Complex ϵ, complex n, dispersion and beyond

Let's review the story: 1) start w/ ϵ, μ. 2) imagine a conducting medium, $J = \sigma E \rightarrow$ replacement $\epsilon \rightarrow \epsilon + \frac{i\sigma}{\omega} \equiv \epsilon(\omega)$

Can think of this as a complex dielectric constant - or a complex index of refraction ($n \sim \sqrt{\epsilon}$) which is frequency dependent, $\epsilon(\omega) = \epsilon + \frac{i\sigma}{\omega}$, ~~complex part of~~ Imaginary part of ϵ or $n \rightarrow$ absorption.

~~Let's generalize further~~: constant ϵ, σ is just a model. Real materials have complicated $\epsilon(\omega)$ or $n(\omega)$. Can we ~~understand~~ build a better model and does it have unexpected consequences? Yes + Yes!

The real "better model" will involve quantum mechanics, of course. But classical models came first and, with careful interpretation, can tell us things about the real world. In fact, the passage from classical models to QM once was the problem Heisenberg was working on when he invented QM - there is a long story there, a mixture of history plus a lot of physics we still care about (interaction of light w/ matter...)

And unexpected consequences ~~both~~ all over the place.

The classical model: an electron on a ~~damped~~ spring!

$$m[\ddot{x} + \gamma\dot{x} + \omega_0^2 x] = -eE(x,t)$$

$\llcorner$ damping term

Assume $E(x,t) = \vec{E}_0 e^{-i\omega t}$, $\vec{x} = \vec{x}_0 e^{-i\omega t}$,

look for steady state solution ~~[illegible]~~

$$-e\vec{E}_0 = m[\omega_0^2 - \omega^2 - i\omega\gamma]\vec{x}_0$$

$\vec{x}_0 \rightarrow$ dipole moment $\quad \vec{p} = -e\vec{x}_0$

$$\vec{p} = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \vec{E}_0 e^{-i\omega t}$$

Now recall your dielectric equations: For N electrons/unit volume the Polarization vector is $\vec{P} = N\vec{p} = \epsilon_0 \chi_e \vec{E}$, χ_e the electric susceptibility, and

$$\frac{\epsilon}{\epsilon_0} = 1 + \chi_e \implies \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \chi_e(\omega) \text{ is a}$$

complex, frequency dependent $\begin{cases} \text{dielectric constant} \\ \Rightarrow \end{cases}$

$\Rightarrow$ " " " index of refraction $n(\omega)$.

The complete formula assumes Z electrons/molecule
f_i electrons/molecule in state i,

$$\sum f_i = Z$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

or

$$\left.\begin{aligned} \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} &= 1 + \frac{Ne^2}{\epsilon_0 m} \sum_i \frac{f_i(\omega_i^2 - \omega^2)}{[\omega_i^2 - \omega^2]^2 + \gamma_i^2\omega^2} \\ \mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} &= \frac{Ne^2}{\epsilon_0 m} \sum_i \frac{\omega\gamma_i f_i}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2\omega^2} \end{aligned}\right\} \text{save}$$

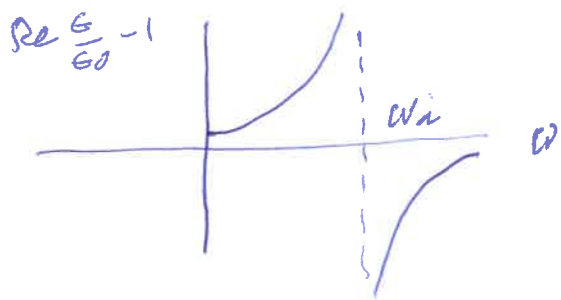
γ + atom $\Leftrightarrow$ γ + atom

In QM: $\omega_i \to$ some $\frac{\Delta E}{\hbar}$, $\gamma_i \Leftrightarrow$ lifetime of excited states

Formula looks complicated – what's going on?

Single freq. first. And assume $\gamma_i \ll \omega_i$.

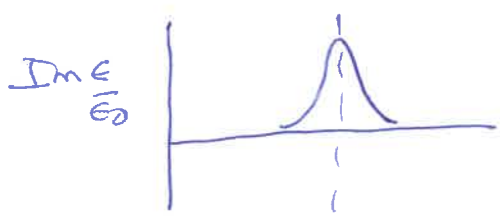
Then $\epsilon(\omega)$ is nearly real, $\frac{\epsilon(\omega)}{\epsilon_0} - 1 \sim \frac{1}{\omega_i^2 - \omega}$



