Vector and Scalar Potentials

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \Rightarrow \nabla\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\Rightarrow E + \frac{\partial A}{\partial t} = -\nabla\Phi$$

or $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

<u>Gauge invariance</u>: Maxwell's equations are invariant under the reparameterization (≡ gauge transformation)

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\Phi' = \Phi - \frac{\partial\chi}{\partial t}$$

[see Jackson & Okun RMP <u>73</u> 663 (2001)]

For <u>any</u> (differentiable) scalar function χ.

proof: $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}$

$$\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial A'}{\partial t} = -\nabla\Phi + \nabla\frac{\partial\chi}{\partial t} - \frac{\partial A}{\partial t} - \frac{\partial}{\partial t}\nabla\chi$$

~~What is the wave equation in terms of Φ, $\vec{A}$?~~

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = -\nabla^2\Phi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) \quad (1)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial A}{\partial t}\right) \qquad c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$= \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A =$$

or $$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\nabla\cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\vec{J} \quad (2)$$

~~Simplify by making a gauge choice~~

What is the wave equation in terms of Φ, A?

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \Rightarrow \nabla \cdot \left\{ -\nabla \Phi - \frac{\partial A}{\partial t} \right\} = \frac{\rho}{\epsilon_0}$$

keep

$$-\nabla^2 \Phi - \frac{\partial}{\partial t}(\nabla \cdot A) = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times A) = \mu_0 J + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla \Phi - \frac{\partial A}{\partial t}\right)$$

$$\nabla(\nabla \cdot A) - \nabla^2 \vec{A} = \mu_0 J - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)$$

$$(2) \qquad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} - \nabla\left(\nabla \cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)$$

keep

hmm - not nice - but we have the freedom to make a gauge choice

Lorentz gauge: $\nabla\cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

$$\left.\begin{array}{ll} 2) & \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \\ 1) & \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \end{array}\right\} \text{Wave eqn again}$$

Btw, solutions of wave eqn in free space are waves

in 1-d $\left(\frac{d^2}{dx^2} - \frac{1}{c^2}\frac{d^2}{dt^2}\right)\left[f_1(x-ct) + f_2(x+ct)\right] = 0$

"retarded" "advanced"

i.e. location of pulse $\Delta x = \pm c\Delta t$

Suppose you have a Φ and $\vec{A}$ which do not satisfy $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$. How to "gauge fix" to Lorentz gauge?

$A' = A + \nabla\chi$, $\Phi' = \Phi - \frac{\partial \chi}{\partial t}$. Find χ so

$$\nabla\cdot A' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0 = \nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} + \nabla\cdot A + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

i.e. $\nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = -\left(\nabla\cdot A + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)$

χ is solution of wave eqn w/ known source.

Also this says Lorentz gauge is not unique – if $\nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0$ then both $\begin{pmatrix}\vec{A}, \Phi \\ A', \Phi'\end{pmatrix}$ are in Lorentz gauge, soln's of homogeneous wave eqn for same source.

Another useful gauge is { Coulomb / radiation / transverse / physical } gauge

$$\vec{\nabla}\cdot\vec{A} = 0$$

In this gauge $\nabla^2 \Phi = \rho/\epsilon_0$

i.e. Instantaneous Coulomb potential

$$\Phi(x,t) = \frac{1}{4\pi\epsilon_0}\int d^3x \, \frac{\rho(x',t)}{|x-x'|}$$

[keeps this eqn] and $\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \left\{-\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t}\right\}$

Wave eqn: $\vec{A}(r - ct)$ but funny source.

Let's break $\vec{J}$ apart $\vec{J} = \vec{J}_t + \vec{J}_\ell$

Transverse current $\vec{\nabla}\cdot\vec{J}_t = 0$

Longitudinal current $\vec{\nabla}\times\vec{J}_L = 0$

Identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{J}) = \nabla(\nabla\cdot J) - \nabla^2 J$

or $\nabla^2 J = \vec{\nabla}(\vec{\nabla}\cdot\vec{J}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{J})$

Now if $\nabla^2\vec{V} = -4\pi\vec{a}(x)$

$$\vec{V}(x) = \int \frac{a(x')}{|x-x'|}\, d^3x'$$

so (formally)

$$\vec{J}(x) = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}'(\vec{\nabla}'\cdot\vec{J})}{|x-x'|} + \frac{1}{4\pi}\int d^3x \, \frac{(\nabla'\times(\nabla'\times J))}{|x-x'|}$$

Now a parts integral and $\nabla' \frac{1}{|x-x'|} = -\nabla \frac{1}{|x-x'|}$

give

$$\vec{J}(x,t) = -\frac{1}{4\pi}\vec{\nabla}\int d^3x' \frac{\vec{\nabla}'\cdot\vec{J}}{|x-x'|} + \frac{1}{4\pi}\nabla\times\left(\nabla\times\int\frac{\vec{J}(x)\,d^3x'}{|x-x'|}\right)$$

The first term has no curl, it is J_L

The 2nd term has no divergence, it is J_T

Now in continuity equation $\nabla\cdot J = \frac{\partial\rho}{\partial t}$

<u>This</u> is $\vec{J}_L$ - only $\vec{J}_L$ participates in $\frac{\partial\rho}{\partial t}$

Finally $\frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t} = \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\vec{\nabla}\int d^3x' \frac{\frac{\partial\rho(x',t)}{\partial t}}{|x-x'|}$

$\frac{1}{\epsilon_0 c^2} = \mu_0$ so this is

$\mu_0\vec{J}_L$

so $$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0\left[\vec{J}-\vec{J}_L\right] = -\mu_0\vec{J}_t$$

$\vec{A}$ is given by the transverse current. $\vec{A}$ itself is transverse, $\vec{\nabla}\cdot\vec{A}=0$ so if $\vec{A}\sim e^{i\vec{k}\cdot\vec{r}}$, $\vec{\nabla}\cdot\vec{A} = \vec{k}\cdot\vec{A} = 0$

In this gauge, electrodynamics consists of and in terms of potentials

- transverse radiation fields only from $\vec{A}$
- instantaneous Coulomb interaction

This is both useful and odd!

