Plane waves in Nonconducting Medium

$$\nabla\cdot\vec{B}=0 \qquad \nabla\cdot\vec{D}=0,\quad \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,\quad \nabla\times\vec{H}-\frac{\partial\vec{D}}{\partial t}=0$$

$$E, B, D, H \sim e^{-i\omega t}$$

$$\vec{\nabla}\times\vec{E}-i\omega\vec{B}=0$$

$$\vec{\nabla}\times\vec{H}+i\omega\vec{D}=0$$

assume linear medium: $\vec{B}=\mu\vec{H}$, $\vec{D}=\epsilon\vec{E}$

$\nabla\times$ (either ~~eqn~~ $\nabla\times$)

$$(\nabla^2+\mu\epsilon\omega^2)\begin{bmatrix}\vec{E}\\ c\vec{B}\end{bmatrix}=0$$

$$\Longrightarrow \begin{pmatrix}\vec{E}\\ c\vec{B}\end{pmatrix}\sim \exp(i(\vec{k}\cdot\vec{r}-\omega t)) \quad \text{plane wave}$$

$$k^2=\mu\epsilon\omega^2$$

"phase velocity" $v=\dfrac{\omega}{k}=\dfrac{1}{\sqrt{\mu\epsilon}}=\dfrac{c}{n}=c\sqrt{\dfrac{\mu_0\epsilon_0}{\mu\epsilon}}$

$$n=\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{"index of refraction"}$$

convention: $\vec{E}(x,t)=$ (real part of) $\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

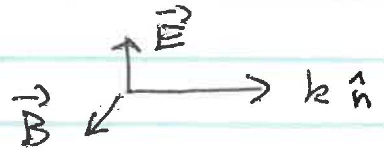
$c\vec{B}=\mathrm{Re}\; c\vec{B}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

Jackson writes $\vec{k}=k\hat{n}=n\dfrac{\omega}{c}\hat{n}$ (assumed k real for now)

$$\nabla\cdot\begin{pmatrix}E\\ cB\end{pmatrix}=0 \Longrightarrow \left.\begin{array}{l}\vec{E}_0\cdot\hat{n}=0\\ c\vec{B}_0\cdot\hat{n}=0\end{array}\right\}\ \text{transverse wave}$$

$$\vec{\nabla}\times\vec{E}=i\omega B \Longrightarrow ik\,\hat{n}\times\vec{E}=i\omega\vec{B}$$

$$\underbrace{n}_{\text{index}}(\hat{n}\times\vec{E})=c\vec{B}$$



$$|cB| = |E|\, n$$

(in MKS, free space $|cB| = |E|$, $n=1$)

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{n}{\mu c}\,\hat{n}\times\vec{E} = \frac{\sqrt{\mu\epsilon}}{\mu}\,\hat{n}\times\vec{E} = \underbrace{\sqrt{\frac{\epsilon}{\mu}}}_{\frac{1}{Z}}\,\hat{n}\times\vec{E}$$

$$Z = \text{impedance},\quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$$

$(H \sim I,\ E \sim V)$

Time ave. Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\hat{n}$$

T.A. Energy density $u = \frac{1}{4}\left(\epsilon\, E\cdot E^* + \frac{1}{\mu} B\cdot B^*\right)$

$$= \frac{\epsilon}{2}|E_0|^2$$

$$\left(\frac{S}{u} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}\right)$$

Polarization

~~Consider the following cases~~

~~a) $\vec{E}_0$ real $\Rightarrow$ $\vec{E}$ ...)~~

Recall that $\vec{E}(x,t) = \vec{E}_0\, e^{i(k\cdot x - \omega t)}$
really means $\vec{E}(x,t) = \mathrm{Re}\left[\vec{E}_0\, e^{i(k\cdot x-\omega t)}\right]$
Consider the following cases:

a) $\vec{E}_0$ real $\Rightarrow$ $\vec{E}(x,t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$
$\Rightarrow$ $\vec{E}$ always points in the same direction
$\Rightarrow$ "linear polarization"

Note: 2 possible (but arbitrary) states of linear pol.
$\vec{\epsilon}_1\cdot\vec{\epsilon}_2 = 0,\quad \vec{k}\cdot\vec{\epsilon}_i = 0$

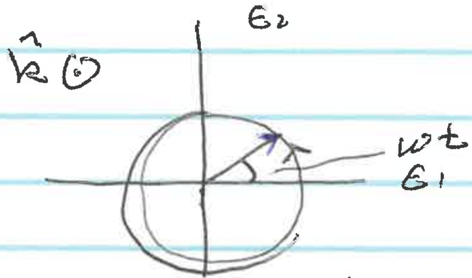
b) $\vec{E}_0 = E_0(\hat{\epsilon}_1 + i\hat{\epsilon}_2)$ where ϵ_1 & ϵ_2 are real, E_0,

$\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 = \vec{\epsilon}_i \cdot \vec{k} = 0$

$\vec{k} = \vec{\epsilon}_1 \times \vec{\epsilon}_2$

means $\vec{E}(x,t) = E_0\left[\hat{\epsilon}_1 \cos(k\cdot x - \omega t) - \hat{\epsilon}_2 \sin(k\cdot x - \omega t)\right]$

Examine time dependence at $\vec{x} = 0$



$\vec{E}(t) = E_0(\hat{e}_1 \cos\omega t + \hat{\epsilon}_2 \sin\omega t)$

this is a "positive helicity state,"
associating spin $\vec{s}$ with $\hat{k}$ via right hand rule
$k \cdot s = +1$

