Magnetism

Historically, one could have introduced the magnetic field $\vec{B}$ from

- the torque on a permanent magnetic dipole

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

- the force on a charged particle

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

- the force on a segment of wire $d\vec{l}$ carrying a current I

$$dF = I\, d\vec{l} \times \vec{B}$$

Here current = charge going past, per unit time down the wire

More convenient for us is to define a current density
$\vec{J}$ = charge crossing unit area per unit time

$J = \frac{coul}{m^2\, sec}$

We assume conservation of charge, then

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$\frac{atoms}{m^2}$

$= e \times velocity$

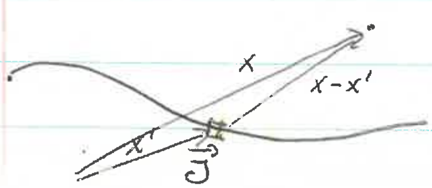
$= \frac{coul}{m^3}\frac{m}{sec}$

and magnetostatics is characterized by $\frac{\partial \rho}{\partial t} = 0$, so

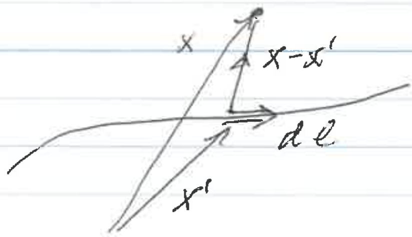
$$\vec{\nabla} \cdot \vec{J} = 0$$

Probably the best way to begin is with the Biot-Savart law

$$\vec{B}(x) = \frac{\mu_0}{4\pi} \int d^3x'\ \vec{J}(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|x - x'|^3}$$



or for wires $\quad d\vec{B}(x) = \frac{\mu_0 I}{4\pi} \oint d\vec{l} \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$



We can "derive" Maxwell's differential equations from the Biot-Savart equation. Write it as

$$\vec{B}(x) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(x')\, d^3x'}{|\vec{x}-\vec{x}'|}$$

then $\vec{\nabla} \cdot \vec{B} = 0$ immediately

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \vec{\nabla} \times \int \frac{J(x')}{|x-x'|} d^3x'$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \nabla^2 \vec{V}$$

so $$\vec{\nabla} \times B = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \vec{\nabla}\Big(\vec{J} \cdot \underbrace{\vec{\nabla} \frac{1}{|x-x'|}}_{-\nabla_{x'} \frac{1}{|x-x'|}}\Big) - \vec{J}(x') \underbrace{\nabla^2 \frac{1}{|x-x'|}}_{-4\pi \delta^3(x-x')} \right\}$$

$$\vec{\nabla} \times \vec{B} = -\frac{\mu_0}{4\pi} \vec{\nabla} \underbrace{\int d^3x' \vec{J}(x') \cdot \nabla_{x'} \frac{1}{|x-x'|}}_{\text{parts}} + \mu_0 J(x)$$

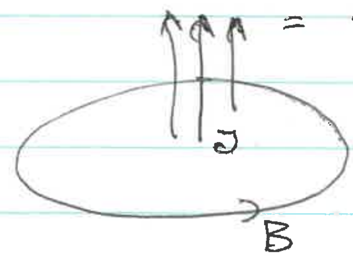
$$-\frac{\mu_0}{4\pi} \vec{\nabla} \int d^3x' \big(\underbrace{\nabla_{x'} \cdot J(x')}_{=0}\big) \frac{1}{|x-x'|} + \mu_0 \vec{J}(x)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(x)$$

Ampere's law is the integral form:

$$\int [\nabla \times B]\cdot n\, dA = \mu_0 \int J \cdot n\, dA = \int_S \vec{B}\cdot d\vec{l}$$

$$= \mu_0 I_{encl}$$



– Methods of solution for Magnetic Problems –

1) direct integration of Ampere's law – better have lots of symmetry
2) " " " Biot – Savart eqn "
3) vector potential : $\vec{B} = \vec{\nabla}\times\vec{A}$ because $\nabla\cdot B = 0$
4) "scalar potential" if $\vec{J} = 0$, $\nabla\times B = 0$, $B = -\nabla\Phi_M$
$\nabla\cdot B = 0 \Rightarrow \nabla^2 \Phi_M = 0$

3) $\vec{A}$ is not unique: any two A's differing by $\vec{\nabla} S(x)$ for any scalar function S, has the same $\vec{B}$ ($\nabla\times\nabla S = 0$). This is a deep statement (particularly when combined with the fact that in QM it is $\vec{A}$ that enters the Schrodinger Eqn) but for the time being we'll just use this gauge freedom to "choose a gauge" where $\vec{A}$ is simple. One such gauge is Coulomb gauge,

$$\vec{\nabla}\cdot\vec{A} = 0$$

for which

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \Rightarrow -\nabla^2\vec{A}$$

or

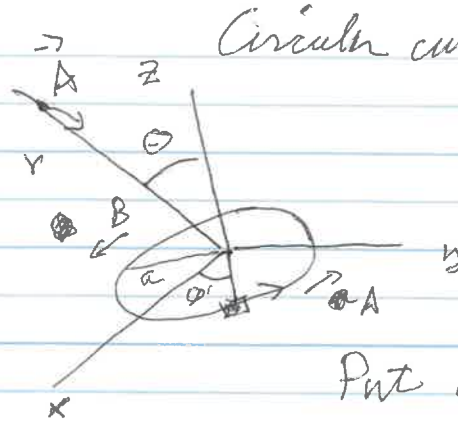
$$\vec{\nabla}^2\vec{A} = -\mu_0 \vec{J}$$

or in unbounded space

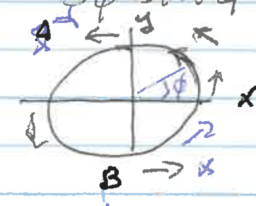
$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(x')}{|x-x'|}$$

only in Coulomb gauge

Circular current loop via $\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(x')}{|x-x'|}$



$\vec{J} = -J_\varphi \sin\varphi \, \hat{i} + J_\varphi \cos\varphi \, \hat{j}$ **



x top view

observer here

Put observer in x-z plane. Then the x-components ~~A & B~~ α + β cancel, $A = A_y$ only.

By symmetry, $A_y = A_\varphi$ everywhere

pick $\vec{x}' = a\,[\sin\theta'\cos\varphi', \; \sin\theta'\sin\varphi', \; \cos\theta']$
$= a\,[\cos\varphi', \; \sin\varphi', \; 0]$ at $\theta' = \frac{\pi}{2}$

$X = r\,[\sin\theta, \; 0, \; \cos\theta]$

keep only $\vec{J}_y$ — away from 90°

Jackson takes J = current/area $= \hat{\varphi}\, \frac{I\,\delta(r'-a)}{a}\, \delta(\cos\theta') \times \sin\theta'$ redundant

$\frac{\hat{\varphi}}{I}$ — $[J] = 1/\text{length}^2$ — to force J in x-y plane

$$A_\varphi = \frac{\mu_0 I}{4\pi a}\int r'^2 \, dr' \, \delta(r'-a) \int d\Omega' \sin\theta' \, \delta(\cos\theta') \times \int \frac{d\varphi' \cos\varphi'}{|x-x'|}$$

J_y **

(***)

$$|x-x'|^2 = r^2 + a^2 - 2ra(\sin\theta\cos\varphi')$$

$$A_\varphi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{\cos\varphi' \, d\varphi'}{[a^2 + r^2 - 2ar\sin\theta\cos\varphi']^{1/2}}$$

Exact answer is an elliptic integral - not so useful!

