Macroscopic Electrodynamics

~~Introduction to Dielectrics~~

Think about the interaction of the electromagnetic field with matter. The quanta of the EM field are photons, which can be emitted or absorbed or scattered by charged particles such as electrons or protons (or atoms - made of charged particles). The passage of EM radiation through a medium like a solid is a complicated process involving the continual, absorption, scattering, and reradiation of photons.

Even when we think about this process classically, it remains quite complicated. The solid body contains charges and current loops which interact with the EM waves, absorbing, scattering, or radiating them.

are the ~~[illegible]~~ electric and magnetic permeabilities ϵ and μ, and it also involves a new set of fields, $\vec{P}$ the polarization and $\vec{M}$ the magnetization, which describe the space-time evolution of the electrodynamic properties of (coarse-grained) matter. In a formal sense this effective theory is different from the kind of electrodynamics we have heretofor considered - we can call it "macroscopic electrodynamics." We treat all our fields ($\vec{E}$, $\vec{B}$, $\vec{P}$, $\vec{M}$, and the two useful derived fields $\vec{D}$, $\vec{H}$) as if they were garden variety fields, defined at a point, continuous, differentiable ... (occasionally you have to be careful) and they encode effects of physics at short distance scales as seen by long distance scales. All done by averaging over short distance physics.

2nd ingredient - <u>accuracy</u> - How accurate is EFT? depends on how many terms you keep, how accurately we do averaging. Example follows..

"integrating out"

And yet - over distances large compared to atomic sizes, this complicated process can be parameterized by a few numbers (possibly frequency dependent) - a dielectric constant $\epsilon(\omega)$, permeability μ, conductivity σ ... We can short-circuit the hard problem of scattering etc by ~~measuring~~ computing these numbers, and expressing the propagation in terms of them. Or - if the calculation is too hard - we perform experiments to measure these auxilliary quantities once, then use these numbers to make predictions for other ~~propagation~~ processes.

There is a modern language to describe this procedure. We say that we are constructing an "effective field theory" which is not a fundamental theory of Nature valid on all length scales, instead it has a built-in cutoff, in energy ~~a momentum~~ scale Λ or ~~a~~ space cutoff $\sim \hbar c/\Lambda$, and it is not supposed to describe scales shorter than $1/\Lambda$. Often these effective theories ~~inherit~~ inherit properties (typically symmetries) ~~parameters~~ from a "more fundamental theory," but they also contain parameters which within the effective field theory are supposed to be fundamental, but which might represent ~~some~~ ~~a~~ the long distance/low momentum tail of the "real physics" (whatever it is.) In ~~the~~ classical electrodynamics, ~~and~~ ~~μ are~~ examples of these second class of parameters.

~~Typically these theories break down as $\langle k \rangle / \Lambda \to 1$ and one's experiments probe distances close to the cutoff~~

$p < \Lambda/c$

$E \ll \Lambda$

$p < \frac{\Lambda}{c}$

$\Delta x \, \Delta p \sim \hbar$

$\Delta x \sim \frac{\hbar c}{\Lambda}$

$\frac{GM^2}{R} = E = \frac{\hbar c}{\lambda}$

$\lambda = R$

$M^2 = \frac{\hbar c}{G}$

$(Mc^2)^2 = \frac{\hbar c^5}{G}$

Examples of "effective field theories"

the effective theory	Λ	the more fundamental theory
classical general relativity	10^{19} GeV (planck scale) $M_{Pl}^2 = \hbar c / G_N$	~~string~~ quantum gravity
~~the standard model~~ ~~Grand unified theories~~ ?	~~10^{16} 10^{19} GeV~~	"
the standard model	100 to 1000 GeV	?? ~~SUSY?~~ ~~GUTS?~~
Weak interactions in isolation / QED in isolation	~100 GeV or M_W	electroweak
pions and nucleons	~1 GeV	QCD
⋮	⋮	⋮
~~classical macroscopic electrodynamics~~	~~$\frac{1}{[\text{hundreds of Å}]}$~~	~~QED~~

In fact, classical macroscopic electrodynamics is effective on two levels:

classical electrodynamics	atomic scale ~Å or $m_e \sim \frac{1}{2}$ MeV *	QED
classical macroscopic electrodynamics	$\frac{1}{[\text{hundreds of Å}]}$	CED