AKA left ~~hand~~ circularly polarized light
(rotation looking at wave)

dir of rotation of pol is to left

c) $\vec{E}_0 = E_0(\hat{\epsilon}_1 - i\hat{\epsilon}_2)$ rotation in opposite sense

negative helicity
right ~~hand~~ pol. light

d) in general $\vec{E} = E_1\hat{\epsilon}_1 + E_2\hat{\epsilon}_2$ $\left(= \sum_j E_j |j\rangle\right)$ (un-indexed)

or $\vec{E} = \sum_{i=\pm 1} (\vec{E} \cdot \hat{\epsilon}_i^*)\hat{\epsilon}_i$ $\left(= \sum_j |j\rangle\langle j|E\rangle\right.$

$\vec{E} = E_+\hat{\epsilon}_{+1} + E_-\hat{\epsilon}_{-1}$

where $\hat{\epsilon}_{\pm 1} = \dfrac{(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2)}{\sqrt{2}}$ (transverse spherical unit vector)

$\hat{\epsilon}_i^* \cdot \hat{\epsilon}_j = \delta_{ij}$

expansion in helicity basis

then

$$E = E_+ \left(\frac{\hat{\epsilon}_1 + i\hat{\epsilon}_2}{\sqrt{2}} \right) + E_- \left(\frac{\hat{\epsilon}_1 - i\hat{\epsilon}_2}{\sqrt{2}} \right)$$

special cases

1) $E_+ = \pm E_-$ (same phase for E's) - linear polarization in x or y

$$\left(\frac{\hat{\epsilon}_1 + i\hat{\epsilon}_2}{\sqrt{2}} \right) \pm \left(\frac{\epsilon_1 - i\epsilon_2}{\sqrt{2}} \right)$$

2) $E_+ / E_- = \pm i$; linear again along $\frac{\hat{x} \pm \hat{y}}{\sqrt{2}}$

$+i$: $\frac{\hat{\epsilon}_1 + i\hat{\epsilon}_2}{\sqrt{2}} + i\left(\frac{\hat{\epsilon}_1 - i\hat{\epsilon}_2}{\sqrt{2}} \right) = \hat{\epsilon}_1 \left(\frac{1+i}{\sqrt{2}} \right) + \hat{\epsilon}_2 \left(\frac{1+i}{\sqrt{2}} \right)$

$$= \left(\frac{\hat{e}_1 + \hat{e}_2}{\sqrt{2}} \right)(1+i)$$

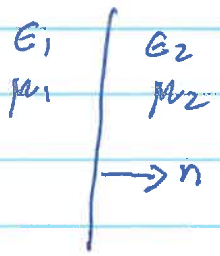
3) E_+ or $E_- = 0$ - circular polarization

4) Otherwise Elliptically polarized light

$E_+ \cdot E_-$
different magnitudes

Measurement of polarization can be done via "Stokes parameters" - pp 301-302

At interface, if no charges, currents



a) $\vec{D}\cdot\hat{n}$ continuous

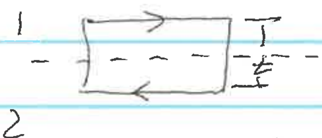
b) E_{tan} "

c) $\vec{B}\cdot\hat{n}$ "

d) H_{tan} "

a) $\nabla\cdot D = \rho = 0$ c) $\nabla\cdot B = 0$

b) $\nabla\times E - \frac{\partial B}{\partial t} = 0$, $\nabla\times H = \frac{\partial D}{\partial t} + J$



$\int \vec{E}\cdot d\vec{l} = \int dA \frac{\partial \vec{B}_t}{\partial t}$

as the thickness is taken to zero LHS is

$E_{1t} - E_{2t}$ while RHS $\to 0$ if

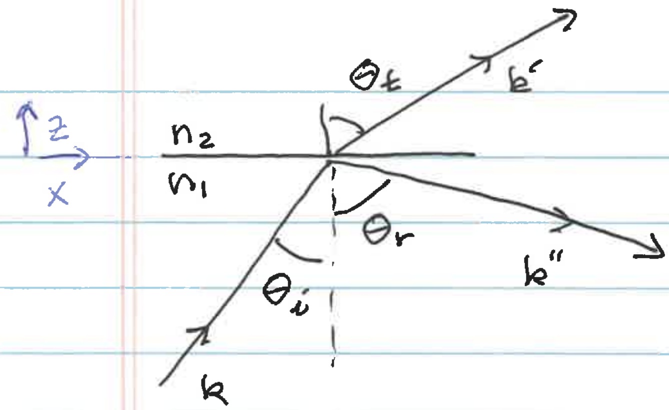
B does not diverge at the interface

d) $\int H\cdot dl = \int dA \left[\frac{\partial D}{\partial t} + J\right]$

If there is no surface current $K = \lim_{A\to 0}\int dA\, J \to 0$

then H_{tan} is also continuous.

Plane interface (at z=0)



$$\vec{E}_2' = \vec{E}_0' e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$$

$$\vec{E}_1 = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} + \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{x} - \omega t)}$$

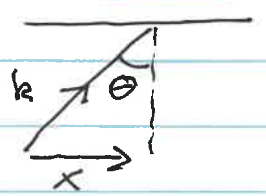
$$|k| = |k''| = n_1\frac{\omega}{c}, \quad |k'| = n_2\frac{\omega}{c}$$

Suppose surface is $z=0$: boundary conditions must match at all x & y

$\Rightarrow$ phase factors must match

$$(\vec{k}\cdot\vec{x})_{z=0} = (\vec{k}''\cdot\vec{x})_{z=0} = (\vec{k}'\cdot\vec{x})_{z=0}$$

$$k\sin\theta_i = k''\sin\theta_r = k'\sin\theta_t$$



$$\sin\theta_i = \sin\theta_r$$

(angle of incidence = angle of reflection)

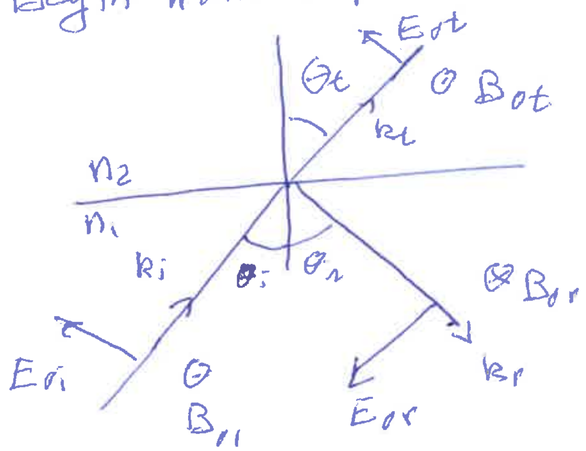
$n_1 \sin\theta_i = n_2 \sin\theta_t \Rightarrow$ Snell's law

Now to actually address b.c.'s.