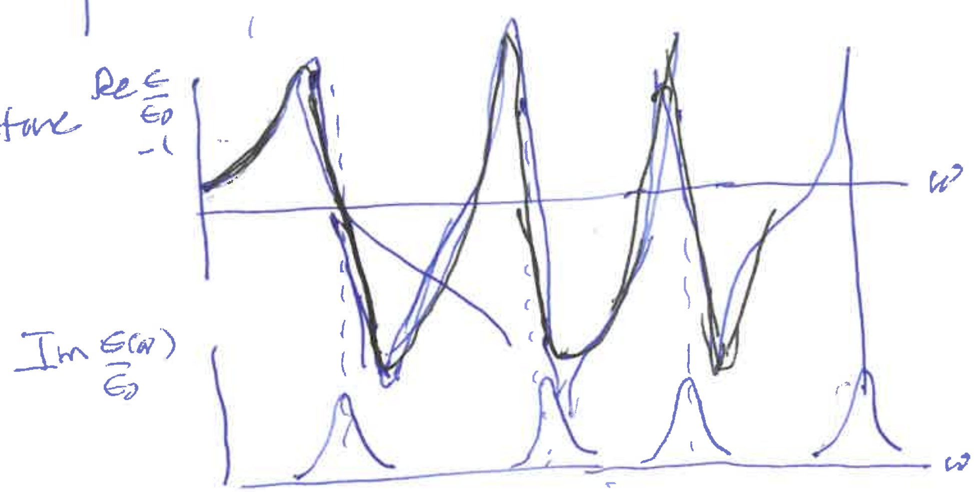
note slope is mostly positive

Next, look at Im $\frac{\epsilon(\omega)}{\omega} \sim \frac{\omega\gamma}{(\omega_i^2 - \omega^2)^2 + \gamma_i^2 \omega^2}$

peaks at $\omega \sim \omega_i$



Full picture

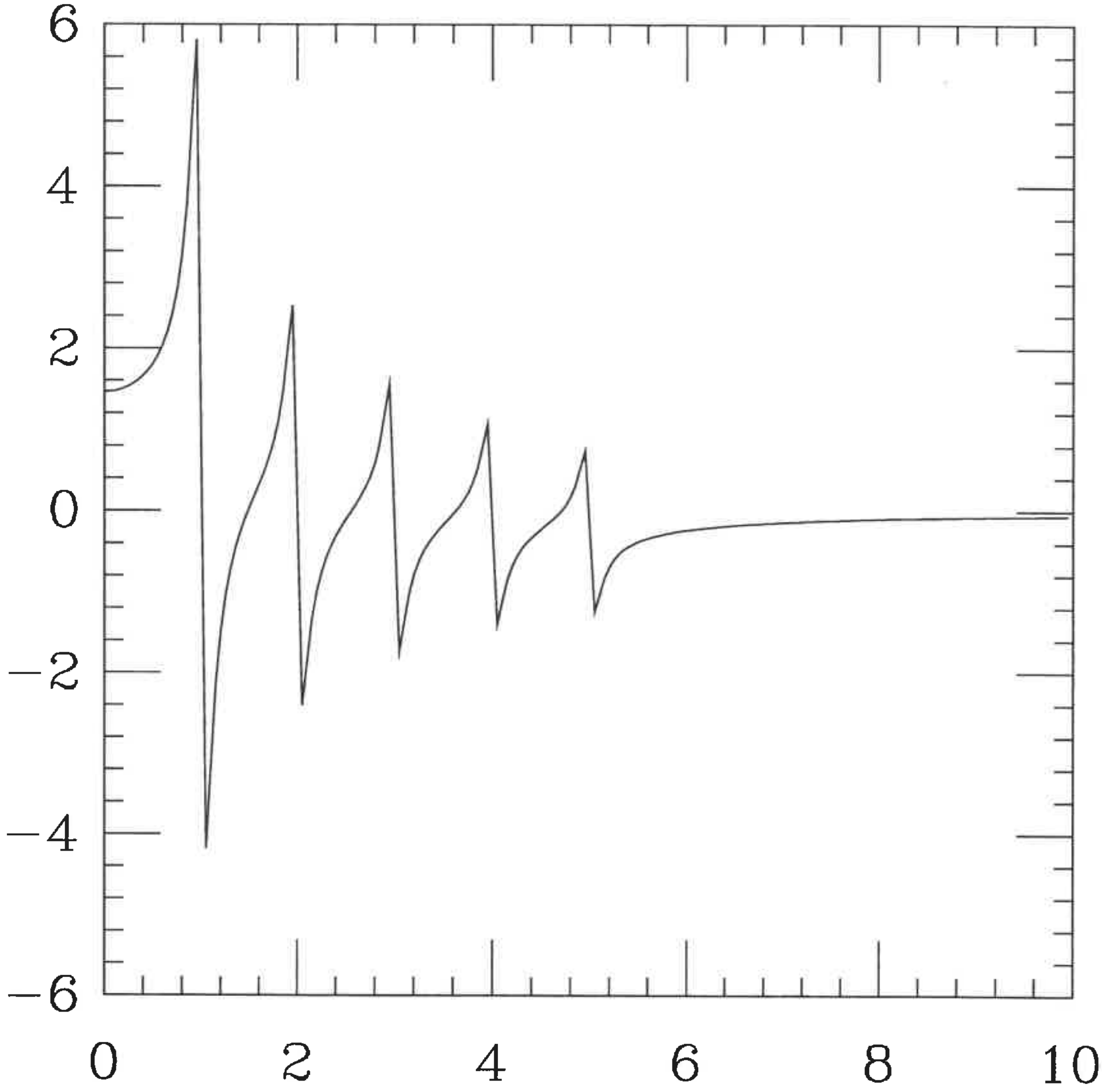


Now recall: $\epsilon \sim n^2$. Normal dispersion ($\omega_g \lesssim \omega_{ph}$) is $\frac{dn}{d\omega} > 0$ or $d\,\mathrm{Re}\,\epsilon / d\omega > 0$. Anomalous dispersion is $\omega_g > \omega_{ph}$, $\frac{dn}{d\omega} < 0$.
Over most of ω, $dn/d\omega > 0$ – "normal dispersion is normal".
Anomalous dispersion always accompanied by (resonant) absorption

$$\sum_{i=1}^{5} \frac{[\omega_i^2-\omega^2]}{[(\omega_i^2-\omega^2)^2+\gamma^2\omega^2]}$$