* plus a long ~~story~~ incompletely story

From Microscopic to Macroscopic Electrodynamics

Following Jackson's notation, sec 6.6,

Maxwell's eqns are

$$\vec{\nabla}\cdot\vec{e} = \eta/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{b} = 0$$

$$\vec{\nabla}\times\vec{e} + \frac{\partial \vec{b}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{b} - \frac{1}{c^2}\frac{\partial \vec{e}}{\partial t} = \mu_0 \vec{j}$$

$b, e \equiv$ microscopic electric + magnetic fields

$\eta, \vec{j}$ " charge + current density

all fluctuating over distances 10^{-8} cm

times $10^{-13} - 10^{-17}$ sec

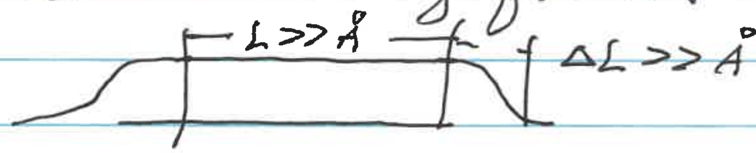
$10^{23\pm}$ atoms/cm^3

Keeping track of all these DoF's is overkill.

Instead, do a spatial average over $L \sim$ few hundred Å. (Want to deal with light in range of UV, so no finer ave.)

$$\langle F(\vec{x}, t)\rangle = \int d^3x' \, f(x') \, F(x-x', t)$$

$f(x)$ a smearing function



$$\frac{\partial}{\partial t}\langle F(x,t)\rangle = \left\langle \frac{\partial F}{\partial t}\right\rangle$$

$$\frac{\partial}{\partial x_i}\langle F(x,t)\rangle = \int d^3x' \, f(x') \frac{\partial F(x-x', t)}{\partial x_i}$$

$$= \left\langle \frac{\partial F}{\partial x_i}\right\rangle$$

Now define macroscopic fields (by smearing with f)

$$\vec{E}(x,t) = \langle \vec{e}(x,t) \rangle$$

$$\vec{B}(x,t) = \langle \vec{b}(x,t) \rangle$$

The homogeneous eqns are simple

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

inhomogeneous ones less so!

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} = \langle \eta(x,t) \rangle$$

$$\frac{1}{\mu_0}\nabla\times B - \frac{\partial E}{\partial t} = \langle \vec{J}(x,t) \rangle$$

Let's work on the two RHS's. We make a distinction between "bound charges" (which stay put, more or less in the material) and "free charges" which move around – and can be injected or removed from the material

$$\eta = \eta_{Free} + \eta_{bound}$$

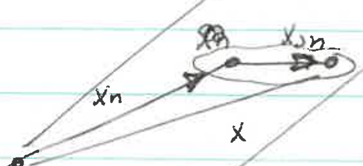
$$\eta_{free} = \sum q_i \delta(x - x_i)$$

$$\eta_{bound} = \sum_{n \text{ molecules}} \eta_n(x,t) = \sum_n \sum_j q_j \delta(x - x_j(n))$$

Let's average molecular charges

$$= \int d^3x' f(x') \eta_n(x - x', t)$$

$$\langle \eta_n(x,t) \rangle = \int d^3x' f(x') \sum_j q_j \delta(x - x' - x_{jn} - x_n)$$



interatomic spacing – could be Å or hundreds of Å...

Å (atoms in molecules)

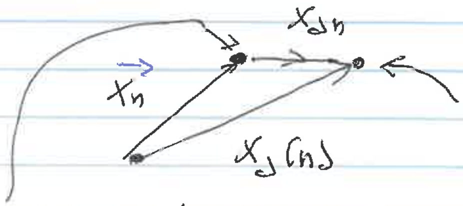
$$= \sum_{j(n)} q_j f(x - x_n - x_{jn})$$

$$\eta = \eta_{free} + \eta_{bound}$$

$$\eta_{free} = \sum_i q_i \, \delta(x - x_i)$$

$$\rho_{free} = \langle \eta_{free} \rangle = \langle \sum_j q_j \, \delta(x - x_j) \rangle$$

$$\eta_{bound} = \sum_{n \text{ molecules}} \eta_n(x,t) = \sum_n \sum_{j \in n \text{ or } j(n)} q_j \, \delta(x - x_j(n))$$