Fresnel equations – reflection & refraction at a plane interface

It's most convenient to consider linear polarization – and separately treat "in-plane" & "out-of-plane" polarization.

Begin with in-plane



$$c\vec{B} = n(\hat{k}\times\vec{E}) \qquad (a)$$

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

$$n_2 \sin\theta_t = n_1 \sin\theta_i$$

B.C.

1) $E_\parallel$ continuous $\qquad (E_{0i}+E_{0r})\cos\theta_i = E_{0t}\cos\theta_t$

2) $B_\perp$ " $\qquad$ nothing! B ∥ surface

3) $D_\perp$ " $\qquad \epsilon_1[E_{0i}-E_{0r}]\sin\theta_i = \epsilon_2 E_{0t}\sin\theta_t$

4) $H_\parallel$ $\qquad \frac{1}{\mu_1}(B_{0i}-B_{0r}) = \frac{1}{\mu_2}B_{0t}$

from & 4) is $\qquad \frac{n_1}{c\mu_1}(E_{0i}-E_{0r}) = \frac{n_2}{c\mu_2}E_{0t}$

$n^2 = \frac{\epsilon\mu}{\epsilon_0\mu_0}$ from $\frac{n_1}{\mu_1} = \frac{\epsilon_1}{n_1}\frac{1}{\mu_0\epsilon_0}$, $\qquad \frac{n_2}{\mu_2} = \frac{\epsilon_2}{n_2}\frac{1}{\mu_0\epsilon_0}$

4) is $\qquad \frac{\epsilon_1}{n_1}(E_{0i}-E_{0r}) = \frac{\epsilon_2}{n_2}E_{0t}$.

$n_1/n_2 = \frac{\sin\theta_t}{\sin\theta_i}$ => 4) is $\epsilon_1\sin\theta_i(E_{0i}-E_{0r}) = \epsilon_2\sin\theta_t E_{0t}$

4) = 3)!

Define $\mathcal{R}_{in} = \dfrac{E_{0r}}{E_{0i}}$ $\quad \mathcal{T}_{in} = \dfrac{E_{0t}}{E_{0i}}$:

1) is $\quad (1+\mathcal{R}_{in})\cos\theta_i = \mathcal{T}_{in}\cos\theta_t$

$$1+\mathcal{R}_{in} = \mathcal{T}_{in}\frac{\cos\theta_t}{\cos\theta_i} \quad *$$

4) $\quad 1-\mathcal{R}_{in} = \dfrac{n_2}{n_1}\dfrac{\mu_1}{\mu_2}\mathcal{T}_{in} \quad *$

$$\mathcal{T}_{in} = \frac{2}{\dfrac{\cos\theta_t}{\cos\theta_i} + \dfrac{n_2}{n_1}\dfrac{\mu_2}{\mu_1}} \qquad \mathcal{R}_{in} = \frac{\dfrac{\cos\theta_t}{\cos\theta_i} - \dfrac{n_2}{n_1}\dfrac{\mu_1}{\mu_2}}{\dfrac{\cos\theta_t}{\cos\theta_i} + \dfrac{n_2}{n_1}\dfrac{\mu_1}{\mu_2}}$$

because $\vec{S} = \frac{1}{2}(\vec{E}\times\vec{H}^*) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\hat{n}$,

$|\mathcal{R}|^2 =$ ratio of $\dfrac{\text{reflected power}}{\text{incident power}}$.

Algebra for "out of plane" similar.

Physics, first: $\mathcal{R}_{in}$ can vanish!

Set $\mu_1 = \mu_2 = \mu_0$ because that's true for most materials

$$\mathcal{R}_{in} = 0 \text{ if } \cos\theta_t = \frac{n_2}{n_1}\cos\theta_i$$

This θ_i is called "Brewster's angle," θ_B

Define $\dfrac{n_2}{n_1} = \dfrac{1}{r}$ for algebra

$$\cos\theta_t = \frac{1}{r}\cos\theta_B$$

$+$ Snell $\quad \sin\theta_t = r\sin\theta_B$

so $\quad \cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-r^2\sin^2\theta_B} = \frac{1}{r}\cos\theta_B$

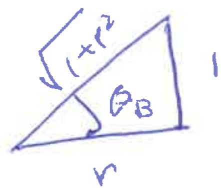
$$1 - r^2\sin^2\theta_B = \frac{1}{r^2}\cos^2\theta_B$$

$$1 - r^2\left[1-\cos^2\theta_B\right] = \frac{1}{r^2}\cos^2\theta_B$$

$$1 - r^2 = \left[\frac{1}{r^2} - r^2\right]\cos^2\theta_B = \frac{1-r^4}{r^2}\cos^2\theta_B$$

$$= \frac{(1-r^2)(1+r^2)}{r^2}\cos^2\theta_B$$

$\therefore \quad \dfrac{r^2}{1+r^2} = \cos^2\theta_B$:

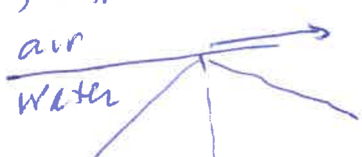


$$\therefore \quad \boxed{\tan\theta_B = \frac{1}{r} = \frac{n_2}{n_1}}$$

example :

$\tan\theta_B = 1.35$, $\theta_B = 53°$.