$\omega_i = i$

$\gamma = 0.1$



Model results: $\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$, $\underline{\sum_i f_i = Z}$ ↑

Suppose ω_i is bounded. At $\omega \gg \omega_{max}$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{Ne^2 Z}{\epsilon_0 m \omega^2} \quad \text{is } \underline{\text{real}} \equiv \cancel{\frac{Ne^2 Z}{\epsilon_0 m \omega^2}}$$

$$\equiv 1 - \frac{\omega_p^2}{\omega^2} \qquad 1 - \frac{NZ}{\epsilon_0}\frac{e^2}{m\omega^2}$$

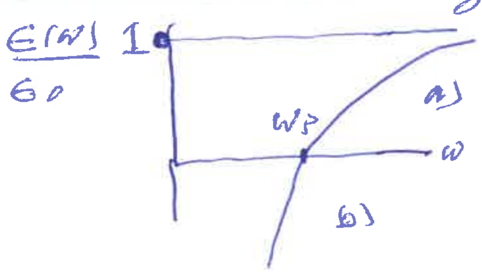
$\omega_p \equiv$ "plasma frequency"

Note $ck = \omega\sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\omega^2 - \omega_p^2}$

k is real if $\omega > \omega_p$ but k is imaginary if $\omega < \omega_p$, radiation w/ $\omega < \omega_p$ can't propagate in material.

Special cases: a) Dielectric - sum is complicated, only $\omega \gg \omega_p$ is simple

b) Free electrons AKA "thin plasma" $\omega_i = \gamma_i = 0$ - asymptotic formula always true $\epsilon(\omega)/\epsilon_0 = 1 - \frac{\omega_p^2}{\omega^2}$



a) $k^2 > 0$: propagation

b) $k^2 < 0$ - damping - $\epsilon \sim -\omega_p^2/\omega^2$

$$k = \sqrt{\frac{\epsilon}{\epsilon_0}}\frac{\omega}{c} = i\omega_p, \quad I \sim e^{-2\omega_p \frac{x}{c}}$$

Metals: $\frac{\epsilon(\omega)}{\epsilon_0} \sim \text{constant} - \frac{\omega_p^2}{\omega^2}$, ω_p in eV

$\omega << \omega_p$ – reflection (metals are shiny)

$\omega > \omega_p$ transparency

"why are metals shiny?"

Back to conductivity: $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = J - i\omega\epsilon\vec{E}$

Contrast a) $J = 0$, ϵ complex, RHS $= -i\omega[\epsilon_R + i\epsilon_I]\vec{E}$

b) $J = \sigma E$, ϵ real RHS $= \cancel{(\sigma - i\omega\epsilon)\vec{E}}$ $-i\omega[\epsilon + i\sigma/\omega]E$

$\Rightarrow \sigma(\omega) = \omega\epsilon_I$, DC conductivity is $\lim_{\omega\to 0} \omega\,\mathrm{Im}\,\epsilon = \sigma$

Once more, look at

$$\epsilon(\omega) = \epsilon_0 + \frac{Ne^2}{m}\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

If all $\omega_i > 0$, then as $\omega \to 0$ $\mathrm{Im}\,\epsilon(\omega) \to 0$.

No conductivity, just an insulator.

But suppose one $\omega_i = 0$ – no spring – a free particle – a free electron. Material should conduct.

Suppose a fraction f_0 of the electrons are free

$$\lim_{\omega\to 0} \omega\epsilon(\omega) = \underbrace{\omega\epsilon_0}_{\text{vanishes}} + \frac{Ne^2}{m}\frac{f_0\,\omega}{-\omega^2 - i\omega\gamma_0} + \underbrace{\omega\sum_{i\neq 0} \text{etc}}_{\text{vanishes}}$$

$$\sigma(\omega) = \frac{f_0 N e^2}{m}\frac{1}{[\gamma_0 - i\omega]}$$

A frequency-dependent conductivity – the Drude model (1900)

Note at $\omega \ll \gamma_0$ σ is real (J + E in phase)
$\omega \gg \gamma_0$ σ is imaginary (J + E out of phase)

Where is crossover?