Useful: $\Phi \sim \frac{1}{r}$, $\underset{\nabla\Phi}{\overset{\vec{E}}{\cancel{\vec{\Phi}}}} \sim \frac{1}{r^2}$ from Φ while radiation fields $\underset{}{\overset{\vec{E},\vec{B}}{\text{are}}}$ $\frac{1}{r}$ far away (still have to show this): $\vec{E}$, $\vec{B}$ only from $\vec{A}$, $\vec{J}_t$
$\vec{A}$, $\vec{E}$ & $\vec{B}$ transverse

Furthermore, in the quantum theory of radiation $\vec{A}$ is what becomes the photon field. 2 components of $\vec{A}$. Happens, canonical momentum conjugate to $\vec{A}$ is $\vec{E}$

$$H = E^2 + B^2 = E^2 + (k \times A)^2 \sim E^2 + k^2 A^2$$

oscillator $H = \qquad p^2 + x^2$

oscillator $\rightarrow \sum_k \hbar\omega_k (n_k + \frac{1}{2})$, n = integer

$\Rightarrow$ photon! (No oscillator description of Φ)

Odd: Instantaneous Coulomb interaction. What became of speed of light?
What happens to $\vec{\nabla}\cdot\vec{A}$ under Lorentz transf.?

Also $\quad \vec{J}_t = \vec{\nabla} \times \nabla \times \int \frac{J(x')\, d^3x'}{|x - x'|}$

is not local in space, even if $\vec{J}$ is local

Lorentz gauge also useful and odd:

Useful: single wave equation – everything (ϕ, A) is causal, $\Phi(r-ct)$

Easy to keep track of Lorentz transformation properties

Odd: 3 A's, 1 ϕ : 4 ~~[illegible]~~ radiation potentials.
How do you see light has 2 polarizations?

See Jackson & Okun RMP 73 663 (2001)

Can say several things to summarize

1) you can "pick a gauge" to simplify your problem. Sometimes can make a gauge choice with a tunable parameter in it. $\vec{A}(\xi)$, $\Phi(\xi)$. At end (~~[illegible]~~ antenna pattern, power radiated) should be no ξ dependence. Useful to check calculations.

2) Something deep is going on in electrodynamics. A peculiar reparameterization invariance which seems not to respect the fundamental degrees of freedom (2 A's vs 3 A's + Φ)

Could say: It's all $\vec{E}$ & $\vec{B}$, they are gauge inv.

But – Quantum mechanics involves $\vec{A}$ $(\vec{p} - e\vec{A}/c)$ interesting observed interference effects (Aharonov-Bohm)

– in Generalizations of E&M – electroweak, QCD not just A, Φ are gauge dependent, so are $\vec{E}$ & $\vec{B}$.

Only GI quantities are energy densities, Poynting vector, cross sections, etc.

Field energy and momentum

Power $= \vec{F}\cdot\vec{v}$: rate E & B fields do work on point charge

is $q\vec{v}\cdot\vec{E}$ = or

1) gain of mechanical energy

2) rate of decrease of EM energy $= \dfrac{dE_{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x$

3) rate fields do work on sources

Write RHS $\int_V J\cdot E\, d^3x = \int_V \left[\vec{\nabla}\times\vec{H} - \dfrac{\partial \vec{D}}{\partial t}\right]\cdot E\, d^3x$

$$\nabla\cdot(E\times H) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

$$= -H\cdot\frac{\partial B}{\partial t} - E\cdot(\nabla\times H)$$

$$\int J\cdot E\, d^3x = -\int \Big(\nabla\cdot(E\times H) + \underbrace{H\frac{\partial B}{\partial t} + E\frac{\partial D}{\partial t}}_{*}\Big)\, d^3x$$

Assume linear medium - (Caution - this is a special case - dispersive systems with $n(\omega)$ are different)

$$* = -\frac{\partial}{\partial t}\left[\frac{1}{2}H\cdot B + \frac{1}{2}D\cdot E\right] = -\frac{\partial u}{\partial t} \qquad u = \text{field energy density}$$

so $$-\int J\cdot E\, d^3x = \int_V d^3x\left(\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(\vec{E}\times\vec{H})\right)$$

or $$\frac{\partial u}{\partial t} + \vec{J}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{S}$$

$\vec{S} = \vec{E}\times\vec{H}$ = Poynting vector = energy flux

$= \dfrac{\text{energy}}{\text{area-time}}$

$$\frac{d}{dt}\left(E_{field} + E_{mech}\text{ in } V\right) = -\int \vec{S}\cdot\hat{n}\, dA$$

= - rate of flow of energy out of V

Momentum conservation uses $F = q(\vec{E} + \vec{v}\times\vec{B})$

$$\frac{d\vec{P}_{mech}}{dt} = \int_V d^3x\,[\,\rho\vec{E} + \vec{J}\times\vec{B}\,]$$

$$\rho = \epsilon_0\,\vec{\nabla}\cdot\vec{E} \qquad \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\rho E + J\times B = \epsilon_0(\vec{\nabla}\cdot E)\vec{E} + \epsilon_0 c^2(\nabla\times B)\times B - \epsilon_0\frac{\partial E}{\partial t}\times B \qquad a$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}\underset{c}{(E\times B)} - \underset{c}{E\times}(\nabla\times E)$$

zero, just to be pretty

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\Big[\overset{(a)}{E(\nabla\cdot E)} + c^2 B(\nabla\cdot B) - \underset{c}{E\times}(\nabla\times E) - c^2\underset{(a)}{B\times(\nabla\times B)} - \frac{d}{dt}(E\times B)\Big]$$