By the way, total internal reflection is $\theta_t = \pi/2$



$$n_1 \sin\theta_I = n_2 \sin\theta_t = n_2$$

$\sin\theta_I = \dfrac{n_2}{n_1}$ needs $n_1 > n_2$

~~air~~

There's more!

$$\tan\theta_B = \frac{n_2}{n_1} = \frac{\sin\theta_B}{\cos\theta_B} \leftarrow$$

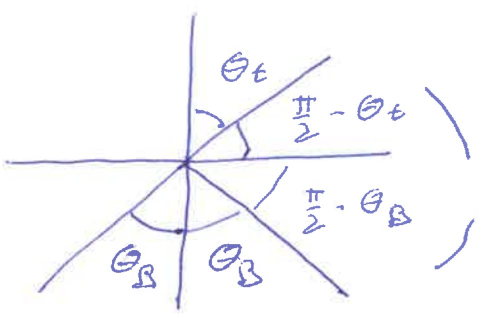
Snell

$$\frac{n_2}{n_1} = \frac{\sin\theta_B}{\sin\theta_t} \leftarrow$$

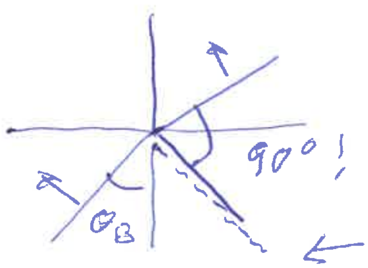
or $\sin\theta_t = \cos\theta_B$

or $\theta_t = \frac{\pi}{2} - \theta_B$

check: $\sin\theta_t = \underset{1}{\sin\frac{\pi}{2}}\cos\theta_B - \underset{0}{\cos\frac{\pi}{2}}\sin\theta_B$ ✓



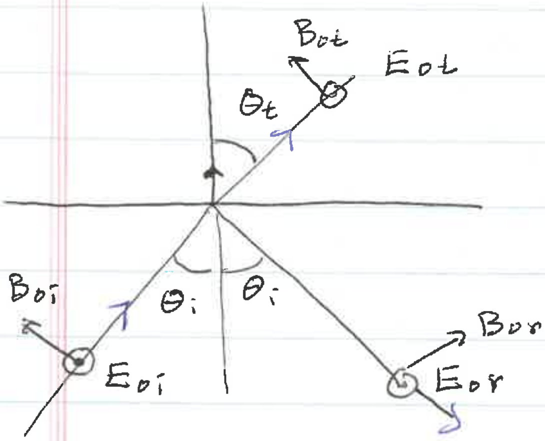
$$\theta = \pi - \theta_t - \theta_B$$
$$= \pi - \left(\frac{\pi}{2} - \theta_B\right) - \theta_B = \frac{\pi}{2}$$



angle between transmitted wave and (absent) reflected wave is 90°

← no in-plane reflected wave.

Algebra for "out of plane" similar:



only draw picture!

omit derivation in lecture

$$E_{\parallel} \quad E_{0i} + E_{0r} = E_{0t} \quad \Leftarrow \quad 1 + \mathcal{R} = \mathcal{T}$$

$$B_{\perp} \quad (B_{0i} + B_{0r})\sin\theta_i = B_{0t}\sin\theta_t$$

$$(E_{0i} + E_{0r})\frac{n_1}{c}\sin\theta_i = E_{0t}\frac{n_2}{c}\sin\theta_t$$

$D_{\perp}$ nothing

$$H_{\parallel} \quad \frac{1}{\mu_1}(B_{0i} - B_{0r})\cos\theta_i = \frac{1}{\mu_2}B_{0t}\cos\theta_t$$

$$\frac{n_1}{\mu_1}(E_{0i} - E_{0r})^{\cos\theta_i} = \frac{n_2}{\mu_2}$$

$$1 - \mathcal{R} = \frac{n_2}{n_1}\times\frac{\mu_2}{\mu_1}\frac{\cos\theta_t}{\cos\theta_i}\mathcal{T}$$

$$1 + \mathcal{R} = \mathcal{T}$$

$$\mathcal{T} = \frac{2}{1 + \frac{n_2}{n_1}\frac{\mu_2}{\mu_1}\frac{\cos\theta_t}{\cos\theta_i}} \xrightarrow[\mu=1]{\theta=0} \frac{2n_1}{n_1+n_2}$$

$$\mathcal{R} = \frac{1 - \frac{n_2}{n_1}\frac{\mu_2}{\mu_1}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \frac{n_2}{n_1}\frac{\mu_2}{\mu_1}\frac{\cos\theta_t}{\cos\theta_i}} \rightarrow \frac{n_1 - n_2}{n_1 + n_2}$$

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i}$$

What does $|R_{out}|^2$, $|R_{in}|^2$ look like?

set $\mu = \mu_0$

$$R_{out} = \frac{1 - \frac{n_2}{n_1}\frac{\cos\theta_t}{\cos\theta_i}}{1 + \frac{n_2}{n_1}\frac{\cos\theta_t}{\cos\theta_i}} \qquad R_{in} = \frac{1 - \frac{n_2}{n_1}\frac{\cos\theta_i}{\cos\theta_t}}{1 + \frac{n_2}{n_1}\frac{\cos\theta_i}{\cos\theta_t}}$$