offset of jth ~~electrons~~ charges on n'th molecule

$$\vec{x}_j(n) = \vec{x}_n + \vec{x}_{jn}$$

nucleus of nth molecule

For nth molecule (no sum on n yet)

$$\langle \eta_n(x,t) \rangle = \int d^3x' \, f(x') \, \eta_n(x - x', t)$$

$$= \int d^3x' \sum_{j(n)} q_j \, \delta^3(x - x' - x_{jn} - x_n) \, f(x')$$

$$= \sum_{j(n)} q_j \, f(\underbrace{x - x_n}_{\text{nonzero over some big range}} - \overset{\text{Å}}{x_{jn}})$$

$$= \sum_{j(n)} q_j \Big[f(x - x_n) - \vec{x}_{jn} \cdot \vec{\nabla}_x f(x - x_n) + \frac{1}{2} \sum_{\mu\nu} x_{jn}^{\mu} x_{jn}^{\nu} \frac{\partial^2 f}{\partial x_\mu \partial x_\nu} + \cdots \Big]$$

Now sum over n:

The first term contributes

$$\langle \eta_{bound} \rangle = \langle \sum_n \underbrace{\sum_j q_j^{(n)}}_{\text{total charge } q_n} f(x - x_n) \rangle$$

$$= \langle \sum_n q_n \, \delta(x - x_n) \rangle$$

$$\equiv \rho_b \qquad \text{total bound charge}$$

The 2nd term

$$- \sum_n \sum_{j(n)} q_j \vec{x}_{jn} \cdot \vec{\nabla}_x f$$

exchange Σ & ∇

$$= - \nabla_x \cdot \sum_n \sum_{j(n)} \vec{x}_{jn} q_j \cdot f$$

$$= - \vec{\nabla}_x \cdot \langle \sum_n \vec{P}_n \rangle$$

$$\langle \eta_b \rangle \mathrel{+}= - \vec{\nabla} \cdot \vec{P}(x)$$

New field $\equiv$ polarization $\vec{P} \equiv$ volume average dipole moment

$$\vec{P}_n = \sum_{j(n)} \vec{x}_{jn} q_{jn} \equiv \text{molecular dipole moment}$$

The 3rd term involves the molecular quadrupole moment

$$(Q_n)_{\mu\nu} = \langle \sum_{j(n)} q_j \, x_{jn}^{\mu} x_{jn}^{\nu} \rangle$$

and $$\langle \eta_b \rangle = \cdots + \frac{\partial^2}{\partial x_\mu \partial x_\nu} Q_{\mu\nu}(x,t)$$

and we re-arrange the $\nabla \cdot E$ equation to write with the charge on one side

$$\nabla \cdot \Big[\epsilon_0 \vec{E} + \vec{P} - \sum_{\nu} \frac{\partial}{\partial x_\nu} Q_{j\nu} + \cdots\Big] = \rho_{Macro}$$

$$\boxed{\rho_{Macro} = \rho_f + \rho_{bound}}$$

generally negligible for atomic systems - or larger - which is why we usually dont see them.

so we have the "usual" equation (note it's not fundamental!)

$$\vec{\nabla} \cdot \vec{D} = \rho_{Macro}$$

where

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

ρ_f = "macroscopic" charge density (the "free" charge density of undergrad E&M)

Now for magnetism ...

$$\vec{J}(x,t) = \sum_j q_j \vec{v}\, \delta(\vec{x} - \vec{x}_n(t))$$

Molecular currents average into (for nth molecule)

$$\langle \vec{J}_n(\vec{x},t) \rangle = \sum_{j(n)} q_j (\vec{v}_{jn} + \vec{v}_n)\, f(\vec{x} - \vec{x}_n - \vec{x}_{jn})$$

A long story, similar to what we just did

~~I dont have the strength to grind this out ...~~

For component l

$$\langle J_l(\vec{x},t) \rangle = \langle J_l(x,t) \rangle + \frac{\partial}{\partial t}\left[D_l - \epsilon_0 E_l\right] + (\vec{\nabla} \times \vec{M})_l + \cdots$$

where

$$\vec{m}_n = \sum_{j(n)} \frac{q_j}{2} \left(\vec{x}_{jn} \times \vec{v}_{jn} \right)$$