Go to far field: $\frac{1}{[a^2+r^2-2ar\sin\theta\cos\varphi']^{1/2}} = \frac{1}{r} + \frac{ar\sin\theta\cos\varphi'}{r^3} + \cdots$

$$\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\cos\varphi'\,d\varphi'\Big[\underbrace{\frac{1}{r}}_{\text{zero}} + \underbrace{\frac{a r \sin\theta\cos\varphi'}{r^3}}_{\frac{1}{2}\cdot 2\pi} + \cdots\Big]$$

$$A_\varphi = \frac{\mu_0 I a^2}{4\pi}\frac{1}{r^2}\cdot\left(\frac{1}{2}\cdot 2\pi\right)\sin\theta + \cdots = \frac{\mu_0}{4\pi}\frac{(\pi I a^2)}{r^2}\sin\theta$$

Look up the curl, $B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\varphi = \frac{\mu_0}{4\pi}(I\pi a^2)\frac{\cos\theta}{r^3}$

$B_\varphi = 0$, $B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r A_\varphi) = \frac{\mu_0}{4\pi}(I\pi a^2)\frac{\sin\theta}{r^3}$

This is a dipole!

Define Magnetic Dipole Moment for current loop.

$$m = I\cdot\pi a^2 = I \times \text{area}.$$

Partial wave version is also useful

$$\frac{1}{|x-x'|} = 4\pi\sum_{lm}\frac{Y_l^m(\theta',\varphi')^* Y_l^m(\theta,\varphi)}{2l+1}\frac{r_<^l}{r_>^{l+1}}$$

$\cos\varphi' \to \mathrm{Re}\, e^{i\varphi'}$ x-z plane

$$A_\varphi = \frac{\mu_0 I}{a}\,\mathrm{Re}\sum_{lm}\frac{Y_l^m(\theta,0)}{2l+1}\int r'^2 dr'\, d\Omega'\, \delta(\cos\theta') \times \delta(r'-a)\, e^{i\varphi'}\, Y_l^m(\theta',\varphi')^* \times \frac{r_<^l}{r_>^{l+1}}$$

$e^{i\varphi'} \to m=1$ only so $l \geq 1$

$$A_\varphi = 2\pi\mu_0 I a\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{Y_l^1(\theta,0)\,Y_l^1(\frac{\pi}{2},0)^*}{2l+1}$$

remove φ' dep

$Y_\ell^1(\frac{\pi}{2}, 0) = 0$ if ℓ is even, so we actually only have odd ℓ.

Again, far away, keep $\ell = 1$, $Y_1^1 = \sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$

$$A_\varphi = 2\pi \mu_0 I a \cdot \frac{a}{r^2} \cdot \frac{1}{3} \cdot \frac{3}{8\pi} \sin\theta = \frac{\mu_0}{4\pi}(\pi a^2 I)\frac{\sin\theta}{r^2}$$

The partial wave expansion has $m \neq 0$ because the current goes around in a closed loop.

We can generalize to an arbitrary $\bar{J}(x)$. This will be the magnetic dipole term in a multipole expansion (higher orders get quite messy ---). Write

$$\frac{1}{|x - x'|} = \frac{1}{|x|} + \frac{x \cdot x'}{|x|^3} + \cdots$$

$$A_i(x) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int J_i(x')\,d^3x' + \frac{x_j}{r^3}\int x'_j J_i(x')\,d^3x' + \cdots\right]$$

We must kill the first term. We know $\nabla \cdot J = 0$, J is local.

Consider identity $\int \left(f \vec{J} \cdot \vec{\nabla}' g + g \vec{J} \cdot \nabla' f\right) d^3x' = 0$

why - 2nd term $\quad - f \nabla' \cdot (g\vec{J})$ parts

$$- f(g \cancel{\nabla' \cdot J} + \vec{J} \cdot \nabla' g)$$

0

$$= \int (f J \cdot \nabla' g - f J \cdot \nabla' g) = 0$$

set $f=1,\ g=x_i'$ $\qquad 0=\int\left[\vec{J}\cdot(\vec{\nabla}' x_i') + x'\,\vec{J}\cdot\underbrace{\vec{\nabla}' 1}_{0}\right] d^3x'$

$0=\int J_i(x')\,d^3x'$ $\qquad$ no first term in A eqn, $\vec{\nabla}' x_i' = \hat{i}$

set $f=x_i'$ $\quad g=x_j'$ $\qquad$ in $f\,\vec{J}\cdot\vec{\nabla}' g + g\,\vec{J}\cdot\nabla' f$

$$\int (x_i' J_j + x_j' J_i)\,d^3x' = 0 \qquad (*)$$

so attack 2nd term in A eqn

$$\sum_j x_j \int x_j' J_i(x')\,d^3x'$$

$$= -\frac{1}{2}\sum_j x_j \int (x_i' J_j - x_j' J_i)\,d^3x' \qquad \text{use } *$$

$$= -\frac{1}{2}\sum_{jk} x_j\,\epsilon_{ijk}\int (\vec{x}'\times\vec{J})_k\,d^3x$$

$$= -\frac{1}{2}\left(\vec{X}\times\int \vec{x}'\times\vec{J}\,d^3x'\right)$$

$$= -\vec{X}\times\vec{m}$$

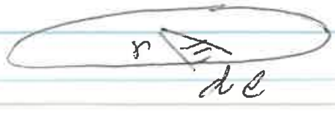
$$\boxed{\text{Magnetic moment } \vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J}(x')\,d^3x'}$$

so
$$A(r) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}$$
$$\vec{B}(r) = \frac{\mu_0}{4\pi}\left[\frac{3\vec{r}(\vec{m}\cdot\vec{r}) - \vec{m}r^2}{r^5}\right]$$
} dipole field

If current lives in a wire
$$\vec{m} = \frac{I}{2}\int \vec{r}\times d\vec{l}$$

If wire lives in plane $\quad \frac{1}{2}\vec{r}\times d\vec{l}$ = area of triangle

$|\vec{m}| = I\cdot$ area



$A = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$ $m = \frac{1}{2}\int d^3x\,(\vec{x} \times \vec{J}(x))$

Other simple magnetic moments are for orbiting particles

$$\vec{J} = \sum_i q_i \vec{V}_i \delta^3(\vec{x}-\vec{x}_i)$$

$$\vec{m} = \frac{1}{2}\sum_i q_i \vec{x}_i \times \vec{V}_i = \sum \frac{q_i}{2M_i} \vec{L}_i \qquad L = r \times p$$

$$\left(\vec{m} = \frac{q\vec{L}}{2Mc} \text{ in cgs } \right)$$

$$\text{spin: } \quad \vec{m} = \frac{gq\vec{S}}{2Mc}, \quad g = 2 \text{ for Dirac particle (electron)}$$

And the formula for B isn't quite right. There is an exact result,

$$\int_{\text{sphere of radius } R} \vec{B}(x)\,d^3x = \frac{\mu_0}{4\pi} \cdot \frac{8\pi}{3}\vec{m}$$

To make our approximate formula agree with this, write it as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3\vec{r}(\vec{m}\cdot\vec{r}) - \vec{m}r^2}{r^5} + \frac{8\pi}{3}\vec{m}\,\delta^3(\vec{r}) \right]$$

Complete dependence of B on an isolated dipole moment, treat dipole as <u>point</u>.