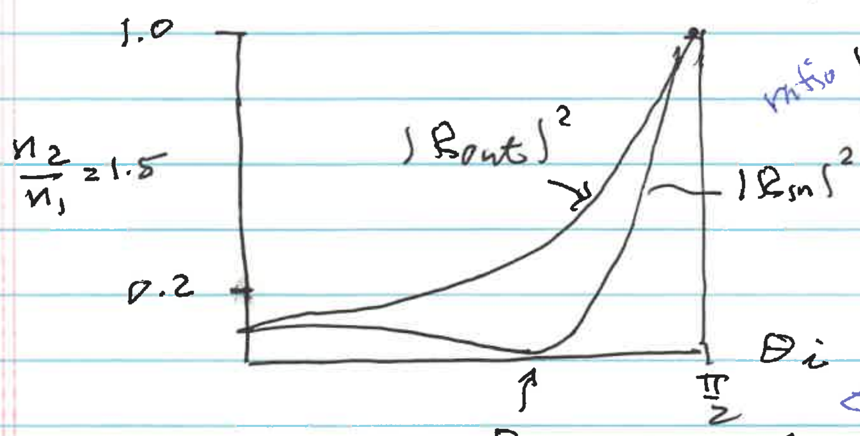
Also $n_1 \sin\theta_i = n_2 \sin\theta_t$

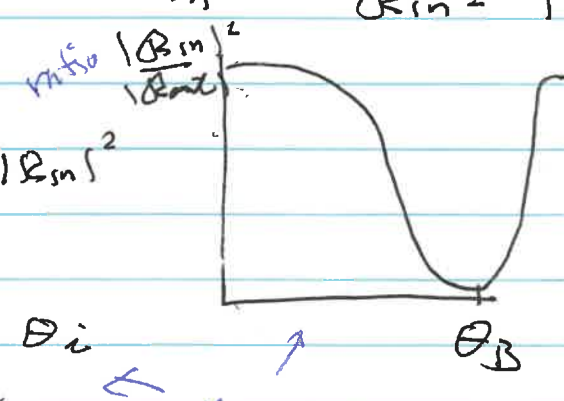
a) $\theta = 0$ $\quad R = \dfrac{n_1 - n_2}{n_1 + n_2}$ for either, of course

if $n_2 < n_1$ R is + (water into air) no phase change

if $n_2 > n_1$ R is − (air into water) phase change of π.

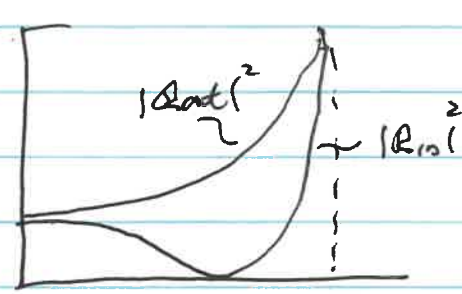
b) $\theta_i \to \frac{\pi}{2}$, $\cos\theta_i \to 0$. If $\frac{n_2}{n_1} > 1$ $\quad R_{out} = -1$, $R_{in} = 1$





$\frac{n_2}{n_1} > 1$

$\frac{n_2}{n_1} = \frac{2}{3}$, $\sin\theta_I = \frac{2}{3}$

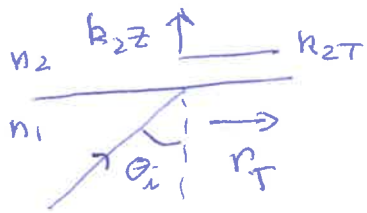


$\frac{n_2}{n_1} < 1$

$\theta_i : \theta_t = 90°$, $\cos\theta_t = 0$

Evanescent waves.

Snell's law comes from matching $\exp i k r_T$ on the interface



$$\frac{\omega}{c} n_1 \sin\theta_i = k_{2T} \quad \text{always.} \quad E \sim e^{i k_{2T} x_T}$$

$$k_2^2 = n_2^2 \frac{\omega^2}{c^2} = k_{2z}^2 + k_{2T}^2 \quad \text{always} \qquad (2)$$

$\theta < \theta_I$ Below the critical angle $k_{2T} < k_2$, $k_{2z} > 0$

$\theta = \theta_I$ At the critical angle $\theta_i = \theta_I$

$$\left\{ \begin{array}{l} n_1 \sin\theta_I = n_2 \\ k_{2T} = \frac{\omega}{c} n_2 \\ k_{2z} = 0 \end{array} \right.$$

$\theta > \theta_I$ Above the critical angle $k_{2T} = \frac{\omega}{c} n_1 \sin\theta_i > \frac{\omega}{c} n_2$

and from (2), $k_{2z}^2 < 0$

Always true that
$$k_{2z}^2 = k_2^2 - k_{2T}^2 = \frac{\omega^2}{c^2}\left[n_2^2 - n_1^2 \sin^2\theta_i \right]$$

Write $n_1 = \dfrac{n_2}{\sin\theta_I}$ to eliminate n_1

above critial angle
$$k_{2z} = i\, n_2 \frac{\omega}{c} \left[\frac{\sin^2\theta_i}{\sin^2\theta_I} - 1 \right]^{1/2} \equiv i D k_2$$

$$\exp(i \vec{k}_2 \cdot \vec{x}) = \exp i k_1 x_T \left(\frac{\sin\theta_i}{\sin\theta_I} \right) \exp(-k_2 D z)$$

↑ damping → propagation

Wave penetrates a distance
$$z \sim \frac{1}{k_2 D} = \frac{\lambda_2}{2\pi D}$$

Also $\quad R = e^{2i\delta}, \quad \tan\delta = -\dfrac{b}{a}$

$$\tan\delta = -\frac{n_2}{n_1}\frac{|\cos\theta_t|}{\cos\theta_i}$$

$$= \frac{-n_2\left[\dfrac{\sin^2\theta_i}{\sin^2\theta_I}-1\right]^{1/2}}{\left(\dfrac{n_2}{\sin\theta_I}\right)\cos\theta_i}$$

$$\tan\delta = -\frac{\left[\sin^2\theta_i - \sin^2\theta_I\right]^{1/2}}{\cos\theta_i}$$

$$\left.\begin{array}{ll} \text{At } \theta_i = \theta_I & \delta = 0 \\ \text{At } \theta_i \to \frac{\pi}{2} & \delta \to -\frac{\pi}{2} \end{array}\right\} \text{phase shift of reflected wave}$$

