Generally, need $\gamma_\lambda^2 > 0$ to satisfy b.c.

The physics of dispersion is easiest to see by comparison with a (cylindrical) cavity.



k_z is quantized by b.c. on E, B

$k_z \sim \ell\pi/L$

$\vec{E}, \vec{B} \sim \cos k_z z$ or $\sin k_z z$

$$\left(\nabla_\perp^2 + \gamma_\lambda^2\right)\vec{E}_{k,\lambda}(x,y) = 0 \; ; \quad \gamma_\lambda^2 = \frac{\omega^2}{c^2} - k_z^2$$

or $\frac{\omega^2}{c^2} = \gamma_\lambda^2 + k_z^2$

This is an eigenvalue problem - usually γ_λ^2 is discrete.

example: square cavity, TM ($E_z = 0$ on surface)

transverse soln $(E_z)_{k,\lambda} \sim \sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{a} \times \cancel{\cos\frac{\ell\pi z}{L}}$

$$\gamma_\lambda^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2 = \frac{\omega^2}{c^2} - k_z^2 = \frac{\omega^2}{c^2} - \left(\frac{\ell\pi}{L}\right)^2$$

$\therefore$ ω^2 is quantized (no surprise!)

General solution in the cavity is a superposition of eigenfunctions

$$\psi(r,t) = \sum_\lambda c_\lambda \psi_\lambda(r)\, e^{-i\omega_\lambda t}$$

A waveguide is different. ω is input (inject a particular frequency into the waveguide). γ_λ is quantized. The general solution is still a superposition of eigenfunctions each eigenfunction is

$$\psi_{k,\lambda}(x,y,z) = e^{\pm i k_z z}\,\psi_\lambda(x,y)$$

$$\psi(r,t) = e^{-i\omega t}\sum_\lambda c_\lambda\,\psi_\lambda(x,y)\,e^{\pm i k_\lambda z} \qquad (*)$$

$k_z^2 = k_\lambda^2 = \dfrac{\omega^2}{c^2} - \gamma_\lambda^2$ — wave number depends on λ, ω.

$$\omega^2 - c^2 k_z^2 = (c\gamma_\lambda)^2 \equiv \omega_\lambda^2 \text{ given by b.c.}$$

The dispersion relation is like that of a massive relativistic particle

$$E^2 - c^2p^2 = (mc^2)^2.$$

For propagating waves, you need $k_z^2 > 0$ or $\omega > \omega_\lambda$.

$\therefore$ $\omega_\lambda \equiv$ "cutoff frequency" $\equiv$ lowest allowed frequency for propagating mode

$\omega < \omega_\lambda$ — this is a non-propagating evanescent mode

For TEM, $\omega^2 = c^2k^2$ — no mode cutoff, all frequencies propagate

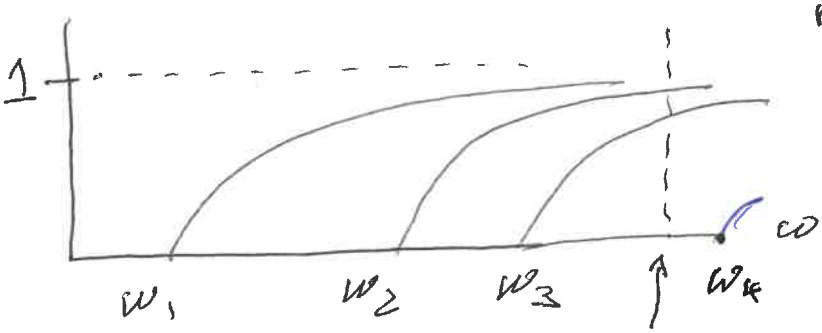
Note: all modes (propagating or not) contribute to the sum — insert superposition $(*)$ at injection point

So to send a signal down a waveguide

1) Choose a mode (TE or TM) - this fixes ω_λ.

2) choose ω

then $\frac{ck_\lambda}{\omega} = \sqrt{1-\left(\frac{\omega_\lambda}{\omega}\right)^2}$ = wave number of propagation down the waveguide - must be real for propagation



3 modes propagate

At finite ω, only a finite number of modes propagate - each at its own phase velocity

With several propagating modes, signal disperses.