$\equiv$ molecular magnetic moment in MKS

CGS $\quad \vec{m}_n = \sum_{j(n)} \frac{q_j}{2c} \left(\vec{x}_{jn} \times \vec{v}_{jn} \right)$

$$\vec{M}(x,t) = \left\langle \sum_n \vec{m}_n \, \delta(x - x_n) \right\rangle \equiv \text{magnetization}$$

so plug into

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J} + \frac{\partial}{\partial t} \left[\vec{D} - \epsilon_0 \vec{E} \right] - \vec{\nabla} \times \vec{M}$$

or

$$\vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \quad \text{or} \quad \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

$\vec{J} =$ macroscopic current density

- Derivation assumes medium isn't moving
- Says nothing about spin (must add by hand to $\vec{M}$)

Recap: Macroscopic Electrodynamics

$\nabla \cdot B = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_{macro}$ or ρ_{free}

$= \vec{\nabla} \cdot \vec{D}$

or bound

$\underline{P}$ = polarization = $\frac{\text{electric dipole moment}}{\text{unit volume}}$

$\vec{J}(x,t) = \sum_n \sum_{j(n)} q_j (\vec{v}_n + \vec{v}_{jn})$

For component ℓ

$\langle J_\ell(x,t) \rangle = J_M(x,t)_\ell + \frac{\partial}{\partial t} \underline{P}_\ell(x,t)$

$+ (\vec{\nabla} \times \vec{M})_\ell$

$\vec{M}(x,t)$ = Magnetization = $\frac{\text{magnetic dipole moment}}{\text{unit vol}}$

$\vec{M} = \langle \sum_n \vec{m}_n \delta(\vec{x} - \vec{x}_n) \rangle$

$\vec{m}_n = \sum_{j \in n} \frac{q_j}{2} (\vec{x}_{jn} \times \vec{v}_{jn})$ MKS

$= \sum_{j \in n} \frac{q_j}{2c} (\vec{x}_{jn} \times \vec{v}_{jn})$ CGS

says nothing about spin - put in by hand!

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t} = \vec{J}_M + \frac{\partial}{\partial t}(\vec{D}-\epsilon_0\vec{E})$$
$$+\vec{\nabla}\times\vec{M}$$

or

$$\vec{\nabla}\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J}_M$$

on board $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ or $\vec{B} = \mu_0(H+M)$

so 2 new fields $(\vec{P}, \vec{M})$

2 useful constructs $(\vec{D}, \vec{H})$

$e, \vec{J}$ macro scopic.

Back to electrostatics!

Dielectric Possibilities

$$\nabla \times \vec{E} = 0, \quad \nabla \cdot \vec{E} = \rho(\vec{r})/\epsilon_0$$

Recall that

$$\nabla \cdot \vec{D} = \rho_{free} \quad \text{but} \quad \vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Need relationships between $\vec{D}$ and $\vec{E}$ to proceed...

Common ones are...

a) Ferroelectricity: $\vec{P} \neq 0$ when $\vec{E} = 0$

Typically $\vec{P}$ specified external to problem

or

analog of "hard ferromagnet"

b) linear isotropic material $\vec{P} \propto \vec{E}$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

χ_e = electric susceptibility

E = total field (external + induced)

i.e. $\vec{P}$ vs $\vec{E}$ is linear, $\vec{P} \parallel \vec{E}$ } not true in general

then $\vec{D} = \epsilon_0 (1 + \chi_e) \vec{E} \equiv \epsilon \vec{E}$

$\dfrac{\epsilon}{\epsilon_0}$ = "dielectric constant"

"relative electric permittivity"

Note if entire medium filled with material of dielectric constant ϵ, independent of position

$$\nabla \cdot D = \rho_f \Rightarrow \vec{\nabla} \cdot \vec{E} = \rho_f/\epsilon$$

Same as free space, but $\rho_f \to \rho_f/(\epsilon/\epsilon_0)$ is renormalized by polarization of medium

$\Rightarrow q' = q/\left(\frac{\epsilon}{\epsilon_0}\right) < q$

c) Anisotropic media (crystals)

$D_i = \epsilon_{ij} E_j \qquad \epsilon_{ij}$ = dielectric tensor, $\epsilon_{ij} = \epsilon_{ji}$

d) "other"

~~Jackson 5.23 = the solution~~ -

Always

$$\nabla \cdot E = \frac{\rho_{tot}}{\epsilon_0}$$

$$\nabla \times E = 0 \leftarrow \vec{E} = -\vec{\nabla}\Phi$$

$$\vec{\nabla} \cdot \vec{D} = \rho_{macro}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