<u>Let's show the identity</u>

$$\int_{\text{sphere } r<R} B(x)\,d^3x = \int_{r<R} \nabla \times A\, d^3x = \int dA\,(\hat{n} \times \vec{A}) = R^2 \int_{\text{surface } r=R} d\Omega\, \hat{n} \times \vec{A}$$

$$= -R^2 \int d^3x'\, \vec{J}(x') \times \int_{r=R} \frac{d\Omega\, \hat{n}}{|\vec{x}-\vec{x}'|} \qquad n \times A = -(A \times n)$$

$$A = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{r}}{r^3}\,, \qquad B = \nabla\times A$$

$$A_k = \frac{\mu_0}{4\pi}\,\epsilon_{klm}\,\frac{m_l r_m}{r^3} \qquad B_i = \epsilon_{ijk}\,\partial_j A_k$$

$$B = \frac{\mu_0}{4\pi}\,\epsilon_{ijk}\,\epsilon_{klm}\,\partial_j\,\frac{m_l r_m}{r^3}$$

$$\epsilon_{klm} = \epsilon_{lmk} \quad \text{(cyclic)}$$

$$B_i = \frac{\mu_0}{4\pi}\,\underbrace{\epsilon_{ijk}\,\epsilon_{lmk}}\,\times \cdots$$

$$\frac{\mu_0}{4\pi}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\,\frac{m_l r_m}{r^3}$$

$$= \frac{\mu_0}{4\pi}\left(\overset{\mathrm{I}}{\delta_{il}\delta_{jm}} - \overset{\mathrm{II}}{\delta_{im}\delta_{jl}}\right)\left[m_l\left(\overset{a}{\frac{\delta_{jm}}{r^3}} - \overset{b}{\frac{3 r_j r_m}{r^5}}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\left[\overset{\mathrm{I}a}{\frac{m_l\,\delta_{il}}{r^3}\,\underbrace{\delta_{jm}\,\delta_{jm}}_{3}} \quad \overset{\mathrm{I}b}{-\frac{3 r^2}{r^5}\,m_i}\right.$$

$$\left.\overset{\mathrm{II}a}{-\frac{m_i}{r^3}} \quad \overset{\mathrm{II}b}{+\,3 r_i\left(\frac{m\cdot r}{r^5}\right)}\right]$$

$$B_i = \frac{\mu_0}{4\pi}\left[\frac{3 r_i (m\cdot r)}{r^5} - \frac{m_i}{r^3}\right] = \text{dipole}$$

Showing the identity

1) $\int (\vec{\nabla}\times\vec{A})\, d^3x = \int d_s S\ \hat{n}\times\vec{A}$

Do it for a cube: $(\nabla\times A)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$

$(n\times A)_z = n_x A_y - n_y A_x$

$$\int dx\,dy\,dz \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$= \int dy\,dz \left[A_y(x_{max}) - A_y(x_{min})\right]$$

$$- \int dx\,dz \left[A_x(y_{max}) - A_x(y_{min})\right]$$

$$= \int d_s S \left[n_x(\text{on } S) A_y - n_y(\text{on } S) A_x\right]$$

Then $\int_{\substack{\text{sphere}\\ r<R}} B(x)\, d^3x = \int_{r<R} (\nabla\times A)\, d^3x = \int d_s S\ \hat{n}\times\vec{A}$

$$= R^2 \int_{\text{surface } r=R} d\Omega\ \hat{n}\times\vec{A}$$

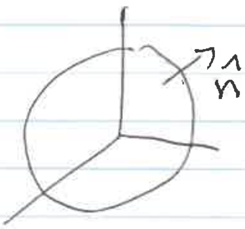
$\hat{n}\times\vec{A} = -\vec{A}\times\hat{n}$

$\vec{A}(x) = \int \frac{\vec{J}(x')\, d^3x'}{|x-x'|}$

~~$\int_{\text{sphere}} B(x)\, d^3x = -R^2 \int$~~

$$\int B(x)\, d^3x = -R^2 \int d^3x'\ \vec{J}(x') \times \int_{r=R} d\Omega\ \frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$

$$I = \int_{|x|=R} \frac{d\Omega\,\hat{n}}{|x-x'|} = \int_{|x|=R} d\Omega\,\hat{n}\,4\pi \sum \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{Y_\ell^{m*}(\Omega')Y_\ell^m(\Omega)}{2\ell+1}$$



$$\hat{n} = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$= (\hat{i} - i\hat{j})\sin\theta\, e^{i\varphi} + (\hat{i} + i\hat{j})\sin\theta\, e^{-i\varphi} + \hat{k}\cos\theta$$

Each term $\propto$ a $Y_{\ell=1}$! sum collapses! and $r_> = r$, $r_< = r'$

$$I = \frac{4\pi}{3}\frac{r'}{r^2}\cdot\hat{n}' = \frac{4\pi}{3}\frac{\vec{x}'}{R^2}$$

$$\int \vec{B}(x)\,d^3x = -R^2 \int d^3x' \,\frac{4\pi}{3R^2}\left[\vec{J}(x')\times\vec{x}'\right]$$

$$= \frac{8\pi}{3}\cdot\frac{1}{2}\int d^3x' \left(\vec{x}'\times\vec{J}(x')\right)$$

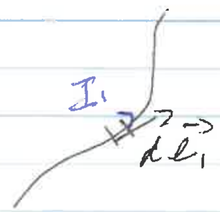
$$= \frac{8\pi}{3}\vec{m} \qquad (!)$$

δ fn for exact result.

Force and related quantities

Force on a current element due to $\vec{B}$ is (of course)

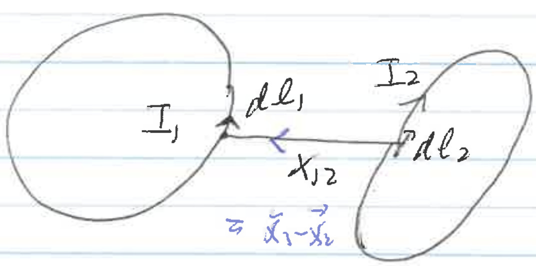
$$dF = I_1 \, d\vec{l}_1 \times \vec{B}$$



Suppose $\vec{B}$ comes from another current loop –

$$\vec{B}(r) = \frac{\mu_0}{4\pi} I_2 \int d\vec{l}_2 \times \frac{(\vec{r} - \vec{r}_2)}{|r - r_2|^3}$$

so the force of loop #2 on loop #1 is



$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint d\vec{l}_1 \times \left(\frac{d\vec{l}_2 \times \vec{x}_{12}}{|x_{12}|^3} \right)$$

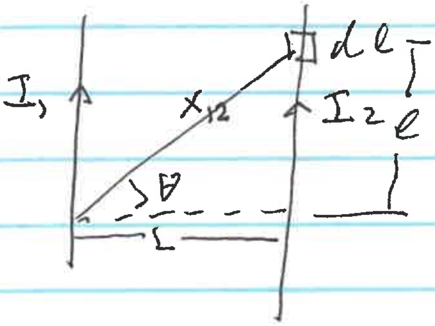
BAC-CAB rule $\quad d\vec{l}_1 \times (d\vec{l}_2 \times \vec{x}_{12}) = -(dl_1 \cdot dl_2)\,\vec{x}_{12} + d\vec{l}_2\,(d\vec{l}_1 \cdot \vec{x}_{12})$

but $\displaystyle \oint \frac{d\vec{l}_1 \cdot \vec{x}_{12}}{|x_{12}|^3} = \oint dl_1 \cdot \vec{\nabla} \frac{1}{x_{12}} = 0,$