Where does the energy go? Look at the time averaged Poynting vector

$$\vec{S}' = \frac{1}{2} \mathrm{Re}\, \vec{E} \times \vec{H}^* \quad \text{and} \quad \vec{H} = \frac{1}{\mu\omega} \vec{k} \times \vec{E} \quad \Bigg\} \quad \vec{S} = \frac{1}{2\mu\omega} \mathrm{Re}\, \vec{E} \times (\vec{k} \times \vec{E})^*$$

Flow in direction $\hat{n}'$: $\vec{S} \cdot \hat{n}' = \frac{1}{2\mu\omega} \mathrm{Re}(\hat{n} \cdot \vec{k}^*) |E|^2 \quad (*)$

In vacuum, $\hat{n} \parallel \vec{k}$, $\omega = ck$, $\mu = \mu_0$, $c^2 = \frac{1}{\mu_0 \epsilon_0}$

$$\mathrm{Re}\, \vec{S}' \cdot \hat{n}' = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} |E|^2 (\hat{n} \cdot \hat{n}')$$

In the medium, if $\theta_i > \theta_I$, k_T is real, k_z pure imaginary

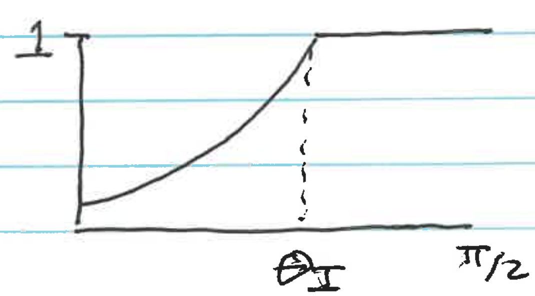
so - note * in $(*)$ no flow into wall!

recall $R_{out} = \dfrac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$

If $\theta_i > \theta_I$, cosine θ_t is pure imaginary

$R_{out} = \dfrac{a - ib}{a + ib}$. This is a pure phase -

$|R|^2 = 1$ ~~or $R = e^{2i\delta}$, $\tan\delta = b/a$~~



Now want to begin to consider frequency dependent $n(\omega)$, absorption --- formulas similar, but new physics: Superposition of Scalar Waves

Lets look at superpositions of solutions of the wave equation, for some externally imposed dispersion relation $\omega = \omega(k)$

(i.e. k = independent variable). This follows from wave eqn.

We'll work in 1 space, 1 time dimension

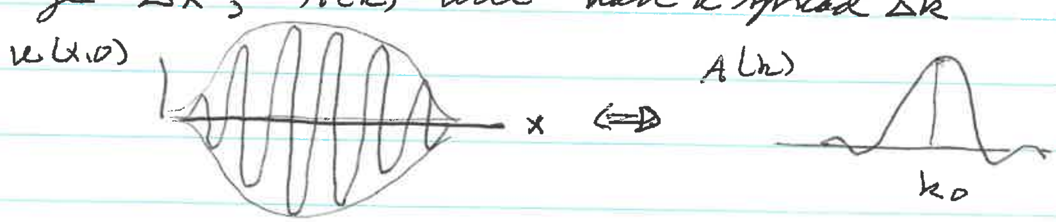
We can write a general solution of the scalar wave eqn as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i(kx - \omega(k) t)}\, dk$$

invert at $t=0$ - Fourier amplitude

keep $\Rightarrow$
$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ikx}\, dx$$

Monochromatic waves ($A(k) = \delta(k-k_0)$) occur if $u(x,0) = \exp ik_0x$ - an infinitely long wave train. But if $u(x,0)$ cuts off over a range Δx, $A(k)$ will have a spread Δk



and $\quad \Delta k \, \Delta x \gtrsim \frac{1}{2}$ (~~uncertainty relation~~)

(usual uncertainty principle argument).

$\nabla^2 + \omega^2(k)^2$ Now "phase velocity" - $v_{ph} = \dfrac{\omega(k)}{k}$

modes w/ different $\omega(k)$ might have different v_{ph} - maybe wave train will distort - ~~[illegible]~~?

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int A(k)\, e^{i(kx - \omega(k)t)}\, dk$$

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int A(k)\, e^{ikx}\, dk$$

come back to this

$$\omega(k) = \omega_0 + \left.\frac{d\omega}{dk}\right)_{k=k_0} (k - k_0) + \cdots$$

$$u(x,t) = \exp i\left[k_0 \left.\frac{d\omega}{dk}\right)_0 - \omega_0 \right] t$$

$$\times \frac{1}{\sqrt{2\pi}} \int dk\, A(k) \exp i\left[\left(x - \left.\frac{d\omega}{dk}\right)_{k_0} t\right) \right] k$$

$$= \text{phase factor} \times u\left(\left(x - \left.\frac{d\omega}{dk}\right)_{k_0} t\right), 0\right)$$

i.e. pulse travels unchanged but with "group velocity" $v_g = \left.\frac{d\omega}{dk}\right)_{k_0}$

For light $\omega(k) = \frac{ck}{n(k)} \approx \frac{ck}{n(\omega)}$, $k = \frac{\omega \cdot n(\omega)}{c}$

$$v_{ph} = \frac{\omega}{k} = \frac{c}{n(\omega)}$$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega} \frac{\omega \cdot n(\omega)}{c} = \frac{1}{c}\left[n + \omega \frac{dn}{d\omega} \right]$$

$$\frac{1}{v_g} = \frac{1}{v_{ph}} + \frac{\omega}{c} \frac{dn}{d\omega};$$ 2 ~~possibilities~~

a)

$$\frac{1}{v_g} = \frac{1}{v_{ph}} + \frac{\omega}{c}\frac{dn}{d\omega}$$

2 possibilities

a) $\frac{dn}{d\omega} > 0$: $v_g < v_{ph}$

This is called "normal dispersion"
pulse moves slower than components

Anomalous dispersion - $\frac{dn}{d\omega} < 0$, $v_g > v_{ph}$

pulse ~~wave~~ goes faster than its components
($v_g > c$? yes, but pulse doesn't have a sharp edge)

~~Example 2 ...~~

There is more, of course:

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}\cdot(k-k_0) + \cdots$$

The ... causes the pulse to distort

In QM $\quad \hbar\omega = \frac{\hbar^2 k^2}{2m}$

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m}$$

but

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int A(k)\exp\left[ikx - i\frac{\hbar k^2}{2m}t\right]dk$$

(homework problem: $u(x,0) = e^{-\alpha x^2}$ etc.

long-ish Jackson example...