LHS

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\int_V \epsilon_0(\vec{E}\times\vec{B})\,d^3x \overset{?}{=} \int dA\;\hat{n}\cdot\text{"momentum flow" into } V$$

2nd term is momentum in E+M field

$$\vec{g} \equiv \epsilon_0(\vec{E}\times\vec{B}) = \mu_0\epsilon_0(\vec{E}\times\vec{H}) = \frac{1}{c^2}(\vec{E}\times\vec{H})$$

$$= \frac{1}{c^2}\vec{S}$$

The other terms — let's look at

$$\big[E(\nabla\cdot E) - E\times(\nabla\times E)\big]_{\text{direction } i}$$

$$= E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right)$$

$$+ E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}\left(E_1^2\right) + \frac{\partial}{\partial x_2}\left(E_1 E_2\right) + \frac{\partial}{\partial x_3}\left(E_1 E_3\right)$$

$$- \frac{1}{2}\frac{\partial}{\partial x_1}\left(E_1^2 + E_2^2 + E_3^2\right)$$

$$\epsilon_0 [\;]_i = \sum_j \frac{\partial}{\partial x_j} \epsilon_0 \left(E_i E_j - \frac{1}{2} E^2 \delta_{ij}\right)$$

a divergence! the object with 2 indices is a 2nd rank tensor, called the Maxwell Stress Tensor"

$$\epsilon_0 [\ ]_i = \sum_j \frac{\partial}{\partial x_j} \epsilon_0 \left(E_i E_j - \frac{1}{2} E^2 \delta_{ij} \right)$$

Divergence of a 2nd rank tensor, the "Maxwell stress tensor". identical calculation for $\vec{B}$

$$T_{ij} = \epsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} (E^2 + c^2 B^2) \delta_{ij} \right]$$

$$\frac{d}{dt} (P_{\substack{mech \\ in\ V}} + P_{field})_i = \sum_j \int_V d^3x \frac{\partial}{\partial x_j} T_{ij}$$

$$= \int_{S'} dA\ T_{ij} n_j$$

$\hat{n}$ outward normal

∴ Force per unit area transmitted in direction i across the surface, acting on particles & fields inside V $= T_{ij} n_j$

and increasing the momentum in V =

PT 3.1

~~Recap Stress Tensor~~

$$\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{fields} \text{ in } V\right) = \int_{S'} dA\, T_{ij} n_j \qquad (1)$$

$$T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2}\delta_{ij} E^2\right] + \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}\delta_{ij} B^2\right]$$

T_{ij} = ~~Pressure~~ ith component of $\frac{\text{force}}{\text{unit area}}$ = pressure across a surface with normal j

---

use Have objects in EM fields. Calculate forces acting on them by enclosing objects with boundary surface S', add up total EM force on surface, by integrating RHS of (1)

T_{ij} = symmetric rank-2 tensor, part of "energy-momentum tensor" $\equiv T^{\mu\nu}$ $\mu\nu = t, x, y, z$

Conservation of scalar quantity (charge) involves 4-vector $(\rho, \vec{J})$: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$, $J \cdot n = J_i n_i$ = flux of charge across surface
J^μ $\partial_\mu J^\mu = 0$

Conservation of vector quantity p^μ needs another index, $T^{\mu\nu}$, $T_{ij} n_j$ = flux of P_i across surface.

Pressure, etc for field variables?

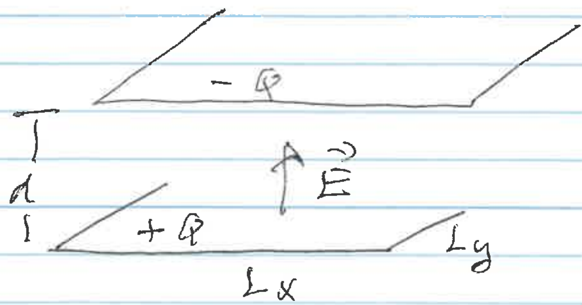
Example:

Stress tensor for constant $\vec{E} = \hat{z} E$ is

$$T = \frac{\epsilon_0}{2} \begin{bmatrix} -E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & E^2 \end{bmatrix} \qquad T_{ij} = \epsilon_0 \left(E_i E_j - \tfrac{1}{2} \delta_{ij} E^2 \right)$$

$$\int F_i \, dV = \text{integrated force in direction } i = \int dA \, T_{ij} n_j$$

Where did this E come from? a capacitor



$$E_z = \frac{Q}{\epsilon_0 L_x L_y}$$

zz 1) plates attract eachother - capacitor experiences negative pressure in z direction - we have to exert a force to hold them apart

xx yy 2) Charges on plates repel - plates want to fly apart, we have to exert a force to hold them together

$$\text{Field energy } U = \frac{\epsilon_0}{2} E^2 \times \text{volume}$$

$$U = \frac{1}{2\epsilon_0} \frac{Q^2 d}{L_x L_y} = \frac{\epsilon_0}{2} \left(\frac{Q}{\epsilon_0 L_x L_y} \right)^2 L_x L_y d$$

$$U = \frac{\epsilon_0}{2}\left(\frac{Q}{\epsilon_0 L_x L_y}\right)^2 L_x L_y d$$

Force in z direction acts to decrease d

$$F_z = - \frac{\partial U}{\partial d} = -\frac{\epsilon_0}{2} E^2 \times \text{area}$$

$$= -T_{zz} \cdot \text{area}$$

~~pressure~~ Need outward pressure to hold plates apart (note T_{zz} +)

Force in x-direction acts to increase L_x

$$F_x = -\frac{\partial U}{\partial L_x} = -\frac{\partial}{\partial L_x}\left[\frac{Q^2 d}{2\epsilon_0 L_x L_y}\right] = \frac{Q^2 d}{2\epsilon_0 L_x^2 L_y}$$

$$\text{Pressure} = \frac{F_x}{\text{area} = L_y d} = \frac{Q^2}{2\epsilon_0 L_x^2 L_y^2} = \frac{\epsilon_0 E^2}{2} = -T_{xx}$$