2nd term goes,

$$F_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint dl_1 \cdot dl_2 \, \frac{\vec{x}_{12}}{|x_{12}|^3}$$

Easy example – two long straight wires



$$\frac{F_\perp}{\text{unit length}} = -\frac{\mu_0 I_1 I_2}{4\pi} \int \frac{dl}{x_{12}^2} \cdot \cos\theta$$

(project force transverse to line of current

$$\frac{L}{x_{12}} = \cos\theta \quad \Rightarrow \quad \frac{1}{x_{12}^2} = \frac{\cos^2\theta}{L^2}$$

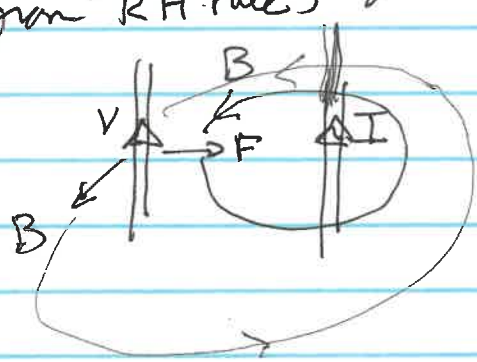
$$\frac{l}{L} = \tan\theta \qquad dl = \frac{L}{\cos^2\theta}\, d\theta$$

$$\frac{F}{\text{length}} = -\frac{\mu_0 I_1 I_2}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{L^2} \cdot \frac{L\, d\theta}{\cos^2\theta} \cdot \cos\theta$$

$$= -\frac{\mu_0}{4\pi} \frac{I_1 I_2}{L} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= -2 \frac{\mu_0}{4\pi} \frac{I_1 I_2}{L}$$

attractive if currents flow in same direction
(easy from RH rule)



$$\vec{v} \times \vec{B} = \vec{F}$$

A more general formula is

$$\vec{F} = \int \vec{J}(x) \times B(x)\, d^3x$$

$$\vec{\tau} = \int [\vec{X} \times (\vec{J} \times \vec{B})]\, d^3x$$

Now assume the current is localized to a small region near the origin, Taylor expand B

$$B_k(x) = B_k(0) + \vec{x} \cdot \vec{\nabla} B_k(0) + \cdots$$

$$F_i = \epsilon_{ijk} \Big[B_k(0) \underbrace{\int J_j(x)\, d^3x}_{\text{zero}} + \int J_j(x)\, \vec{x} \cdot \vec{\nabla} B_k(0)\, d^3x + \cdots \Big]$$

Do a parts $\int$ on 2nd term a la magnetic dipole derivation

$\vec{y} = \vec{\nabla} B_k(0)$ is outside the $\int$,

2nd term is $\quad \vec{y} \cdot \int \vec{x} J_j(x)\, d^3x = -\frac{1}{2}\left[\vec{y} \times \left[\int \vec{x} \times \vec{J}(x)\, d^3x\right]\right]_j$

$= (\vec{m} \times \vec{y})_j$ where $\vec{m}$ = magnetic moment

$$F_i = \epsilon_{ijk} (m \times \nabla)_j B_k(0) = \cancel{(\vec{m} \times \vec{\nabla}) \times \vec{B}}$$

$$\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B}(0)$$

$$= \vec{\nabla}(\vec{m} \cdot \vec{B}) - \vec{m}\,(\underbrace{\vec{\nabla} \cdot \vec{B}}_{0})$$

$$= -\vec{\nabla}[-\vec{m} \cdot \vec{B}]$$

$$\therefore \quad U = -\vec{m} \cdot \vec{B}(0)$$

also $\quad \vec{\tau} = \vec{m} \times \vec{B}(0)$

For 2 dipoles

$$U(m_1, m_2) = -\frac{\mu_0}{4\pi} \left[\frac{3(\vec{m}_1 \cdot \vec{r})(\vec{m}_2 \cdot \vec{r}) - \vec{m}_1 \cdot \vec{m}_2 r^2}{r^5} + \frac{8\pi}{3} \vec{m}_1 \cdot \vec{m}_2\, \delta^3(r) \right]$$

Macroscopic Equations, again

Lets begin with $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0 J_{total}$$

and introduce $\vec{B} = \mu_0(\vec{H}+\vec{M})$

with

$$\vec{\nabla}\times\vec{H} = \vec{J}_{macro}$$

so $\vec{\nabla}\times\vec{B} = \mu_0 J_{macro} + \mu_0 \vec{\nabla}\times\vec{M}$

i.e. the magnetization contributes to $\vec{B}$ like a current density, the "magnetization volume current density"

$$\vec{J}_M = \vec{\nabla}\times\vec{M}$$

We also need a "constitutive relation" between $\vec{B}$ & $\vec{H}$. For many materials it's linear

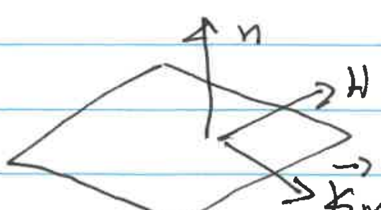
$\vec{B} = \mu\vec{H}$ $\quad \mu =$ permeability

$\mu/\mu_0 =$ rel. permeability

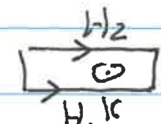
although in general, life gets complicated, especially for ferromagnetics

Boundary conditions for B & H at interfaces are easy!

$\nabla\cdot B = 0 \Rightarrow \vec{B}_1\cdot n = \vec{B}_2\cdot\hat{n}$



$\vec{\nabla}\times\vec{H} = J$



$$\hat{n}\times(\vec{H}_2-\vec{H}_1) = \vec{K}$$

$\vec{K}_{macro}$ - surface current density $\quad\left[\text{if } \vec{K}=0 \text{ or } H_{tan} \text{ continuous}\right]$

Boundary Value Problems in Magnetostatics

The generic situation is $\nabla \cdot B = 0$, $\nabla \times H = \vec{J}_{macro}$

plus given $\vec{B} = \vec{B}(\vec{H})$ or $\vec{B} = \mu_0(\vec{H} + \vec{M})$

with $\vec{B} \cdot \hat{n}$ continuous

$\hat{n} \times \Delta\vec{H}_{tan} = \vec{K}_{tan}$ at interfaces

2 text book cases:

a) linear permeability: $\vec{B} = \mu \vec{H}$

b) specified $\vec{M}$

If $\vec{B} = \mu \vec{H}$ (μ = const) can try using $\vec{A}$ ∋ $B = \nabla \times A$

$$\nabla \times H = \nabla \times \left(\frac{1}{\mu} \nabla \times A\right) = J$$

$$\nabla \times \nabla \times A = \mu \vec{J}$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu J$$

and in Coulomb gauge $\nabla \cdot A = 0$

$$\nabla^2 A = -\mu \vec{J} \quad \text{(as before)}$$

"Hard Ferromagnet": $\vec{M}$ given, $\vec{J} = 0$ has several possibilities, built on $\vec{\nabla} \times \vec{H} = 0$