Waves in a conducting medium - single freq

Suppose that instead of being in free space with $\vec{J}=0$, we are inside a material which obeys Ohm's law,

$$\vec{J} = \sigma \vec{E}, \quad \sigma \equiv \text{conductivity - a constant}$$

and material has dielectric constant ϵ

$$\vec{D} = \epsilon \vec{E} \qquad \epsilon = \text{constant}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}, \quad H, J, D \sim e^{-i\omega t} \Rightarrow$$

$J=0$	$J=\sigma E$
$\vec{\nabla}\times\vec{H} = -i\omega \epsilon \vec{E}$	$\nabla \times H = (\sigma - i\omega\epsilon) E$
	$= -i\omega\epsilon\left[1 + \frac{i\sigma}{\omega\epsilon}\right]\vec{E}$
	$\equiv -i\omega\,\epsilon(\omega)\cdot\vec{E}$

introducing a complex frequency dependent dielectric constant.

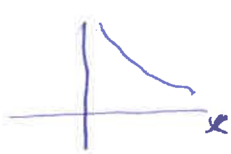
Solve Maxwell's equs...

second $\rightarrow$ $$k = \frac{\omega}{c}\sqrt{\frac{\mu\,\epsilon(\omega)}{\mu_0\epsilon_0}}$$

first $\rightarrow$ or $$k^2 = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)\frac{\omega^2}{c^2}\left[1 + \frac{i\sigma}{\omega\epsilon}\right]$$

k^2 - and k - are complex - Waves are damped! Write $k = \beta + \frac{i\alpha}{2}$

$$\exp i\left[\vec{k}\cdot\vec{x} - \omega t\right] = \exp\left[i\beta\,\hat{n}\cdot x - \frac{\alpha}{2}\hat{n}\cdot\vec{x} - i\omega t\right]$$



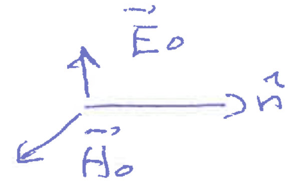
But there's more!

$$\begin{pmatrix}\vec E(x,t)\\ \vec H(x,t)\end{pmatrix} = \begin{pmatrix}\vec E_0\\ \vec H_0\end{pmatrix} e^{i(\beta \hat n\cdot x - \omega t)}\, e^{-\alpha \frac{\hat n\cdot x}{2}}$$

$\vec\nabla\cdot\vec E = 0 \Rightarrow \hat n\cdot\vec E_0 = 0$ = transversality (still)

$\vec\nabla\times\vec E = i\vec k\times\vec E = i\omega\vec B = i\omega\mu\vec H$ (β)

$$\vec H_0 = \frac{1}{\omega\mu}\left[\beta + i\frac{\alpha}{2}\right]\hat n\times\vec E_0$$

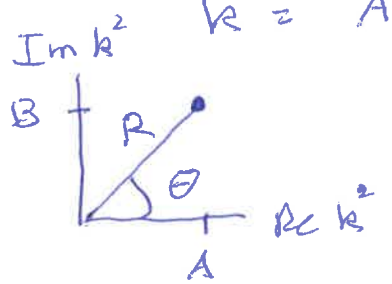


but H and E are out of phase!

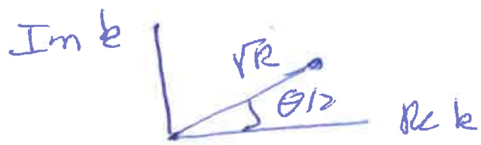
This affects energy flow: $\vec S = \frac{1}{2}\mathrm{Re}\,\vec E\times\vec H^*$!

We need to find β & α ...

$k^2 = A + iB = Re^{i\theta}$



$\to k = \sqrt{k^2} = \sqrt{R}\, e^{i\theta/2}$



general result straightforward but two limiting forms are much easier (and more useful)

$$k = \left[\frac{\omega}{c}\sqrt{\mu\epsilon}\right]\underbrace{\left(1+\frac{i\sigma}{\omega\epsilon}\right)}^{1/2}$$

i) poor conductor: $\frac{\sigma}{\omega\epsilon} \ll 1$

$$\sqrt{1+\frac{i\sigma}{\omega\epsilon}} \simeq 1+\frac{1}{2}\frac{i\sigma}{\omega\epsilon}$$

$$k = \underbrace{\omega\sqrt{\mu\epsilon}}_{\beta \text{ - as before}} + \frac{i\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

β - as before $\operatorname{Re} k = \frac{n\omega}{c}$

attenuation - α is frequency independent $= \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$

$\beta + i\frac{\alpha}{2}$ nearly real : H & E nearly in phase

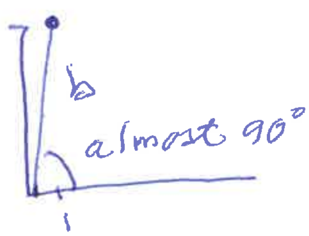
phase (~~... $\sigma = \infty$~~), ~~attenuation frequency independent~~

(all pretty much like $\sigma = 0$ ...)