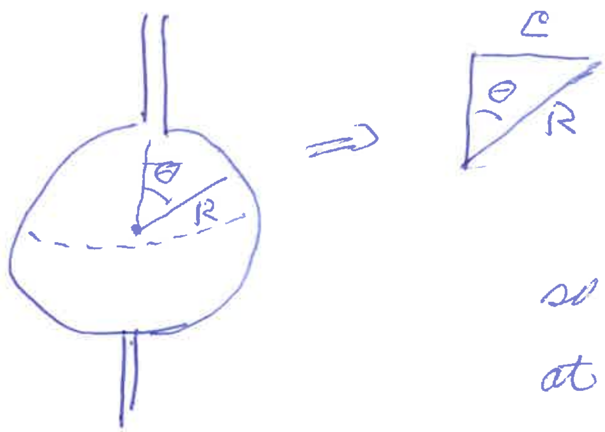
Need inward pressure to hold plates together (note T_{xx} negative)

Alternative set of words for stress tensor

T_{ii} = pressure needed (from some non electromagnetic source) in ith direction to maintain equilibrium

negative T: inward pressure needed
positive T: outward pressure needed

Comps 1991: A long conductor which is a thin cylindrical shell of radius "a" carries a current I into a thin conducting shell of radius R >> a. The current flows uniformly through the shell. Cut the shell apart at the equator, hold it together, current continues to flow. What is the force required to pull the 2 halves apart?



1) need $\vec{B}$: Ampere says

$$\oint B \cdot dl = \mu_0 I_{encl}$$

so $\vec{B} = 0$ inside shell. Just outside, at $\rho = R\sin\theta$, $2\pi\rho B = \mu_0 I$

or $$\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi} = \frac{\mu_0 I}{2\pi R\sin\theta}\hat{\varphi}.$$

$$T_{ij} = \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}\delta_{ij}B^2\right]$$

($\mu_0\epsilon_0 = \frac{1}{c^2}$, $\vec{E} = 0$, $\vec{B}$ = constant)

Force/unit area in direction i on surface w/ outward normal n_j is $= T_{ij} n_j$.

Here we want $i = \hat{z}$ (vertical), $\hat{n} = \hat{r}$ of the lower shell so we want $T_{zr} = \frac{1}{\mu_0}\left[\hat{z}\cdot\vec{B}\,B\cdot\hat{r} - \frac{1}{2}B^2\,\hat{z}\cdot\hat{r}\right]$.

~~Integrate over the surface of the hemisphere~~

~~F_z~~ $= -\frac{1}{2\mu_0}\,\hat{z}\cdot\hat{r}\;B^2$ since $\vec{B} = \hat{\varphi}\,B_0$

Integrate over the surface of a hemisphere to find the total force

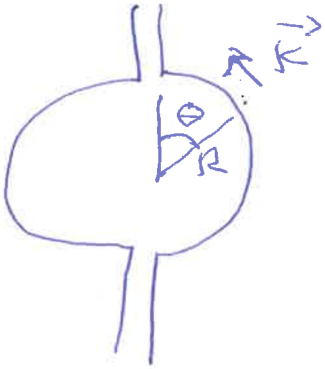
$$F_z = -\frac{1}{2\mu_0}\int_{\theta_0 = \frac{a}{R}}^{\pi/2} 2\pi R^2 \sin\theta\, d\theta \left[\frac{\mu_0 I}{2\pi R \sin\theta}\right]^2 \cos\theta$$

$$= -\frac{\mu_0 I^2}{4\pi}\int_{\theta_0}^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta} = -\frac{\mu_0 I^2}{4\pi}\ln\frac{1}{\sin\theta_0}$$

$$\simeq -\frac{\mu_0 I^2}{4\pi}\ln\frac{R}{a}$$ - the inward force holding the shell together.

An alternate approach, not using the stress tensor - find the Lorentz force acting on the device via the surface currents

PT=6



$$F_z = \hat{z} \cdot \int dA \; \vec{K} \times \vec{B}$$

$$\vec{K} = -\underbrace{\hat{\theta}}_{\text{up!}} \; \frac{I}{\underbrace{2\pi R \sin\theta}_{\text{circumference}}}$$

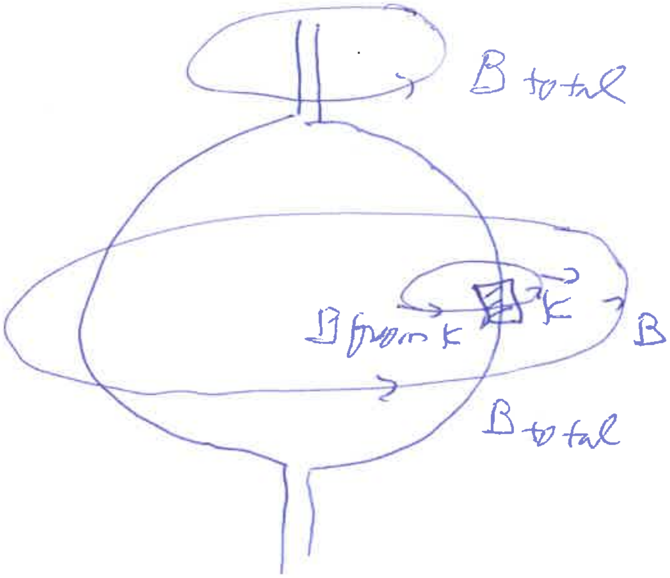
$$F_z = 2\pi \int_{\theta_0}^{\pi/2} R^2 \sin\theta \, d\theta \; \overbrace{\left[\frac{I}{2\pi R \sin\theta} \right]}^{K} \overbrace{\left[\frac{\mu_0 I}{2\pi R \sin\theta} \right]}^{B}$$

$$\times \left[\hat{z} \cdot \underbrace{(-\hat{\theta} \times \hat{\varphi})}_{-\cos\theta} \right]$$

$-\hat{\theta} \times \hat{\varphi}$ is $-$radial

$$F_z = -\frac{2\mu_0 I^2}{4\pi} \int_{\theta_0}^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta}$$

An extra 2! ~~The problem~~ It's an overcount – the B field has to come from all the currents except the little piece of K we are integrating over



B from K extends inside the shell.