1) $B = \mu_0(H + M) \rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times \vec{M} \equiv \mu_0 \vec{J}_M$

Volume magnetization current density

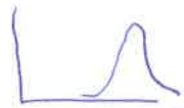
$$\vec{J}_M = \vec{\nabla} \times \vec{M}$$

With no boundaries

$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int d^3x \frac{(\vec{\nabla}' \times \vec{M}(x'))}{|x-x'|}$$

If M has a sharp boundary, you can

$\Rightarrow \vec{\nabla} \times M =$



$\frac{1}{\delta}\vec{M}\times\hat{n}$

separate off a

surface magnetic current density

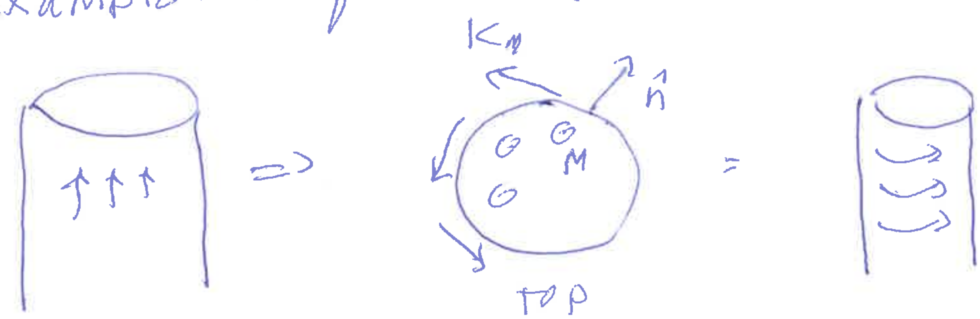
$$K_M(x') = \vec{M}(x') \times \hat{n}'$$

~~$A(x) = \frac{\mu_0}{4\pi}\int$~~ $\int d^3x \to \int \delta\, dA$ cancels $\frac{1}{\delta}$

$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\nabla' \times M(x')}{|x-x'|} + \frac{\mu_0}{4\pi}\int \frac{(\vec{M}(x') \times \hat{n})\, dA}{|x-x'|}$$

(careful - dont over count!)

example: uniform magnet, M constant inside



$\vec{K} = M\hat{\varphi}$

This is the familiar analog of the bar magnet and the solenoid: can do a direct attack

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{K}_M$$

bar magnet → solenoid → Biot-Savart or Ampere

Magnetic Scalar Potential

If $\vec{J}_{macro} = 0$ then $\vec{\nabla} \times \vec{H} = 0$

so we can write $\vec{H} = -\vec{\nabla} \Phi_M$

$\Phi_M \equiv$ "magnetic scalar potential"

ex. 1 $\vec{B} = \mu H$, μ = constant,

$$\nabla \cdot B = 0 \Rightarrow \nabla^2 \Phi_M = 0$$

ex. 2 $\vec{B} = \mu_0 \vec{H} + \vec{M}$, $\nabla \cdot B = 0$ is

$$\nabla \cdot H = -\nabla \cdot M$$

$$\nabla^2 \Phi_M = -\rho_M$$

$\rho_M = -\vec{\nabla} \cdot \vec{M}$ (magnetic pole density / volume !)

$$\Phi_M(x) = -\frac{1}{4\pi} \int d^3x' \, \frac{\nabla' \cdot M(x')}{|x - x'|}$$

example: suppose M is smooth, do a parts $\int$

$$\Phi_M = +\frac{1}{4\pi} \int \vec{M}(x') \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}'|}$$

then $\nabla' \frac{1}{|x-x'|} = -\nabla \frac{1}{|x-x'|}$

$$\Phi_M = -\frac{1}{4\pi} \vec{\nabla} \cdot \int \frac{\vec{M}(x')\, d^3x'}{|x - x'|}$$

Far away, $\frac{1}{|x-x'|} \sim \frac{1}{r}$

$$\Phi_M(\vec{x}) \sim -\frac{1}{4\pi}\left(\vec{\nabla}\frac{1}{r}\right)\cdot\int d^3x'\,\vec{M}(x')$$

$$= \frac{m\cdot r}{4\pi r^3}$$

This is the usual dipole formula for a scalar potential, with $\vec{m} = \int d^3x'\,\vec{M}(x')$

With a sharp boundary for M we can treat it as discontinuous. Then – just as for a dielectric we have a volume pole density

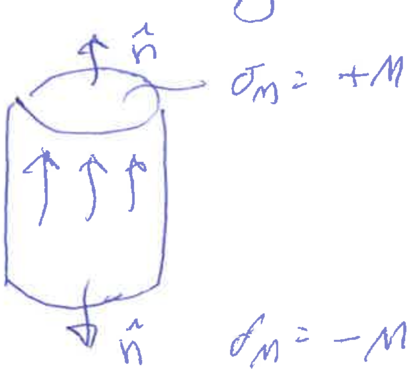
$$\rho_M = -\vec{\nabla}\cdot\vec{M}$$

and a surface pole density

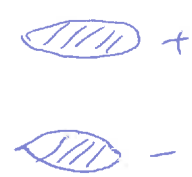
$$\sigma_M = \hat{n}\cdot\vec{M}$$

$$\Phi_M(x) = \frac{1}{4\pi}\int_V \frac{\rho_M(x')}{|x-x'|}\,d^3x' + \frac{1}{4\pi}\int \frac{\sigma_M(x')\,dA'}{|\vec{x}-\vec{x}'|}$$

Bar magnet again – with constant M



$\nabla^2\Phi =$



(Contrast w/ $\vec{K}_M = \vec{M}\times\hat{n}$!)

Magnetized Sphere

Uniformly magnetized sphere of radius "a"

$$\vec{M} = \hat{z} M_0 \qquad 0 < r < a$$

(not true for other shapes)

answer: it's a pure dipole. Many ways to solve

a) Φ_M ; use surface $\sigma_M = \hat{n} \cdot \vec{M} = M_0 \cos\theta$

$$\int \frac{\sigma_M(x')\, d^2x'}{4\pi |x-x'|} = \Phi_M(r,\theta) = \frac{M_0 a^2}{4\pi} \int d\Omega' \cos\theta' \frac{1}{|x-x'|}$$

$\cos\theta'$: Legendre expansion only has $l=1$!

$$\Phi_M = \frac{1}{3} M_0 a^2 \frac{r_<}{r_>^2} \cos\theta \qquad \frac{2}{3} \cdot \frac{2\pi}{4\pi}$$

Inside, $\Phi_M = \frac{1}{3} M_0 r \cos\theta = \frac{1}{3} M_0 z$

$$\vec{H} = -\nabla \Phi_M \Rightarrow \quad H_z = -\frac{1}{3} M_0$$

$$\vec{B} = \mu_0 [\vec{H} + \vec{M}] = +\frac{2}{3} \mu_0 M_0 \hat{z}$$

Outside, a pure dipole, $\vec{m} = \frac{4\pi}{3} a^3 \vec{M}$ = dipole moment

b) $\vec{J}_M = 0$, $\vec{K}_m = \vec{M} \times \hat{n} = M_0 \sin\theta\, \hat{e}_\varphi$