Copper $N = 8 \cdot 10^{28} \frac{\text{atoms}}{m^3}$

$\sigma(\omega = 0) = 5.9 \times 10^7 \frac{1}{\Omega\text{-m}} = \frac{N e^2 f_0}{m \gamma_0}$

$\frac{\gamma_0}{f_0} \sim 4 \times 10^{13} \text{ sec}^{-1}$ $\frac{N e^2}{\hbar c}$

if $f_0 \sim 1$ $\gamma \sim 10^{13} \text{ sec}^{-1}$

Microwaves $\sim 10^{11} \text{ sec}^{-1}$
σ in metals is real at low ω

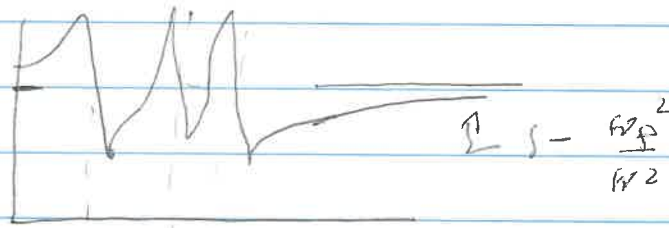
~~[illegible]~~

Dont use!

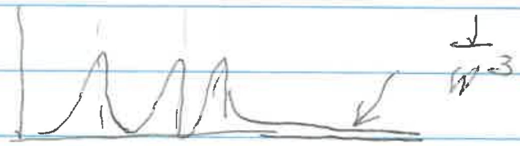
Where we were: "mass on spring"

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{\epsilon_0 m} \sum \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0}$



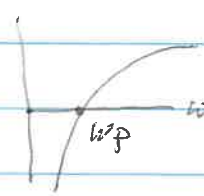
$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0}$



Conductor $\quad \sigma(\omega) = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} = \omega\,\mathrm{Im}\,\epsilon(\omega)$

At big ω $\quad \frac{\epsilon(\omega)}{\epsilon_0} = \frac{\epsilon_b(\omega)}{\epsilon_0} - \frac{\omega_p^2}{\omega^2}$

Free electrons $\quad \omega_p^2 = \frac{e^2 N}{m\epsilon_0} =$



$$= \left[\frac{e^2}{4\pi\epsilon_0 \hbar c}\right] \frac{N}{mc^2} 4\pi(\hbar c)c^2$$

$10^{12}\frac{1}{cm^3} \times \frac{cm^3}{10^{24}\mathring{A}^3}$

$\frac{1}{cm} = \frac{10^{-8}}{\mathring{A}}$

$\mathring{A} = 10^{-8}\,cm$

$$= \frac{1}{137} \frac{10^{12-16}\,cm^{-3} \times 10^{-24}\frac{cm^3}{\mathring{A}^3}\,4\pi}{5\cdot 10^{-5}\,eV}\left[2000\ eV\text{-}\mathring{A}\right]\left(3\cdot 10^{18}\frac{\mathring{A}}{sec}\right)^2$$

$\omega_p^2 \sim 10^{12-24-5+3+36 = 36-14 = 22}$

$\omega_p \sim 10^{12} \sim$ microwaves (homework: $N \sim 10^{12}\,e^-/m^3$, $\omega_p \sim 10\,MHz$)

Is all this physics just a model?

Kramers-Kronig relations

We have been writing

$$\vec{D}(x,\omega) = \epsilon(\omega) E(x,\omega) = \epsilon_0 [1 + \chi_E(\omega)] E(x,\omega) \quad (*)$$

without thinking too deeply – but now lets be careful – both in time domain

To do this, define Fourier transforms

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt$$

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} dt$$

and also for D

and $\chi_E(\omega) = \frac{\epsilon(\omega)}{\epsilon_0} - 1 \equiv \int_{-\infty}^{\infty} G(t) e^{i\omega t} dt \quad (?)$

G is called a "response function". The time domain version of (*) is

$$\vec{D}(x,t) = \epsilon_0 \Big[\vec{E}(x,t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \Big\{ \int_{-\infty}^{\infty} G(t') e^{i\omega t'} dt' \Big\} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(t'') dt'' e^{i\omega t''}$$

$$= \epsilon_0 \Big[\vec{E}(x,t) + \frac{2\pi}{2\pi} \int dt' dt'' \delta(t - t' - t'') G(t') E(t'')$$

$$= \epsilon_0 \Big[E(x,t) + \int_{-\infty}^{\infty} dt' G(t') \vec{E}(t-t') \Big]$$

It seems benign – except that D(t) can't depend on E (time later than t) – the future can't influence the past. This is "causality." To get it, it must be that $\underline{G(t') = 0}$ if $t' < 0$.

(look at $t=0$: $D(0) = \epsilon_0 [E_0 + \int_0 dt' G(t') E(-t')]$.

We need $-t' < 0$. The causal form of (?) is really

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^{\infty} G(t) e^{i\omega t} dt}$$

This seems benign but it is not. To keep going, I have to pause and describe some mathematics which (sadly) many of you don't know – called the "residue theorem."

If you know ~~the~~ about integration in the complex plane, what I am about to say will be extremely over simplified. But this interlude is not for you!

To start: $z = x + iy$ is a complex #, consider a complex function $f(z)$.

$f(z)$ is "analytic" at z if it has a derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

independent of how limit is taken.