Outside:

$B_{TOT} = B_{other} + B_K = B$

inside

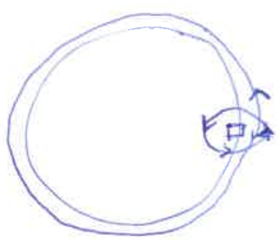
$B_{TOT} = 0 = \cancel{B_{other} - B_K}$

$B_{other} - B_K$

$= \cancel{B_K - B_{other}}$

(CW orientation)

$\Rightarrow \quad B_{other} = \frac{B}{2} \quad (= B_K$



Need

$$\vec{F} = \frac{1}{2} \int \vec{K} \times \vec{B} \, dA$$

Summary: Stress tensor

advantage - very mechanical

disadvantage - unfamiliar

Direct - advantage - physical

disadvantage: 2's

cross products

Maxwell Stress Tensor

The Maxwell stress tensor of the electromagnetic field has a form

$$\sigma_{\alpha\beta} = \frac{1}{4\pi}\left\{E_\alpha E_\beta + H_\alpha H_\beta - \frac{1}{2}\delta_{\alpha\beta}\left(E^2 + H^2\right)\right\}. \tag{1}$$

Here α, β are spacial indices.

In particular, for a constant electric field directed along the z-axis, the stress tensor takes the form

$$\sigma = \frac{1}{8\pi}\begin{pmatrix} -E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix}. \tag{2}$$

Question: what is the meaning of the stress tensor? In particular, why is it that even though the field is along the z-axis, the stress tensor has components along x and y axes, of opposite sign to that along the z-axis.

The meaning of stress tensor, as discussed in elasticity theory, is the following. The integral over a closed surface gives the total force acting on this volume

$$\oint dn^\beta \sigma_{\alpha\beta} = \int dV F_\alpha, \tag{3}$$

where n^β is the vector normal to the surface and pointing in the outward direction.

In particular, the components $-\sigma_{\alpha\alpha}$ can be interpreted as pressure in the α-direction. The minus sign signifies the fact that, at positive pressure, we must apply the force "inward" to keep the material from expanding.

Going back to the electric field case, we recall that it is not just the stress tensor of the electric field which is conserved, but the total of the electric field and of the charges which created it. Thus, we have to consider the combined system of the charges and the electric field.

To create the constant electric field, we consider a capacitor, with large plates of lengths L_x and L_y, and the distance between the places $d \ll L_x, L_y$, with the charge Q and $-Q$ on opposite plates. The electric field inside this capacitor is

$$E = \frac{4\pi Q}{L_x L_y}, \tag{4}$$

and it is pointed along the z-direction. It is clear that the opposite plates of the capacitor attract each other, in other words, the capacitor experiences negative pressure in the z-direction. It is also clear that the charges on a given plate of a capacitor repel each other. In other words, the capacitor should experience positive pressure in the x and y directions, or if the charges of the plates were simply floating in space, they would of course fly apart. This is the qualitative explanation of the form of the stress tensor (2).

To make the argument more quantitative, we calculate the attractive force between the plates. It's easiest to do by calculating the energy of the field

$$\mathcal{E} = \frac{E^2}{8\pi} L_x L_y d = \frac{2\pi d Q^2}{L_x L_y}. \tag{5}$$

The force in the z-direction is given by the derivative of the energy with respect to d, and the pressure is the force divided by the area, thus

$$p_z = \frac{\partial \mathcal{E}}{\partial d}\frac{1}{L_x L_y} = \frac{2\pi Q^2}{\left(L_x L_y\right)^2} = \sigma_{zz}. \tag{6}$$

At the same time, the force trying to tear the capacitor's plates apart is given by the derivative of $\mathcal{E}$ with respect to L_x or L_y while the pressure is again obtained by dividing by the area (dL_y in this case) and is given by

$$p_x = \frac{\partial \mathcal{E}}{\partial L_x}\frac{1}{L_y d} = -\frac{2\pi Q^2}{\left(L_x L_y\right)^2} = \sigma_{xx} \tag{7}$$

Harmonic fields, Lumped circuits

If everything has harmonic time dependence, we can write (for example)

$$\vec{E}(x,t) = \frac{1}{2}\left[\vec{E}_0(x)e^{-i\omega t} + \vec{E}_0^*(x)e^{i\omega t}\right]$$

$$= \mathrm{Re}\,\vec{E}_0(x)\,e^{-i\omega t}$$

where $\vec{E}_0$ is a complex vector. A dot product is

$$\vec{J}\cdot\vec{E} = \frac{1}{4}\left[\vec{J}_0 e^{-i\omega t} + \vec{J}_0^* e^{i\omega t}\right]\cdot\left[\vec{E}_0 e^{-i\omega t} + \vec{E}_0^* e^{i\omega t}\right]$$

$$= \frac{1}{4}\left[\vec{J}_0\cdot\vec{E}_0\, e^{-2i\omega t} + J_0^*\cdot E_0 + E_0^*\cdot J_0 + \vec{J}_0^*\cdot E_0^*\, e^{2i\omega t}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\vec{J}_0^*\cdot\vec{E}_0 + \vec{J}_0\cdot\vec{E}_0\, e^{-2i\omega t}\right]$$

Time average of $\vec{J}\cdot\vec{E} = \frac{1}{2}\mathrm{Re}\, J_0^* E_0$.