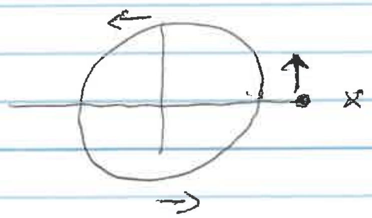
~~$\vec{A}(x) = \hat{\varphi} A_\varphi(x) = \frac{\mu_0 M_0 a^2}{4\pi} \int d\Omega' \sin\theta'$~~

Integrate current loops, $\vec{K}(\theta) = M_0 \sin\theta' [-\hat{x} \sin\varphi' + \hat{y} \cos\varphi']$

next page

$$= A_y \hat{y}$$

At $\varphi = 0$: $$\vec{A}(x) = A_\varphi(x)\hat{\varphi} = \frac{\mu_0 M_0 a^2}{4\pi}\int d\Omega' \frac{\sin\theta'\cos\varphi'}{|\vec{x}-\vec{x}'|}$$



(recall cancellation of x-currents)

$$\sin\theta'\cos\varphi' = -\sqrt{\frac{8\pi}{3}}\,\mathrm{Re}\,Y_1^1 \Rightarrow l=1,\ m=1$$ in harmonic expansion

$$A_\varphi = \underbrace{\frac{\mu_0}{4\pi}\left(\frac{4\pi}{3}\right)}_{\text{have to chase carefully!}} M_0 a^2 \underbrace{\frac{r_<}{r_>^2}\sin\theta}_{\text{dipole again -- just have to recognize it.}}$$

$$\cancel{\sqrt{\frac{8\pi}{3}} \times [Y_1^1 + Y_1^{-1}]\ \frac{1}{3}}$$

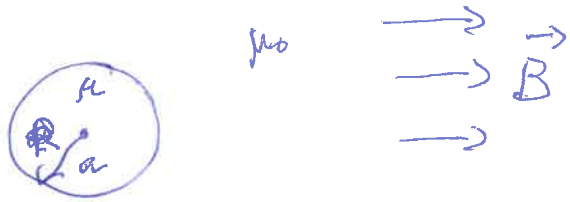
$$= \cancel{\sqrt{\frac{8\pi}{3}} \times \sqrt{\frac{3}{8\pi}} \times 2}$$

Permeable sphere in external B-field

We've seen the math before! In each region

$\vec{J}=0$, $\vec{\nabla}\times\vec{H}=0$

$\vec{B}\propto\vec{H}$ ($\vec{B}=\mu\vec{H}$ or $\mu_0\vec{H}$)



and $\nabla\cdot B=0\rightarrow$

$\nabla^2\Phi_M=0$, $\vec{H}=-\nabla\Phi_M$

so $$\Phi_M^{out}=\left[-\frac{B_0}{\mu_0}r+\frac{C}{r^2}\right]\cos\theta$$

$$\Phi_M^{in}=Dr\cos\theta$$

all other partial waves vanish - then

1) Match Φ_M: $D\cdot a=-\frac{B_0 a}{\mu_0}+\frac{C}{a^2}\rightarrow\frac{C}{a^3}=D+\frac{B_0}{\mu_0}$

2) match $\vec{B}\cdot\hat{n}$: $\mu\frac{\partial\Phi_M^{in}}{\partial r}=\mu_0\frac{\partial\Phi^{out}}{\partial r}$ at $r=a$.

$$\mu D=-B_0-\frac{2\mu_0 C}{a^3}$$

$$\mu D=-B_0-2\mu_0\left[D+\frac{B_0}{\mu_0}\right]$$

$$(\mu+2\mu_0)D=3B_0\Rightarrow D=\frac{3B_0}{\mu+2\mu_0}$$

so inside the sphere $\Phi_M=\frac{3B_0}{\mu+2\mu_0}r\cos\theta=\frac{3B_0}{\mu+2\mu_0}z$

$\vec{H}=\hat{z}\cdot\frac{3B_0}{\mu+2\mu_0}$ $=\frac{\vec{B}}{\mu}$ and $\vec{B}=\mu_0(\vec{H}+\vec{M})\Rightarrow\vec{M}=\frac{\vec{B}}{\mu_0}-\vec{H}$

$$\vec{M}=\hat{z}\cdot\frac{3B_0}{\mu+2\mu_0}\left[\frac{\mu}{\mu_0}-1\right]=\hat{z}\,\frac{3}{\mu_0}\left[\frac{\mu-\mu_0}{\mu+2\mu_0}\right]B_0$$

similar to $\vec{P}$ for dielectric sphere

Magnetic Shielding

Want Φ_M on the interior of the sphere

Yet again only $\ell = 1$.

$$\Phi_{I} = \frac{1}{\mu_0}\left[-B_0 r + \frac{E b^3}{r^2}\right]\cos\theta$$

$$\Phi_{II} = \frac{1}{\mu}\left[B r + \frac{C b^3}{r^2}\right]\cos\theta$$

$$\Phi_{III} = \frac{1}{\mu_0} D r \cos\theta$$

coefficients chosen with malice!

H_t or Φ	B_r ($B = \mu H$)
$\frac{1}{\mu_0}\left[-B_0 + E\right] = \frac{1}{\mu}(B + C)$	$B_0 + 2E = -B + 2C$
$\frac{1}{\mu}\left[B + C\frac{b^3}{a^3}\right] = \frac{1}{\mu_0} D$	$-B + 2C\frac{b^3}{a^3} = -D$

or

$$\begin{bmatrix} 1 & -\frac{\mu_0}{\mu} & -\frac{\mu_0}{\mu} & 0 \\ -2 & -1 & 2 & 0 \\ 0 & 1 & b^3/a^3 & -\frac{\mu}{\mu_0} \\ 0 & -1 & 2\frac{b^3}{a^3} & 1 \end{bmatrix} \begin{matrix} E \\ B \\ C \\ D \end{matrix} = \begin{pmatrix} B_0 \\ B_0 \\ 0 \\ 0 \end{pmatrix}$$

4 equations, 4 unknowns - awful! but interesting case is $\mu >> \mu_0$ - easy to get $\frac{\mu}{\mu_0} \sim 10^{3-6}$. Then easy to get an approximate solution:

1) top line: $E = B_0$. Then

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & r & -\hat{\mu} \\ -1 & 2r & 1 \end{bmatrix} \begin{matrix} B \\ C \\ D \end{matrix} = \begin{pmatrix} 3B_0 \\ 0 \\ 0 \end{pmatrix}$$

setting $\hat{\mu} = \frac{\mu}{\mu_0}$ $\quad r = \frac{b^3}{a^3}$

top line: $-B + 2C = 3B_0$

$$B = -3B_0 + 2C$$

2nd line: $-3B_0 + (r+2)C - \hat{\mu} D = 0$

3rd line $3B_0 + (2r-2)C + D = 0$

or $\left[(r+2)C - \hat{\mu} D = 3B_0\right] \frac{1}{r+2}$

$\left[-(2r-2)C - D = 3B_0\right] \frac{1}{2r-2}$

$$\left[\frac{-\mu}{r+2} - \frac{1}{2r-2}\right] D = 3B_0 \left(\frac{1}{r+2} + \frac{1}{2r-2}\right)$$

$$\frac{-\mu(2r-2) - (r+2)}{(r+2)(2r-2)} D = 3B_0 \frac{3r}{(r+2)(2r-2)}$$

~~keep $\hat{\mu}$~~

$$D = \frac{-9 \cdot B_0 r}{\cancel{} + 2\hat{\mu}(r-1) + (r+2)}$$

drop the $r+2$ if $\hat{\mu} = \frac{\mu}{\mu_0} >> 1$

$$r = \left(\frac{b}{a}\right)^3$$

If $b = a + \delta$ $\quad r = \left(1 + \frac{\delta}{a}\right)^3 = 1 + \frac{3\delta}{a}$