"non-analytic" can be singular: pole: $\frac{c}{z-z_0}$, $\frac{c}{(z-z_0)^2}$ etc

or multiple valued: ex. $\sqrt{z}$: $z = \rho e^{i\theta}$

$\sqrt{z} = \sqrt{\rho}\, e^{i\theta/2}$. On the real axis

$\theta = 0, 2\pi, \ldots$

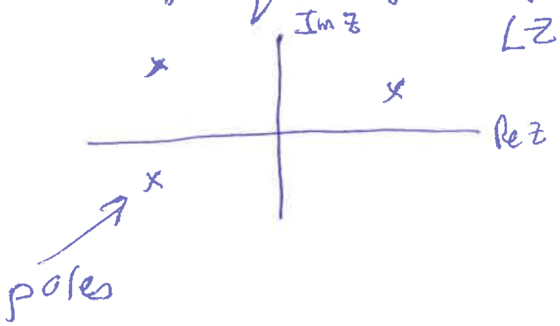
$\sqrt{z} = e^{i0}$ or $e^{i\pi}$ on real axis – it is double valued

or $\log z = \log \rho + i\theta \ldots$

analytic, single valued = "regular"

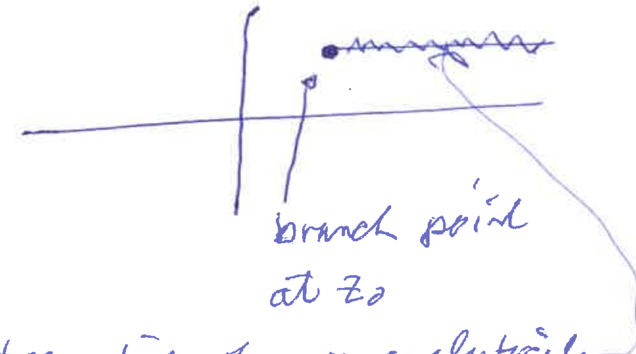
KK-3

Map of $f(z)$ in complex plane



or branch point, branch cut

$$f(z) = \sqrt{z-z_0} = \sqrt{\rho}\, e^{i\theta/2}$$

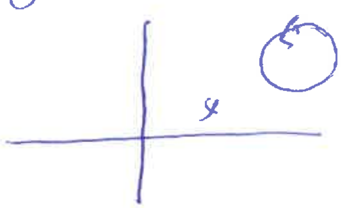


branch cut: arbitrary line of non-analyticity

starting to get complicated - but there are only 1 (or 2) things you have to know:

Cauchy's theorem: If $f(z)$ is regular in a region, any integral around <u>any</u> closed path is zero.

use top picture



$$\oint_c f(z)\,dz = 0$$

<u>Residue theorem</u> $\oint_c f(z)\,dz = 2\pi i \sum$ residues of $f(z)$ inside c

Residue - if $f(z)$ has a simple pole at z_0

residue $R = (z-z_0) f(z) \big|_{z=z_0}$

<u>or</u> $f(z) = \dfrac{A}{z-z_0}$, $R = A$.

~~Check for~~

KK-4

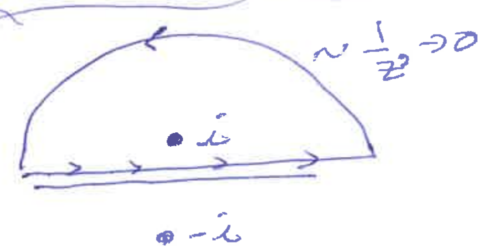
Check for a circle: $z - z_0 = \varrho e^{i\varphi}$

$\cancel{\oint f(z)\,dz} =$ Integrate in φ: $dz = i\varrho e^{i\varphi} d\varphi$

$$\oint f(z)\,dz = \int_0^{2\pi} \frac{A}{\varrho e^{i\varphi}} \cdot i\varrho e^{i\varphi} d\varphi = 2\pi i A$$

$$= \oint dz \frac{A}{z - z_0}$$

Use $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} \to \oint \frac{dz}{z^2+1}$



$$= \oint dz \left[\frac{1}{z-i} - \frac{1}{z+i} \right] \frac{1}{2i} = \frac{2\pi i}{2i} = \pi$$

KK5

Hold that thought, recall

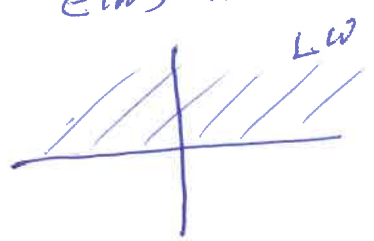
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^\infty G(t)\, e^{i\omega t}\, dt$$

Think of ω as a complex number, $\omega = \alpha + i\beta$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^\infty e^{i\alpha t}\, e^{-\beta t}\, G(t)\, dt$$

If $e^{-\beta t} G(t) \to 0$ as $t \to \infty$ for any $\beta > 0$, the integral is finite - so $\epsilon(\omega)$ is never singular, never blows up when $\omega = \alpha + i\beta$ and $\beta > 0$.