Convention is to work with complex fields, try to ignore $e^{i\omega t}$'s,

Maxwell's eqns: drop the "0", $\frac{\partial}{\partial t} = -i\omega$

$$\nabla\cdot D = \rho \qquad \nabla\cdot B = 0 \qquad \vec{\nabla}\times\vec{E} = i\omega\vec{B} \qquad \vec{\nabla}\times\vec{H} = \vec{J} - i\omega\vec{D}$$

Power formula uses

$$\frac{1}{2}\int J^*\cdot E\, d^3x = \frac{1}{2}\int\left[\nabla\times H^* - i\omega D^*\right]\cdot E\, d^3x$$

(repeat vector identities from before)

$$= -\frac{1}{2}\int \vec{\nabla}\cdot(\vec{E}\times\vec{H}^*)\, d^3x - i\omega\int d^3x\left[\frac{E\cdot D^*}{2} - \frac{B\cdot H^*}{2}\right]$$

$\Rightarrow$ define complex Poynting vector $\vec{S} = \frac{1}{2}(\vec{E}\times\vec{H}^*)$

complex energy densities $\quad W_e = \frac{1}{4}E\cdot D^*$

$$W_m = \frac{1}{4}B\cdot H^*$$

Recap

$$E(x,t) = \mathrm{Re}\, E_0(x)\, e^{-i\omega t} = \frac{1}{2}\left[E_0(x)\, e^{-i\omega t} + E_0(x)^* e^{i\omega t}\right]$$

etc

$$J \cdot E = \frac{1}{2}\mathrm{Re}\left[J_0^* E_0 + J_0 \cdot E_0\, e^{-2i\omega t}\right]$$

Time average of $J \cdot E = \frac{1}{2}\mathrm{Re}\, J_0^* E_0$

~~energy eqn~~ power eqn

$$\boxed{\frac{1}{2}\int_V \vec{J}^*\cdot E\, d^3x + 2i\omega\int_V [W_e - W_m]\, d^3x = -\int \vec{S}\cdot\hat{n}\, dA}$$

real part: time average energy conservation

imaginary part: "reactive energy" - stored, oscillating energy.

For linear medium, real ϵ, μ perfect conductors

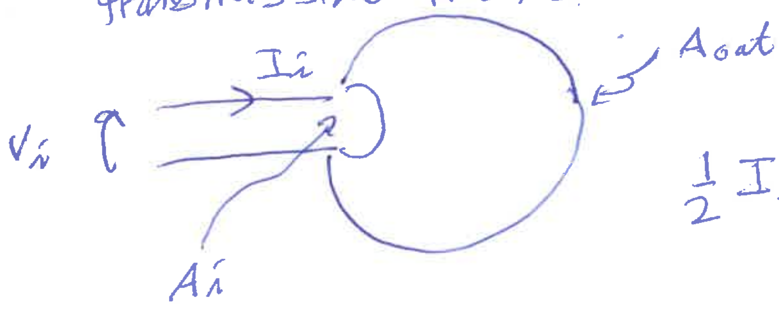
$$\frac{1}{2}\mathrm{Re}\int J^*\cdot E\, d^3x = -\mathrm{Re}\int \vec{S}\cdot\hat{n}\, dA$$

time ave ~~power dissipated~~ =
rate of doing work on ~~sources~~ sources = rate of power flowing in from outside

~~or loss of energy~~ [energy flowing out: $\vec{S}\cdot n > 0$..]

~~resistance~~ (resistivity: 2nd term has real part)

Circuit analogy: imagine radiation entering port on transmission line/coax



$$\frac{1}{2} I_i^* V_i = -\int_{A_i} \vec{S}\cdot\hat{n}\, dA$$

$$= \frac{1}{2}\int_V \vec{J}^*\cdot E\, d^3x + 2i\omega\int_V d^3x\,(W_E - W_M) + \int_{A_{out}} \vec{S}\cdot\hat{n}\, dA$$

Complex Ohm's law $\qquad V_i = I_i Z$

$Z = R - i X$ = impedance $\qquad (R + jX)$

so $\frac{1}{2} I_i^* V_i = \frac{1}{2} |I_i|^2 Z$

2 parts of Z. Real Z = resistance - energy loss.

Typically 2 kinds.

a) Ohmic resistance, $\vec{J} = \sigma \vec{E}$ - device gets warm!

$$R_\Omega = \frac{1}{|I_i|^2} \int \vec{J}^* \cdot \vec{E} \, d^3x = \frac{1}{|I_i|^2} \int_V \sigma |E|^2 d^3x$$

b) radiation resistance - energy goes out antenna

$$R_{rad} = \frac{1}{2|I_i|^2} \int_{A_{out}} \vec{S}' \cdot \hat{n} \, dA$$

~~both represent~~

$$X = \text{"reactance"} = \frac{4\omega}{|I_i|^2} \int [W_m - W_E] d^3x = X_L + X_C$$

inductive & capacitive reactance

$$W_M = \int w_M \, d^3x = \frac{1}{2} L I^2 \; (\text{real}) \longrightarrow \text{time ave} = \frac{1}{4} L |I|^2$$

$$\equiv X_L \frac{|I_I|^2}{4\omega} \qquad X_L = \omega L$$

$$W_E = \int w_E \, d^3x = \frac{1}{2} \frac{Q^2}{C} \xrightarrow[\text{T.A.}]{} \frac{1}{4} \frac{|I_i|^2}{\omega^2 C} = -X_E \frac{|I_i|^2}{4\omega}$$

$$X_C = -\frac{1}{\omega C}$$

(turn field energy into "EE" energy)

Circuit analogy often used, For ex, "radiation resistance" of antenna

$$\vec{J} = \sigma \vec{E}$$
$$\vec{B} = \mu \vec{H} \ \ldots$$

(phenomenological) relations between observables, dynamical variables

$$W_{mag} = \frac{1}{2} \vec{B} \cdot \vec{H}$$

Can we imagine any arbitrary relation between them?