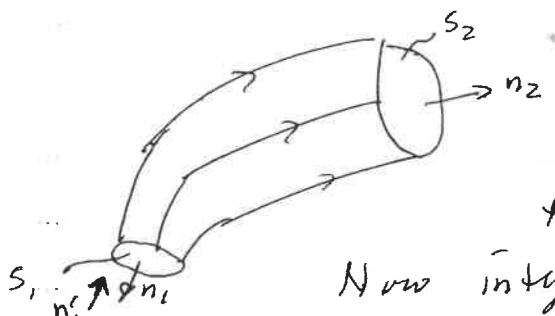
$$D \cong \frac{-9B_0}{6\left[\frac{\mu}{\mu_0}\right]\left(\frac{\delta}{a}\right)}$$

you win with big $\frac{\mu}{\mu_0}$

lose with thin shell!

There are some useful approximations for dealing with the fields from permanent magnets. Let's begin by imagining a region of space filled with magnetic field and construct "magnetic field lines" - lines whose direction shows the direction of $\vec{B}$ at a point. Now imagine a tube of flux, a volume bounded on its side by lines of B but not cut by them.

field lines - line tangent to mag field



The surface areas at the ends of the ~~tube~~ are S_1 and S_2.

Now integrate $\nabla \cdot B$ over the tube

$$\int_V \vec{\nabla} \cdot \vec{B}\, dV = 0 = \int_{S_2} B \cdot n\, dA - \int_{S_1} B \cdot n_1'\, dA$$

$$= \Phi(S_2) - \Phi(S_1)$$

Φ = magnetic flux - i.e. flux lines are conserved. (by ~~construction~~ construction)

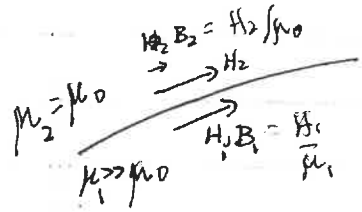
This is not true for H:

$$\int_{S_1} \vec{H} \cdot \hat{n}\, dA - \int_{S_2} \vec{H} \cdot \hat{n}'\, dA = \int \nabla \cdot H\, dV$$

$$= \int \rho_M\, dV$$

= total magnetic pole strength intercepted by the flux tube.

Now if we have a magnet or a permeable material with a large μ, B is (mostly) confined to the magnet, except at the poles. Why? because H_t is continuous at the side



so $H_1 = H_2$

but $B = \mu H$

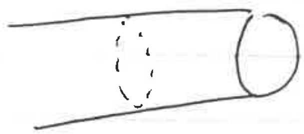
$$\frac{B_1}{\mu_1} = \frac{B_2}{\mu_2}$$

or if $\mu_1 >> \mu_2 \; (= \mu_0)$

$$B_2 = \frac{\mu_2}{\mu_1} B_1 \to 0$$

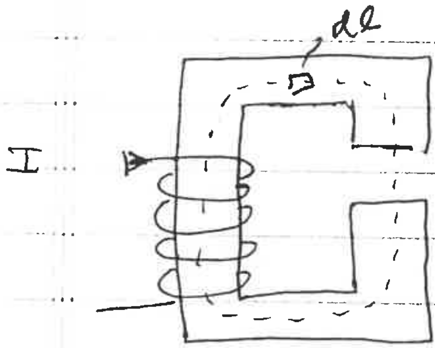
This says that B is inside, and since Φ is conserved,

$$\Phi = B \cdot A \quad \text{or}$$



$$B = \frac{\Phi}{A} \quad ; \; B \propto \frac{1}{\text{area}}$$

Now suppose the magnet is an electromagnet and is driven by N turns of current I



We can integrate

$$\oint H \cdot dl = NI$$

and $H(l) = \dfrac{\Phi}{\mu(l)\, A(l)}$

at a point l on the loop. This gives connection between Φ & I

$$\Phi \left[\frac{1}{\mu} \int_{\text{core}} \frac{dl}{A(l)} + \frac{1}{\mu_0} \int_{\text{gap}} \frac{dl}{A(l)} \right] = NI$$

Now you can see a problem: what is A in the gap? Make some reasonable assumption (linear interpolation?) and go on.

but

Since $\mu \gg \mu_0$ the core term is negligible, then

$$\Phi \int_{\text{gaps}} \frac{dl}{A(l)} = \mu_0 N I$$

NI gives Φ – and then $B = \Phi / A$.

This is why the gaps are small – bigger B!

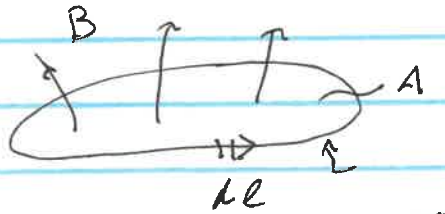
Magnetic Energy - slightly tricky

To create constant I's & B's requires transient time-varying fields. Faraday's law (Maxwell eqn #4) tells us that

$$\vec{\nabla} \times \vec{E} + \frac{d\vec{B}}{dt} = 0$$

i.e. ~~[illegible]~~ time varying B → circulating E

$$\int dA \left[\vec{\nabla} \times \vec{E} + \frac{dB}{dt} \right] \cdot \hat{n} = 0$$



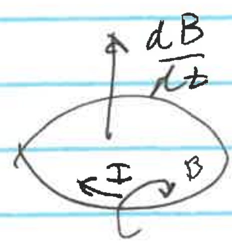
$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n}\, dA$$

voltage ≡ Electro Motive Force ($\Phi_E = -\int E \cdot d\ell$)

$$\mathcal{E} = -\frac{d}{dt} \Phi_B$$

= - time rate of change of magnetic flux.

Minus sign: Lenz' law: induced $\mathcal{E}$ (and current, if wire has resistance) opposes change in flux



Induced EMF causes source of current to do work, which goes into energy budget.

The discussion of magnetic energy starts with the Lorentz force law for a charged particle

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The time rate of change in energy of electron is

$$\frac{dU}{dt} = \vec{F}\cdot\vec{v} = q\vec{v}\cdot\vec{E} \quad ; \; [q\vec{v} = \text{current}]$$

~~$\vec{E}$ corresponds $\dot{\Phi}_B$. Sources do work to maintain the current; $\dot{\Phi}_B$ induces currents opposing.~~

Now we want to create a steady state current. There is an initial transient period as current rises from zero: $\dot{\Phi}_B \neq 0$, there is an induced EMF, sources do work to maintain the current ($\dot{\Phi}_B$ induces currents opposing change in flux).

Work done by source to maintain current =

$$= \frac{dW}{dt} = -I\,\mathcal{E} = -I\oint \vec{E}\cdot d\vec{l} = I\,\frac{d\Phi_B}{dt}$$

Lenz' law — sum over circuit

$\therefore \quad \delta W = I\,\delta\Phi_B$ = work done by external agent
= energy stored in system.

Now compute work expended in building up a steady state current & field distribution. Do this adiabatically, so $\vec{\nabla}\cdot\vec{J} = 0$.