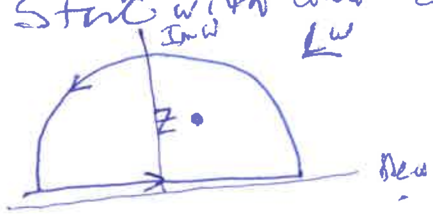
$\Rightarrow$ $\epsilon(\omega)$ is "analytic in the upper half frequency plane"



[$\epsilon(\omega)$ can be singular on the real axis: $\epsilon(\omega) = \epsilon_0 + \frac{i\sigma}{\omega}$ has a pole on the real axis]

$\Rightarrow$ Cauchy's theorem will give a relation between the real & imaginary parts of $\epsilon(\omega)$ at real ω -

Start with contour S



$$\frac{\epsilon(z)}{\epsilon_0} - 1 = \frac{1}{2\pi i} \oint \frac{d\omega'}{\omega' - z} \left[\frac{\epsilon(\omega')}{\epsilon_0} - 1 \right].$$

Contour at infinity gives zero: $\frac{\epsilon(\omega)}{\epsilon_0} - 1 \to 0$ as $\beta \to \infty$

$$\frac{\epsilon(z)}{\epsilon_0} - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - z} \left[\frac{\epsilon(\omega')}{\epsilon_0} - 1 \right]$$

KK6

Of course, we want $z \to$ real valued ω. This involves more thought — $I = \int \frac{d\omega'}{\omega'-\omega} f(\omega') = ?$

A common problem in physics — need to move the pole away from the integration line ~~by~~ — often, there is a physics story. Here the story is a bit abstract, want to keep the pole in the formula in the UHP.

$$\frac{1}{\omega'-z} \to \frac{1}{\omega'-\omega-i\epsilon}, \text{ take } \epsilon \to 0.$$

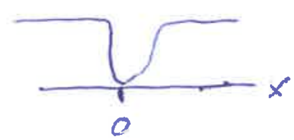
~~Now look at integral on real line, in fact,~~

Temporarily simplify notation, put singularity at zero

$$I_\pm = \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x \pm i\epsilon} = \int_{-\infty}^{\infty} \frac{\mp i\epsilon f(x)\,dx}{x^2+\epsilon^2} + \int_{-\infty}^{\infty} \frac{x^2}{x^2+\epsilon^2}\frac{f(x)}{x}dx$$

$$= I_1 + I_2$$

In I_2, note $\frac{x^2}{x^2+\epsilon^2} \to 1$ for $x \gg \epsilon$, $\to 0$ for $x \ll \epsilon$

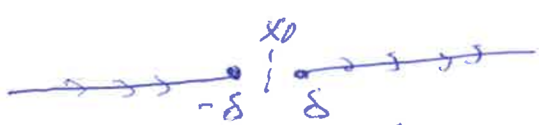


$$I_2 = \int_{-\infty}^{-\epsilon} \frac{f(x)\,dx}{x} + \int_{+\epsilon}^{\infty} \frac{f(x)\,dx}{x}$$

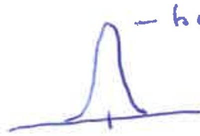
"~~Cauchy~~ called the "Cauchy principal parts $\int$",

$$\mathbb{P}\int_a^b \frac{f(x)}{x-x_0}dx \equiv \lim_{\delta\to 0} \int_a^{x_0-\delta} \frac{f(x)\,dx}{x-x_0} + \int_{x_0+\delta}^{b} \frac{f(x)\,dx}{x-x_0}$$

=



in I_1, $\frac{\epsilon}{x^2+\epsilon^2}$ =



— height $\frac{1}{\epsilon}$ at $x=0$

width in x is $\pm\epsilon$

This is a δ-function of y, nearly

$$I_1 \simeq f(x_0) \int \frac{\epsilon\, dx}{x^2+\epsilon^2} \quad \text{and let} \quad x = \epsilon y$$

$$\int \frac{dy\, \epsilon^2}{\epsilon^2 y^2 + \epsilon^2} = \int \frac{dy}{y^2+1} = \pi \quad \text{from before}$$

$$\int \frac{f(x)\,dx}{x \pm i\epsilon} = \mp i\pi f(0) + \mathbb{P}\int \frac{f(x)\,dx}{x}$$

Books write this formula cryptically; as

$$\lim_{\epsilon \to 0} \frac{1}{\omega' - \omega \pm i\epsilon} = \mathbb{P}\frac{1}{\omega' - \omega} \mp i\pi\,\delta(\omega - \omega').$$

—

End of math aside – back to $\epsilon(\omega)/\epsilon_0$

$$\frac{\epsilon(z)}{\epsilon_0} - 1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - z}\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]$$

Let $z \to \omega + i\epsilon$

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{1}{2\pi i}\left\{ \mathbb{P}\int_{-\infty}^{\infty} d\omega' \frac{\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]}{\omega' - \omega} + \pi i\left(\frac{\epsilon(\omega)}{\epsilon_0} - 1\right)\right\}$$

or

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi i}\mathbb{P}\int_{-\infty}^{\infty} d\omega' \frac{\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]}{\omega' - \omega}$$

or

$$\mathrm{Re}\left[\frac{\epsilon(\omega)}{\epsilon_0} - 1\right] = \frac{1}{\pi}\mathbb{P}\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\left(\frac{\epsilon(\omega')}{\epsilon_0}\right)}{\omega' - \omega}$$

$$\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} = -\frac{1}{\pi}\mathbb{P}\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}\left[\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]}{\omega' - \omega}$$

KIC8

The parts of complex $\epsilon(\omega)$ are not independent!