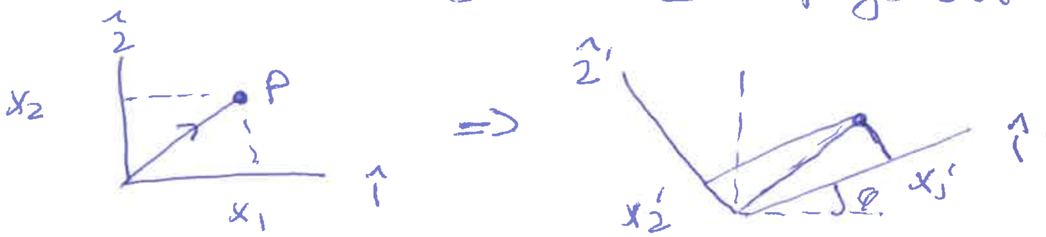
i.e. $\vec{J} = \chi \vec{B}$???

No. Such relations are constrained by observation of "space-time symmetries"

rotational invariance
spatial inversion (parity)
time reversal symmetry
(Lorentz invariance - next semester!)

~~objects~~
We use objects to build physics which automatically encode these symmetries.

Rotations, first. I'll use a "passive picture" — rotate coordinate system, not physical D.o.F



$P = (x_1, x_2, x_3)$ $\qquad$ $P = (x_1', x_2', x_3')$

$$x_i' = \sum_j a_{ij} x_j$$

For rotation about z-axis

$$a_{ij} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $x_1' = x_1 \cos\varphi + x_2 \sin\varphi$

$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$

$x_3' = x_3$

Length of vector is invariant: $x' \cdot x' = \vec{x} \cdot \vec{x}$

$$\sum_i x_i' x_i' = \sum_{ijk} a_{ij} a_{ik} x_j x_k$$

or $\sum_i a_{ij} a_{ik} = \delta_{jk} \Rightarrow \underline{\underline{a}}^T \underline{\underline{a}} = \underline{\underline{1}}$

~~i.e.~~ $= \cancel{\sum_i (a^T)_{ji} a_{ik} = \delta_{jk}}$

i.e. orthogonal transformation

~~Note~~

i.e. orthogonal transformation

Note also $\det a = 1$ for rotation (consider infinitesimal $\begin{bmatrix} 1-\epsilon^2/2 & \epsilon & 0 \\ -\epsilon & 1-\epsilon^2/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$)

Now consider objects with distinct properties under rotations.

a) Scalars: do not change under rotation

ex. $$\Phi(x) = \frac{q}{4\pi\epsilon_0 |x|} = \frac{q}{4\pi\epsilon_0 |x'|} = \Phi(x')$$

or $$\frac{q}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|} = \frac{q}{4\pi\epsilon_0 |x_1' - x_2'|}$$

[We have implicitly assumed q is a scalar!]

b) Vectors. Consider first $\vec{x}$

transforms $x_i' = \sum a_{ij} x_j$

i.e. components depend on rotation (of frame)

vector function

$$V_i'(\vec{x}_i') = \sum_j a_{ij} V_j(x_j) \quad \text{defines a vector}$$

ex. $$\vec{\nabla}\Phi = \frac{1}{4\pi\epsilon_0}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\frac{q}{|x|} = \underbrace{\frac{-q}{|x|^3}}_{\text{scalar}} \underbrace{\vec{x}}_{\text{vector}} = \vec{E}$$

c) Tensor (rank 2 tensor) (mention rank n cartesian tensor)

defined by $$T_{ij}' = \sum_{kl} a_{ik} a_{jl} T_{kl} \equiv \overleftrightarrow{T}$$

Note $$(\vec{A}' \cdot \overleftrightarrow{T}')_j = \sum_i A_i' T_{ij}' = \sum_i \sum_{knm} a_{ik} a_{in} a_{jm} A_k T_{nm}$$

$$= \sum_{knm} \delta_{kn} a_{jm} A_k T_{nm} = \sum_m a_{jm} \left(\sum_n A_n T_{nm}\right) = \sum_m a_{jm} (\vec{A} \cdot \overleftrightarrow{T})_m$$

i.e. $\vec{A} \cdot \overleftrightarrow{T}$ transforms as a vector

Recap

Electrodynamics respects several space-time symmetries:

translational invariance
rotational invariance
spatial inversion (parity)
time reversal

To encode these symmetries in predictions, ~~construct~~ work in terms of dynamical variables with simple transformation properties.

<u>rotation</u>: <u>scalars</u> $S' = S$

<u>vectors</u>

$$V_i' = \sum a_{ij} V_j$$

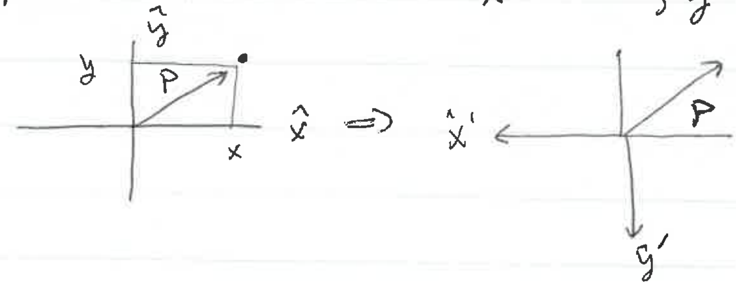
tensors

$$T_{ij}' = \sum a_{ik} a_{jl} T_{kl}$$

$$\sum_i a_{ik} a_{il} = \delta_{kl} \, , \quad \det a = 1$$

Next symmetry:

2) Spatial Inversion $\hat{x}' = -\hat{x}$, $\hat{y}' = -\hat{y}$, $\hat{z}' = -\hat{z}$



obviously $x' = -x$, $y' = -y$, $z' = -z$

or $x'_i = \sum_j a_{ij} x_j$ $\quad a_{ij} \to \underline{\underline{a}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$a a^T = 1$$
$$\det a = -1$$