Consider a little current loop shaped like a doughnut

$\sim S$ = surface area of loop

σ = cross section, $I = J\Delta\sigma$

$$\delta W = I\,\delta\Phi_B$$

$$\Delta(\delta W) = (J\Delta\sigma)\int \hat{n}\cdot\delta\vec{B}\,dS$$

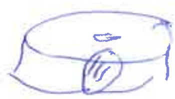
change in energy — change in area

$\vec{B} = \vec{\nabla}\times\vec{A}$ so $\Delta(\delta W) = J\Delta\sigma\int(\vec{\nabla}\times\delta\vec{A})\cdot n\,dS$

$$= J\Delta\sigma\oint\delta A\cdot dl$$

circumference of doughnut

But $\Delta\sigma\times\oint dl$ = volume of doughnut



$$\delta W = \int\delta\vec{A}\cdot\vec{J}\,d^3x$$

and if $\vec{A}\propto\vec{J}$ $\qquad W = \frac{1}{2}\int\vec{J}\cdot\vec{A}\,d^3x$

Now ~~[illegible]~~ consider macroscopic E&M, $\vec{J}$ a macro current: $\vec{\nabla}\times\vec{H} = \vec{J}$

$$\delta W = \int\delta\vec{A}\cdot(\vec{\nabla}\times\vec{H})\,d^3x$$

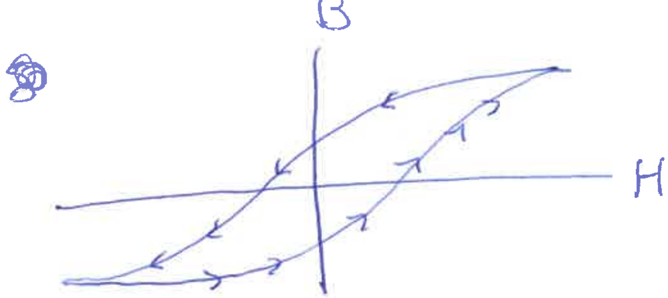
Vector identity $\vec{\nabla}\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\vec{\nabla}\times\vec{P}) - \vec{P}\cdot(\vec{\nabla}\times\vec{Q})$

$\vec{Q} = \delta\vec{A} \qquad \vec{P} = \vec{H}$

$$\delta W = \int\left[\vec{H}\cdot(\vec{\nabla}\times\delta\vec{A}) + \underbrace{\vec{\nabla}\cdot(\vec{H}\times\delta\vec{A})}_{\text{assume vanishes on surfaces}}\right] d^3x$$

$$\delta W = \int \vec{H}\cdot\delta\vec{B}\, d^3x$$ true in general

Example: hysteresis curve for real magnets – M depends on past history of material (magnetic domains ~~move~~ form and move around)



$\cancel{\Delta U = \oint \delta U = \int\oint H\cdot\delta B\, dV \quad = \oint B\cdot\delta H\, dV}$

closed path

$= \cancel{\text{area under}}$ $\qquad \oint \delta B \to \oint dB$

Cycle the magnet :

$$\Delta U = \int dV \oint H\cdot dB = \int dV \oint B\cdot dH$$

energy expended per unit volume = area of hysteresis loop

(Recall similar formula for dielectric
$$\delta W = \int \vec{E}\cdot\delta\vec{D}\, d^3x \;)$$

For linear materials $\vec{H}\cdot\delta\vec{B} = \frac{1}{2}\delta(\vec{H}\cdot\vec{B})$

$$W = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x \longrightarrow \underbrace{\left[\frac{1}{2\mu_0}\int B^2 d^3x + \frac{\epsilon_0}{2}\int E^2 d^3x\right]}_{\text{energies in EM field}}$$

Variation on this story: what is the change in energy when a permeable material ($\mu \neq \mu_0$) is placed in an external field, when currents are held constant?

before	after
μ_0	μ
B_0	B
H_0	H

Copy from the electrostatics problem
$$\vec{E}\cdot\delta\vec{D} \to \vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0$$
$$\vec{H}\cdot\delta\vec{B} \to \vec{H}\cdot\vec{B}_0 - \vec{B}\cdot\vec{H}_0$$

except - copy from case of fixed voltage / fixed sources - there is an extra minus sign

$$W = \frac{1}{2}\int_V d^3x\,[\vec{B}\cdot\vec{H}_0 - \vec{H}\cdot\vec{B}_0]$$
$$= \frac{1}{2}\int_V [\mu-\mu_0]\,\vec{H}\cdot\vec{H}_0\, d^3x$$

$$= \frac{1}{2}\iint\left[\frac{1}{\mu_0}-\frac{1}{\mu}\right]\vec{B}\cdot\vec{B}_0\, d^3x$$

$$B = \mu_0[H+M] \quad \therefore \quad \frac{B}{\mu_0} = H+M, \quad \frac{B}{\mu} = H$$

$$W = \frac{1}{2}\int d^3x\,[H+M-H]\cdot B_0 = \frac{1}{2}\int d^3x\, \vec{M}\cdot\vec{B}_0$$

(contrast w/ $-\frac{1}{2}\int \vec{P}\cdot\vec{E}\, d^3x$)

W_{mag} = total energy change including work done against EMF – like electrostatics at fixed voltage.

Inductance, self-inductance, mutual inductance

Self-inductance L: $\Phi_B = L\cdot I$

Φ_B = flux through circuit ~~...~~ carrying current I

Then $\quad \delta W = I\,\delta\Phi_B = I\,\delta(LI) = \delta\,\frac{1}{2}LI^2$

$$W = \frac{1}{2}LI^2 \quad \left(\text{like } \frac{1}{2}CV^2\right)$$

Mutual inductance: $(\Phi_B)_1 = M_{12}I_2$

$(\Phi_B)_1$ = flux through circuit 1 due to I_2 in circuit 2

Easy to show $M_{12} = M_{21}$

$$\delta W_{12} = I_1(\delta\Phi_B)_1 = I_1\,\delta(M_{12}I_2)\cdot M_{12}I_1I_2$$

$$W_{TOT} = \frac{1}{2}L_1I_1^2 + M_{12}I_1I_2 + \frac{1}{2}L_2^2I_2$$

Displacement current

$$\nabla \cdot \left(\vec{\nabla} \times \vec{H} = \vec{J} \right) \Rightarrow 0 = \vec{\nabla} \cdot \vec{J}$$

but if there is time variation, $\nabla \cdot J = -\frac{\partial \rho}{\partial t} \neq 0$

inconsistent w/ charge conservation

Maxwell's great fix-up

$$0 = \nabla \cdot J + \frac{\partial \rho}{\partial t} = \nabla \cdot J + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D}$$

$$= \nabla \cdot \left[J + \frac{\partial \vec{D}}{\partial t} \right]$$

"displacement current"

then rewrite
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Microscopic: $\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

~~Four eqns~~ Maxwell eqns: $\nabla \times E + \frac{\partial B}{\partial t} = 0 \qquad \nabla \cdot E = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot B = 0$

In free space $J = \rho = 0$ $\quad \nabla \times (\nabla \times B) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times E)$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\nabla^2 \vec{B} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

Waves: $\frac{1}{c^2} = \mu_0 \epsilon_0$

BTW solutions of wave eq are waves – in 1-d $\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) f(x \pm ct) = 0$

Now need machinery to deal efficiently with EM radiation