One last issue to resolve – what is $\omega < 0$?

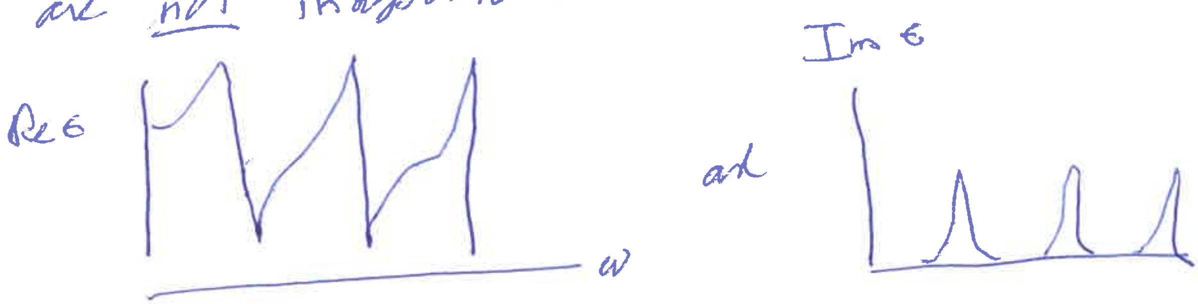
Resolution: $G(t)$ is real, so in the FT,

$$\epsilon(-\omega) = \epsilon(\omega^*)^* \Rightarrow \begin{array}{l} \mathrm{Re}\,\epsilon(\omega) \text{ even in } \omega \\ \mathrm{Im}\,\epsilon(\omega) \text{ odd in } \omega \end{array}$$

Fold the integrals for final result:

$$\left[\begin{array}{l} \mathrm{Re}\,\dfrac{\epsilon(\omega)}{\epsilon_0} - 1 = \dfrac{2}{\pi}\,\mathbb{P}\displaystyle\int_0^\infty \frac{\omega'\,d\omega'}{\omega'^2-\omega^2}\,\mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0} \\[2ex] \mathrm{Im}\,\dfrac{\epsilon(\omega)}{\epsilon_0} = -\dfrac{2\omega}{\pi}\,\mathbb{P}\displaystyle\int_0^\infty \left[\mathrm{Re}\,\frac{\epsilon(\omega')}{\epsilon_0} - 1\right]\frac{d\omega'}{\omega'^2-\omega^2} \end{array}\right.$$

This is called a "dispersion relation." It says absorption ($\mathrm{Im}\,\epsilon$) and dispersion ($\mathrm{Re}\,\epsilon$) are not independent



Each curve uniquely determines the other. And dispersion relations are exact, not model dependent – they come from causality.

There are many applications – let's look at a few

1) $n(\omega) = \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}$ is also analytic in the upper half plane, so it obeys a DR

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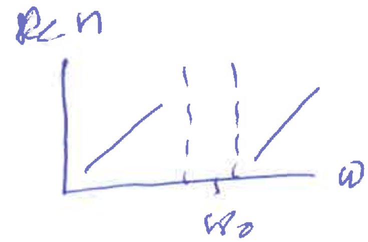
$$\mathrm{Re}\, n(\omega) = 1 + \frac{2}{\pi} \mathcal{P} \int \mathrm{Im}\, n(x) \frac{x\,dx}{x^2-\omega^2}$$

differentiate wrt ω

$$\frac{d}{d\omega} \mathrm{Re}\, n(\omega) = \frac{4\omega}{\pi} \mathcal{P} \int \frac{x\,dx}{(x^2-\omega^2)^2} \mathrm{Im}\, n(x)$$

use: suppose we know a material is strongly absorbing for $\omega \sim \omega_0$: $\cancel{\mathrm{Im}\, n(\omega_0) \sim \delta(x-\omega_0)}$ $\mathrm{Im}\, n(\omega_0)$ big near ω_0

Then we know



① $\frac{d\, \mathrm{Re}\, n}{d\omega} > 0$ on either side $\mathrm{Im}\, n(\omega) = \delta(\omega-\omega_0)$

② $\frac{d\, \mathrm{Re}\, n}{d\omega} \sim \frac{\omega\omega_0}{(\omega_0^2-\omega^2)^2}$ doing the $\int$

③ analyticity = continuity

~~DATO~~
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A sum rule: start with

$$\mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{2}{\pi}\,\mathbb{P}\int_0^\infty \frac{\omega' d\omega'}{\omega'^2-\omega^2}\,\mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0}$$

so

$$\lim_{\omega\to\infty} \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{2}{\pi}\left(-\frac{1}{\omega^2}\right)\int_0^\infty \omega'\,\mathrm{Im}\,\frac{\epsilon(\omega')}{\epsilon_0}\,d\omega'$$

a) $\Rightarrow$ <u>if</u> the integral is finite

$$\lim_{\omega\to\infty} \mathrm{Re}\,\frac{\epsilon(\omega)}{\epsilon_0} = 1 - O\!\left(\frac{1}{\omega^2}\right) \qquad \text{exact!}$$

b) Recall definition of plasma frequency

$$\frac{\epsilon(\omega)}{\epsilon_0} \equiv 1 - \frac{\omega_p^2}{\omega^2}$$

$$\omega_p^2 = \frac{2}{\pi}\int_0^\infty d\omega \cdot \omega\,\mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0}$$