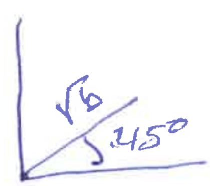
ii) Good conductor: $\frac{\sigma}{\omega\epsilon} \gg 1$

$1+\frac{i\sigma}{\omega\epsilon} \equiv 1+ib$ ~~$\frac{i\sigma}{\omega\epsilon}$~~ $= b$

$b \gg 1$



$\Rightarrow$



$\sqrt{b} = \sqrt{\frac{\sigma}{\omega\epsilon}}$, $e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$ so

$$k = \beta + i\frac{\alpha}{2} = \omega\sqrt{\mu\epsilon}\left\{\sqrt{\frac{\sigma}{\omega\epsilon}}\left(\frac{1+i}{\cancel{2}\sqrt{2}}\right)\right\}$$

$$= \sqrt{\frac{\sigma\mu\omega}{2}}\,(1+i) \quad \Rightarrow \quad \frac{\alpha}{2} = \sqrt{\frac{\sigma\mu\omega}{2}}$$

$\vec{H}_0 = \frac{1}{\omega\mu} k\,(\hat{n} \times \vec{E}_0) \Rightarrow$ E & H out of phase.

$$\frac{|H_0|}{|E_0|}$$

$$\frac{|H_0|}{|E_0|} \sim \frac{1}{\omega\mu}\left[\sqrt{\sigma\omega\mu}\right] \quad \text{in size}$$

Multiply by $\sqrt{\frac{\epsilon}{\epsilon}}$

$$\frac{|H_0|}{|E_0|} = \sqrt{\frac{\epsilon}{\epsilon}\frac{\sigma}{\omega\mu}} = \sqrt{\frac{\epsilon}{\mu}}\sqrt{\frac{\sigma}{\omega\epsilon}}$$

$\sqrt{\frac{\epsilon}{\mu}}$ is usual free space ratio. $\frac{\sigma}{\omega\epsilon} \gg 1$ for good conductor - H is bigger than E in conductor, EM energy in conductor is mostly magnetic.

$$\vec{E}, \vec{H} \sim \exp\left[-\frac{\alpha}{2}\vec{x}\cdot\hat{n}\right] \equiv \exp\left[-\frac{\vec{x}\cdot\hat{n}}{\delta}\right]$$

$$\delta = \frac{2}{\alpha} \equiv \text{"skin depth"} = \sqrt{\frac{2}{\sigma\omega\mu}}$$

oscillating fields only penetrate the "skin" of a good conductor - note δ is frequency dependent. p. 220 - Copper $\delta = \frac{6\text{ cm}}{\sqrt{\nu(\text{Hz})}}$, sea water $\frac{240\text{ m}}{\sqrt{\nu}}$

Often work w/ $\vec{H}$: $\vec{E} = \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)\left[\hat{n}\times\vec{H}\right]$

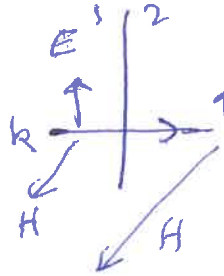
$$S = \frac{1}{2}\text{Re}\,\hat{n}\cdot(\vec{E}\times H^*) = \frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}}|H|^2$$

$$\frac{1}{\sigma} = \frac{\delta^2\omega\mu}{2} \qquad \text{so} \quad \frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}} = \frac{1}{2}\sqrt{\frac{\mu^2\omega^2\delta^2}{4}}$$

$$S \sim \frac{1}{2}\frac{\mu\omega\delta}{4}|H|^2 = \frac{\delta}{4}\left[\frac{\mu\omega}{\omega}\cdot\frac{2}{\sigma\delta^2}\right]|H|^2$$

$$= \frac{1}{2\sigma\delta}|H|^2$$

A new feature – energy loss in the material



$\sim \exp\left[-\frac{x\cdot\hat{n}}{\delta}\right]$

2 (equivalent) ways to see this:

1) energy added to charges in the material

$\frac{\text{energy}}{\text{volume}} = w_{loss} \sim \int \vec{J}\cdot\vec{E}\, d^3x \longrightarrow \frac{1}{2}\,\mathrm{Re}\, J^*\cdot E$ and $\vec{J} = \sigma\vec{E}$

so $w_{loss} = \frac{1}{2}\sigma|E|^2 = \frac{\sigma}{2}|E_{2,\text{surface}}|^2 \exp\left[-\frac{2x\cdot n}{\delta}\right]$

– integrate over region to find energy lost in medium. This is called "ohmic heating"

or (equivalently)

2) $w_{loss} = \mathrm{Re}\left[2i\omega(W_e - W_m)\right] = \frac{1}{2}\mathrm{Re}\left[i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)\right]$

and ~~D = εE, ε real complex~~ ② $\vec{D} = \epsilon\vec{E}$, ϵ complex

① $H = \frac{n(\omega)}{\mu_0 c}\,\hat{n}\times\vec{E}$, n is complex, $W_{loss} \neq 0$!

$\Rightarrow$ get same answer as (1) – see prob 7.6

Poynting vector gives flux of radiation in. Often, this is done with $\vec{H}$: $\vec{E} = \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)\left[\hat{n}\times\vec{H}\right]$

$S' = \frac{1}{2}\mathrm{Re}\ \hat{n}\cdot(\vec{E}\times\vec{H}^*) = \frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}}\,|H|^2$

$\frac{1}{\sigma} = \frac{\delta^2\omega\mu}{2}$, $\frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}} = \frac{1}{2}\sqrt{\frac{\mu^2\omega^2\delta^2}{4}}$,

$S \sim \frac{1}{2}\frac{\mu\omega\delta}{4}|H|^2 = \frac{\delta}{4}\left[\mu\omega = \frac{2}{\sigma\delta^2}\right]|H|^2 = \frac{1}{2\sigma\delta}|H|^2$

~~⇒ Bottom line: ε → ε + iσ/ω → loss of EM energy (absorption)~~