Often want to operate w/ only one propagating mode - so use $\omega_1 < \omega < \omega_2$

Also $v_{phase} = \frac{\omega}{k_\lambda} = \frac{\omega c}{\sqrt{\omega^2-\omega_\lambda^2}} = \frac{c}{\sqrt{1-\left(\frac{\omega_\lambda}{\omega}\right)^2}} > c$

$$v_{group} = \frac{d\omega}{dk_\lambda} = \left(\frac{dk_\lambda}{d\omega}\right)^{-1} = \left(\frac{\omega}{c\sqrt{\omega^2-\omega_\lambda^2}}\right)^{-1}$$

$$= c\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}} < c$$

$$v_{group}\, v_{phase} = c^2$$

Return to math. Consider a TE mode,

$$E_z = 0 \quad , \quad (\nabla_t^2 + \gamma^2) B_z(x,y) = 0$$

$\vec{E}_t$ and $\vec{B}_t$ are (more) interesting. How to find them?

$$\hat{z} \times \left[\vec{\nabla} \times \vec{E} = i\omega \vec{B} \right]$$

$$z \times (\underbrace{\vec{\nabla}_t \times \vec{E}} + \vec{\nabla}_z \times \vec{E}) = i\omega\, \hat{z} \times \vec{B}_t$$

$$\nabla_t (\underset{\overset{\shortparallel}{0}}{z \cdot E}) + (\underset{\overset{\shortparallel}{0}}{z \cdot \nabla_t}) E + \hat{z} \times \left(\hat{z} \times \frac{\partial E_t}{\partial z}\right) = i\omega\, \hat{z} \times B_t$$

$E \sim \exp \pm ikz$

$$\hat{z} \times \vec{E}_t = \pm \frac{\omega}{k} \vec{B}_t \qquad \pm \text{ for } \rightleftarrows$$

$$\hat{z} \times (\vec{\nabla} \times \vec{B} = -i\omega\mu\epsilon \vec{E})$$

$$\nabla_t B_z \pm ik \underbrace{\hat{z} \times (\hat{z} \times B_t)}_{-B_t} = -i\omega\mu\epsilon(\hat{z} \times E_t)$$

$$\nabla_t B_z \mp ik \vec{B}_t = \pm i \omega^2 \frac{\mu\epsilon}{k} \vec{B}_t$$

or $$\vec{\nabla}_t B_z = \pm i \left(k - \frac{\mu\epsilon\omega^2}{k} \right) \vec{B}_t$$

$$\vec{B}_t = \mp i \left(\frac{k}{k^2 - \mu\epsilon\omega^2} \right) \vec{\nabla}_t B_z$$

$$B_z \Rightarrow B_t \Rightarrow E_t$$

Also note $k^2 - \mu\epsilon\omega^2 = -\gamma_\lambda^2 = -\frac{\omega_\lambda^2}{c^2}$

$$B_t = \pm i \frac{k}{\gamma_\lambda^2} \nabla_t B_z$$

$B_t \gg B_z$ for $\omega \gg \omega_\lambda$

□ : $\gamma_\lambda \sim \frac{1}{L}$ $\quad B_z \sim \sin\frac{n\pi x}{L}$

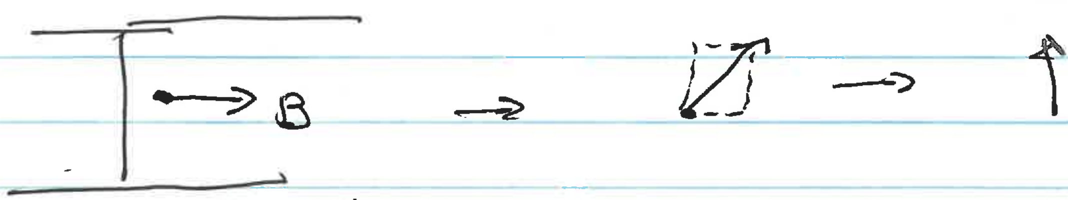
or $\cos\frac{n\pi x}{L}$

$$\nabla_t B_z \sim \frac{1}{L} B_z$$

$$B_t \sim k \underbrace{\left(L^2\right)}_{\frac{1}{\gamma^2}} \frac{1}{L} B_z \sim kL \cdot B_z$$

$B_t = 0$ at $\omega = \omega_\lambda$ or $k = 0$

$B_t \gg B_z$ at big ω



at $\omega = \omega_\lambda$

At $\omega \gg \omega_\lambda$, $kL \gg 1$, confinement region doesn't matter, wave is transverse, $v = c$

$$E \times B^* = \odot \underset{z}{\longrightarrow} = \Uparrow \text{ at } \omega = \omega_\lambda$$

$$= \overset{\uparrow}{\odot} = \underset{z}{\Longrightarrow} \quad \omega \gg \omega_\lambda$$

Energy down the waveguide – see Sec 8.5 for cannel formulas

Find $\psi \equiv E_z$ or H_z, then $(\nabla_t^2 + \gamma^2)\psi = 0$

TM: $\psi = E_z$ $\quad \vec{E}_t = \pm \dfrac{ik}{\gamma^2} \vec{\nabla}_t \psi$

$\psi = 0$ on bndy $\quad \vec{H}_t = \pm \dfrac{\epsilon\omega}{k} \hat{z} \times \vec{E}_t$