Linear isotropic material

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon_0 (1+\chi_e) \vec{E} = \epsilon \vec{E}$$

$$\therefore \vec{P} = (\epsilon - \epsilon_0) \vec{E}$$

Boundary Conditions and Static Boundary Value Problems with Dielectrics

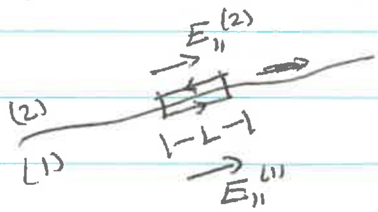
It's easiest to begin with

$$\vec{\nabla}\cdot\vec{D} = \rho_M \qquad (1)$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad (2)$$

Now we consider a boundary between two dielectric material, which possibly contains a free surface charge density σ

$\vec{E}$:

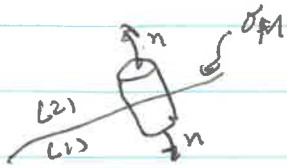


uses $\nabla\times E = 0$
consider a line $\int$ as shown:

$$\int E\cdot dl = 0 = L\cdot(E_\parallel^1 - E_\parallel^2)$$

$\therefore$ $E_\parallel^1 = E_\parallel^2$ = tangential component of E continuous

$\vec{D}$:



consider a Gaussian pillbox, use Gauss' law for $\vec{D}$

$$(\vec{D}_2 - \vec{D}_1)\cdot n = \sigma_M$$

(discontinuity in normal component of D given by macro surface charge density.)

Also: note from (2)

$\vec{E} = -\vec{\nabla}\Phi$. so (if $\vec{D} = \epsilon\vec{E}$) $\vec{\nabla}\cdot[\epsilon\vec{\nabla}\Phi] = -\rho_f$

and if ϵ is piecewise constant, $\epsilon\nabla^2\Phi = -\rho_f$

These 2 b.c. + Poisson's equation get us through most boundary value problems.

$$\nabla\cdot\left[\epsilon_0\vec{E}+\vec{P}+\cdots\right]=\rho_{macro}$$

$$\epsilon_0\nabla\cdot E=\rho_{macro}-\nabla\cdot P$$

suppose $\rho_{macro}=0$, $E=-\nabla\Phi$

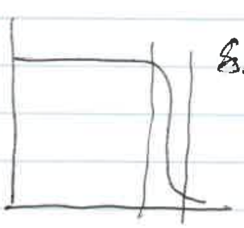
$$\nabla^2\Phi=\frac{\vec{\nabla}\cdot\vec{P}}{\epsilon_0}$$

then (formally)

$$\Phi(x)=-\int_{\text{vol of dielectric}}d^3x\ \frac{1}{4\pi\epsilon_0}\ \frac{\vec{\nabla}\cdot\vec{P}(x')}{|x-x'|}\qquad(*)$$

$\Rightarrow$ $-\vec{\nabla}\cdot\vec{P}$ = volume charge density in analogy with ρ

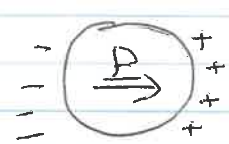
Also, this assumes $\nabla\cdot P$ is smooth. It's sometimes convenient to treat P as if it were a constant but discontinuous—almost

$P(x)$  δx

$$\vec{\nabla}\cdot\vec{P}\sim\frac{1}{\delta}\vec{P}(x)\cdot\hat{n}$$

$$\int d^3x\ \nabla\cdot P=\int\delta\ d^2x\ \frac{1}{\delta}P\cdot\hat{n}$$

i.e. $\sigma_{pol}=\vec{P}\cdot\hat{n}$

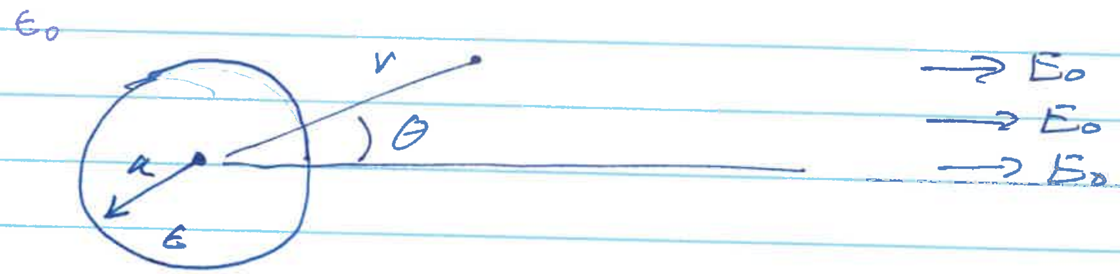


$$\Phi=\frac{1}{4\pi\epsilon_0}\int ds\ \frac{\sigma_{pol}(x')}{|x-x'|}$$

Be careful not to overcount with $(*)$

$\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$
$\vec{E} = -\nabla \Phi$