Contrast pure rotation: $\det a = 1$

inversion $\neq$ rotation. Can combine the two into one a_{ij}

Terminology: if $\det \underline{\underline{a}} = 1$ "proper rotation"

if $\det \underline{\underline{a}} = -1$ "improper rotation" = proper rotation followed by space inversion

note $\sum_i x_i^2 = \sum_i x_i'^2$ in either case

a) Scalar: $\Phi(x') = \Phi(x)$; invariant under inversion

bi) Polar vectors.

$$\vec{A}' = (A'_1, A'_2, A'_3) = -(A_1, A_2, A_3) = \underline{\underline{-\vec{A}}}$$

bii) Axial vectors

$$\vec{A} = \vec{B} \times \vec{C} \qquad A_i = \sum_{jk} \epsilon_{ijk} B_j C_k$$

$$A'_i = \sum_{jk} \epsilon_{ijk} (-B_j)(-C_k) = \underline{\underline{+A_i}}$$

a ii) Pseudoscalar - transforms as a scalar under proper rotations, flips sign under spatial inversion

ex. $\psi = \vec{A} \cdot (\vec{B} \times \vec{C})$ A, B, C all polar vectors

$$\psi' = A' \cdot (B' \times C') = -\psi$$

3) Time Reversal. All the classical laws of physics ~~the laws of physics~~ are invariant to the sense of direction of time

time reversal is $dt \Rightarrow dt' = -dt$

$$\vec{F} = \frac{d\vec{p}}{dt} = -\vec{\nabla} u$$

$$\frac{d\vec{p}'}{dt'} = -\nabla' u' = -\nabla u$$ (does not know anything about time)

$$\Rightarrow \quad \vec{p}' = -\vec{p} \quad \text{under time reversal}$$

Makes sense, since $\vec{p} = m\vec{v} = m\frac{d\vec{x}}{dt}$

$$m\frac{d\vec{x}}{dt} \equiv -m\frac{d\vec{x}}{dt'} \quad \text{under time reversal}$$

(here assume $d\vec{x} = d\vec{x}'$ does not know about time)

Classifying dynamical variables by transformation properties under coordinate changes

rotation

inversion (parity)

	rotation	inversion
Scalar	unchanged $s \to s$	unchanged $s \to s$
Pseudoscalar	unchanged $p \to p$	$p \to -p$
Polar Vector	$V_i' = a_{ij} V_j$	$\vec{V}' = -\vec{V}$
Axial Vector	$A_i' = a_{ij} A_j$	$\vec{A}' = \vec{A}$
Tensor	$T_{ij} = a_{il} a_{jm} T_{lm}$	

(after top of p.4)

Now consider quantities encountered in electrodynamics

Kinematic quantity	Rotation, Spatial inversion	Time reversal
m	scalar (e)	even (E = even = no sign change)
$\vec{r}$	vector (o)	even
$\vec{v} = d\vec{r}/dt$	"	odd
$\vec{p} = m\vec{v}$	"	"
$\vec{L} = \vec{r}\times\vec{p}$	Axial vector (e)	"
$\vec{F} = d\vec{p}/dt$	vector (o)	even
$\vec{\tau} = \vec{r}\times\vec{F} = d\vec{L}/dt$	Axial vector (e)	"
$T = p^2/2m$	scalar (e)	"

Electrodynamic quantity			
e	scalar (e)	even	(assumed - experimental fact - charge is invariant under Lorentz transf; scalar under rotation - assumed for spatial inversion + time reversal)
Φ	"	"	
$\vec{E} = -\nabla\Phi$	vector (o)	"	
$\vec{J} = e\vec{v}$	"	odd	
$\vec{A} = \mu_0 \int \frac{\vec{J}\, d^3x'}{\lvert x - x'\rvert}$	"	"	
$\vec{B} = \vec{\nabla}\times\vec{A}$	axial vector (e)	"	
$\vec{H} = \frac{1}{\mu}\vec{B}$ (μ = scalar)	"	"	
$\vec{S} = \vec{E}\times\vec{B}$	vector (o)	"	
$\overleftrightarrow{T}$ (Maxwell stress tensor)	2nd rank tensor	even	

Check consistency

1) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$: contraction of vectors = scalar; TR even × even = even

2) $\nabla\cdot B = 0$ spatial inv. odd × even = odd but zero

3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ V × V = axial vector
TR e × e = o × o

4) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

The point of this exercise is that the laws of electrodynamics - the equations of electrodynamics should be consistent with respect to space-time symmetries - rotations, space inversion, time reversal. And we see they are.

Now take a step back. Suppose we know we have "physical" DoF's (E, B, ...) but we don't know the equations. How can we constrain ~~more~~ more complicated quantities? Symmetries!

eg. EM energy - U = scalar, T even so

$$U(E, B) = c_1 E^2 + c_2 B^2 + \cdots$$

no $E \cdot B$

but $(E \cdot B)^2$ etc

Suppose we imagine $\underline{P}$ for isotropic material is affected by E, B

$\underline{P}, E$ are polar vectors, T even

B is axial vector T odd.

~~Consider~~ Note $\dfrac{\partial^n}{\partial t^n}(\vec{E} \times \vec{B})$ is polar, T even if n odd

as is $\dfrac{\partial^n}{\partial t^n}(\vec{E} B^2)$ n even

or $\dfrac{\partial^n}{\partial t^n}(\vec{E} \cdot \vec{B})\vec{B}$ n even i.e. $$\underline{P} = \chi_0 \vec{E} + \chi_1 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \chi_2 \frac{\partial^2}{\partial t^2}(\vec{E} B^2) + \chi_3 (\vec{E}\cdot\vec{B})\vec{B} + \cdots$$

No story about χ's (yet!)

but - missing terms: No $\underline{P} \propto \chi_B \vec{B}$

All for now - why talk about it?

Immediate story - ~~and~~ EM waves in ~~med~~ complicated medium, often introduce phenomenological relations. Are they "really true" or is there a back story?