TE: $\psi = H_z$ $\quad \vec{H}_t = \pm \dfrac{ik}{\gamma^2} \vec{\nabla}_t \psi$

$\dfrac{\partial\psi}{\partial n} = 0$ $\quad E_t = \mp \dfrac{\mu\omega}{k} \hat{z} \times H_t$

$\vec{S} = \frac{1}{2} \mathrm{Re}(\vec{E} \times \vec{H}^*) = \vec{S}_z + \vec{S}_\perp$

usually imaginary

$$P = \int S \cdot \hat{z}\, dA = \frac{\omega k}{2\gamma^4} \left\{ \begin{matrix} \epsilon \\ \mu \end{matrix} \right\} \int |\nabla_t \psi|^2 dA$$

TM
TE

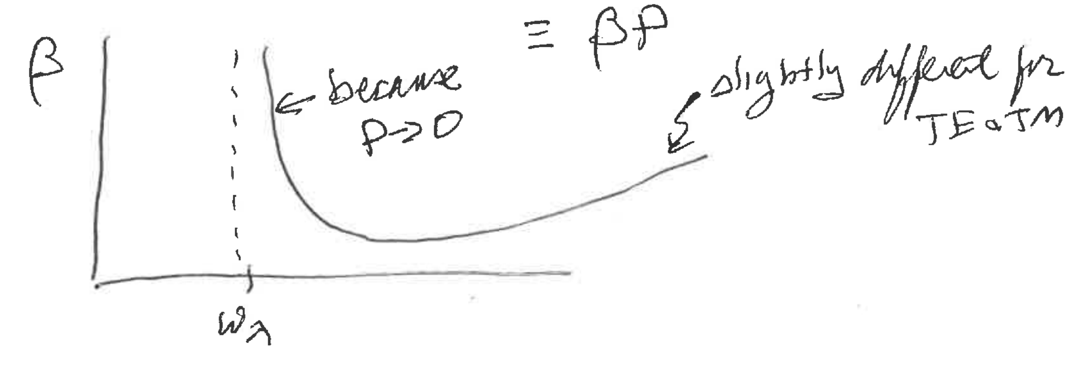
$c^2 k^2 = \omega^2 - \omega_\gamma^2$

so $P = 0$ at $\omega = \omega_\gamma$

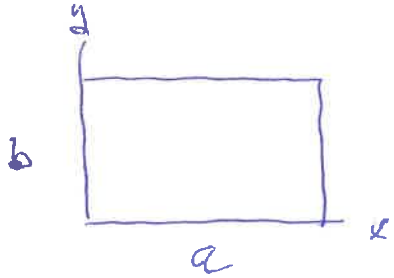
Power loss is still $\quad -\dfrac{dP}{dz} = \dfrac{1}{2\sigma\delta} \displaystyle\int |\hat{n} \times \vec{H}_\parallel|^2 dl$

$\equiv \beta P$

Messy!



Rectangular waveguide



TE: $\psi = H_z$

$$(\nabla_t^2 + \gamma^2)\psi = 0, \quad \frac{\partial \psi}{\partial n} = 0 \text{ on edges}$$

$$\Rightarrow \psi_{n,m}(x,y) = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$\gamma_{mn}^2 = \mu\epsilon\,\omega_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad (*)$$

then $\vec{H}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t \psi \qquad \vec{E}_t = \mp \frac{\mu\omega}{k}\hat{z}\times\vec{H}_t$ *2

You would think from (*), $m, n \geq 0$ but the $\vec{\nabla}_t$ says - you can't have both m & $n = 0$, then $\vec{H}_t = 0$, $\vec{E}_t = 0$

Further - if $a > b$, lowest cutoff freq. is $m = 1, n = 0$, $\omega_{10} = \frac{\pi}{a}\cdot c$

And $H_z \sim e^{ikx}$, $ck = \sqrt{\omega^2 - \omega_{10}^2} = \left[\omega^2 - \left(\frac{\pi c}{a}\right)^2\right]^{1/2}$

(is this the absolute lowest? Have to check TM as well!)