$=$ integral of absorption over all freq. $\Bigg\}$ $\equiv$ "sum rule"

<u>If</u>: in general $\dfrac{\epsilon(\omega)}{\epsilon_0} - 1 = \displaystyle\int_0^\infty dt\, e^{i\omega t}\, G(t)$

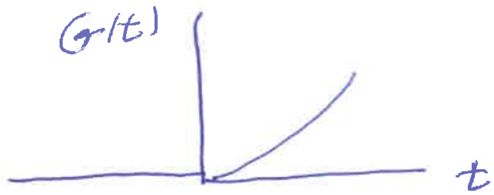
large ω probes small t, so expand

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \int_0^\infty dt\, e^{i\omega t}\left[G(0^+) + t\,G'(0^+) + \cdots\right]$$

Do the integral by shifting $\omega \to \omega + i\epsilon$

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 = \frac{i}{\omega} G(0^+) - \frac{G'(0^+)}{\omega^2} + i\,\frac{G''(0^+)}{\omega^3} + \cdots$$

Now $G(t) = 0$ if $t < 0$ and G should be smooth - $G(0^+) = 0$



so no first term

so $\displaystyle\lim_{\omega \to \infty} \frac{\epsilon(\omega)}{\epsilon_0} - 1 = -\frac{G'(0^+)}{\omega^2}$ $\Rightarrow \mathrm{Re}\,\epsilon \sim$ [illegible]

as before $\left[1 - O\left(\frac{1}{\omega^2}\right)\right]$

And $\displaystyle\lim_{\omega \to \infty} \mathrm{Im}\,\frac{\epsilon(\omega)}{\epsilon_0} \sim -\frac{G''(0^+)}{\omega^3} \sim \frac{1}{\omega^3}$

also exact !

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To check this, go back to single-resonance model -

$$\frac{\epsilon(\omega)}{\epsilon_0} - 1 \equiv \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \leftarrow \text{to set a scale}$$

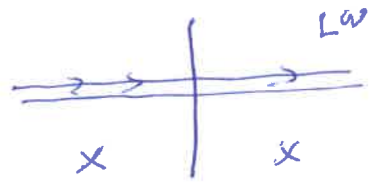
$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[\frac{\epsilon(\omega)}{\epsilon_0} - 1 \right]$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega_p^2 e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Treat as a contour integral, poles at

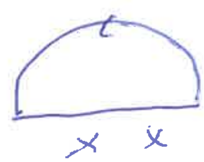
$$\omega^2 + i\gamma\omega - \omega_0^2 = 0$$

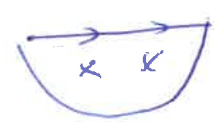
$$\omega = -\frac{i\gamma}{2} \pm \sqrt{-\frac{\gamma^2}{4} + \omega_0^2} \equiv -\frac{i\gamma}{2} \pm \nu_0 \quad (\text{if } \gamma << \omega_0)$$



Note analyticity of $\frac{\epsilon(\omega)}{\epsilon_0}$ in UHP.

If $t<0$, set ~~T~~ $t = -T$, $e^{-i(i\Omega)(-T)} \to 0$ - close contour in UHP, get zero - causality works



If $t<0$, close contour in LHP , pick up 2 poles. $\int d\omega \frac{e^{-i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = -2\pi i \left[\frac{e^{-i\omega_1 t} - e^{-i\omega_2 t}}{\omega_1 - \omega_2} \right]$

$$G(t) = -\frac{2\pi i}{2\pi} \frac{e^{-\gamma t/2}}{2\nu_0} \left[e^{-i\nu_0 t} - e^{i\nu_0 t} \right] \omega_p^2 \,\theta(t)$$

$$= \frac{\omega_p^2}{\nu_0} e^{-\gamma t/2} \sin \nu_0 t \;\theta(t)$$

can expand ... $G(t) = 0 + \frac{\omega_p^2}{\nu_0} \nu_0 t + \ldots$

$D(t) = \epsilon_0 \left[E(t) + \int_0^{t'} dt' G(t') E(-t) \right]$

~~Fitting~~. Note: $\gamma \sim$ width of spectral

~~Many variations~~ - (i.e., $e^{-\gamma t}$ in $G(t)$) -

In time - many variations on this story

Take-away thoughts

1) Complex $\epsilon(\omega)$ or $n(\omega)$ or σ not really part of Maxwell's eqns, but necessary to think about for real-world applications

2) Can construct models for these quantities

a) From first principles (i.e. QM) (still nontrivial problem!)

b) or as a real model (mass on spring, fit to data ...)

Either way, there are general statements which constrain/relate possibilities