Cavity - continue w/ TE. But have to think about the ends. On a conductor, on a boundary, $\vec{H}\cdot\hat{n} = 0$, $E_{\parallel} = 0$ (this is region of

TM b.c. $(\nabla_t^2 + \gamma^2)E_z = 0$, $E_z = 0$ on sides)

$$\psi(x,y)\,e^{\pm ikz} \rightarrow \psi(x,y)\begin{Bmatrix}\cos kz \\ \sin kz\end{Bmatrix}$$

For TE mode $H_z = 0$ at ends $\rightarrow$

$H_z = \psi(x,y)\sin kz$, $k = \frac{p\pi}{L}$, $p = 1, 2, 3$

~~[illegible]~~

TM mode: $\vec{E}_t = 0$ at ends, $E_t \sim \nabla E_z$

$$E_z = \psi(x,y) \cos kz$$

and (TE again) - combine $+kz$ & $-kz$
waveguide b.c's

$$\vec{H}_t = \frac{k}{\gamma^2} \cos kz \, \nabla_t \psi$$

TE

$$\vec{E}_t = -\frac{i\omega\mu}{\gamma^2} \sin kz \, \hat{z} \times \nabla_t \psi$$

$$\omega^2 = c^2 \left[\gamma^2 + \left(\frac{p\pi}{d}\right)^2 \right]$$

All pretty straightforward if you start w/ linear combination of $e^{\pm ikz}$ is.

Then compute $U = \mathrm{Re} \int d^3x \, \frac{1}{4} \left(E \cdot D^* + B \cdot H^* \right)$

$\rightarrow$ energy in mode in terms of H_0

Energy loss $\quad \frac{dU}{dt} = - \int_{\text{surface}} dA \, \frac{dP}{dA}$

$$\frac{dP}{dA} = \frac{1}{2\sigma\delta} \left| \vec{H}_\parallel \right|^2 \quad \text{as before}$$

Loss in cavity : "Q"

Compute $U = \mathrm{Re} \int d^3x \left[\frac{1}{4} E \cdot D^* + B \cdot H^* \right]$

$\rightarrow$ energy in terms of H_0, for example

Energy loss

$$\frac{dU}{dt} = - \int_{\text{surface}} dA \, \frac{dP}{dA}$$

$$\frac{dP}{dA} = \frac{1}{2\sigma\delta} |H_\parallel|^2 \text{ as before}$$

In ~~network~~ microwave/engineering applications energy loss is parameterized by "Q" $\equiv$ "quality factor" - if ~~resonant~~ freq of mode is ω_0

$$\frac{dU}{dt} = -\frac{\omega_0}{Q} U$$

~~Use E & H fields in cav~~

~~Energy loss in cavity parameterized by "Q" ≡ "quality factor"~~

$$\cancel{\frac{dU}{dt} = -\frac{\omega_0}{Q}U \quad \left(= -\int dA \, \frac{dP}{dA}\right)}$$

E+H fields in cavity ~~do~~ ~~eventually~~ decay

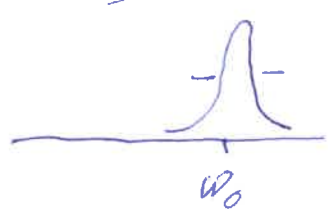
$$E(t) \sim E(t=0)\exp\left\{-i\omega_0 t - \frac{1}{2}\frac{\omega_0}{Q}t\right\}$$

$$E(\omega) \sim \int dt\, E(t)\, e^{i\omega t}$$

$$\sim \frac{1}{i(\omega-\omega_0) - \frac{1}{2}\frac{\omega_0}{Q}}$$

$$|E(\omega)|^2 \sim \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\omega_0}{Q}\right)^2}$$

A Lorentzian again -



Width of spectral line FWHM $\equiv \dfrac{\omega_0}{Q}$.

In QM, $\psi(t) \sim \psi(0)\exp\left\{-i\frac{E_0 t}{\hbar} - \frac{\Gamma}{2}t\right\}$

$$\Gamma = \frac{\text{trans prob}}{\text{unit time}}$$

$$\psi(E) \sim \frac{1}{E - E_0 - i\frac{\Gamma}{2}} \to |\psi|^2 = \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

For the final

- heavily weighted to 2nd half of course
- could ask an electrostatics problem (most likely in context of magnetostatics)
- ~~[illegible]~~ Magnetism
- ~~[illegible]~~ Induction
- stress tensor
- free space solutions of Maxwell eqns
- interfaces, boundaries, refraction, reflection
- waveguides & cavities

Remember - clean copy of Jackson
table of integrals

7320

1) more applications: <u>how</u> is radiation generated from sources?
(antennas, scattering, diffraction

2) special relativity - it's there, but hidden

3) "unification - (go to list of equ's)

and extension - other classical field theories

2 focus points in 7310

1) Maxwell's eqns

$\nabla \cdot E = \rho/\epsilon_0 \qquad \nabla \cdot B = 0$

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$

plus charge conservation $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

plus Lorentz force law $F = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$

(plus stress tensor ...)

plus $D = \epsilon E = \epsilon_0 (E + P)$

$B = \mu H = \mu_0 (H + M)$

focus on physics from many special cases

2) Mathematical techniques - mostly by examples for specific physics problems - nearly all approximations

(Green's functions the one exact method - but practical applications essentially always approximations)