Electrostatics with linear dielectrics is only a small variation on what we've done before. Just solve for Φ within & outside the dielectric, then match b.c., continuity in normal D, tangential E. A classical problem: Dielectric sphere in external E-field



$$\Phi_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \left(B_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$\longrightarrow \left(-E_0 r + \frac{C_1}{r^2}\right)\cos\theta$$

(what happened to the other ℓ's?) and

$$\Phi_{in} = A r \cos\theta \qquad (\vec{E} = -\vec{\nabla}\Phi)$$

continuity in Φ: $\quad -E_0 a + \dfrac{C_1}{a^2} = A a \qquad (1)$

D·n continuous $\quad -\epsilon \dfrac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\epsilon_0 \dfrac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$

$$-\epsilon A = \epsilon_0 \left[E_0 + \frac{2C_1}{a^3}\right] \qquad (2)$$

or $\quad \left[A = -E_0 + \dfrac{C_1}{a^3}\right] \cdot 2\epsilon_0 \quad$ from (1)

$2\epsilon_0 A = -2\epsilon_0 E_0 + 2\dfrac{C_1}{a^3}$

$$(2\epsilon_0 + \epsilon) A = -3\epsilon_0 E$$

$$A = \frac{-3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 \qquad C_1 = a^3 E_0 \cdot \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)$$

$$\Phi_{in} = \frac{-3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 r \cos\theta$$

$$\Phi_{out} = -E_0\left[r - \left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\frac{a^3}{r^2}\right]\cos\theta$$

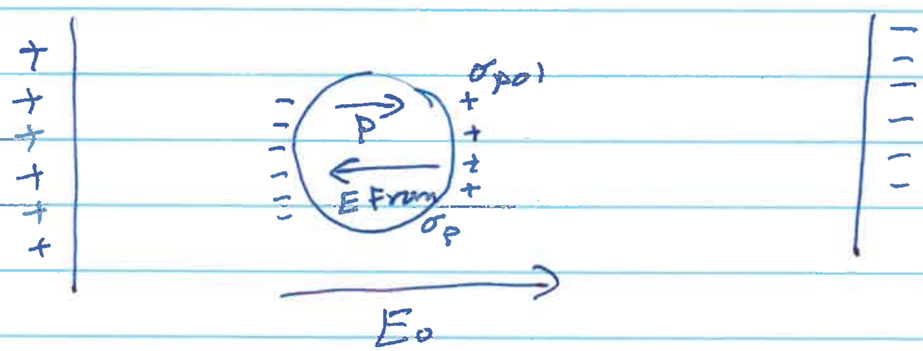
Notice inside $\vec{E}_{in} = \hat{z}\left(\frac{3}{2+\frac{\epsilon}{\epsilon_0}}\right)E_0$

constant and smaller than E_0

Polarization: $\underline{\vec{P}} = (\epsilon-\epsilon_0)\vec{E}_{in} = 3\epsilon_0\left[\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right]\vec{E}_0$

$$\sigma_{pol} = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = 3\epsilon_0\left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right)E_0\cos\theta$$

σ_{pol} generates an E-field which is opposedly oriented to E_0, reducing the internal E



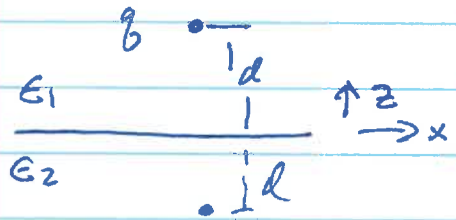
$$\vec{E}_{in} = \vec{E}_{out}\left[\frac{3}{2+\epsilon/\epsilon_0}\right] = -\nabla\Phi$$

$$3\epsilon_0\frac{\epsilon(1+\frac{\epsilon}{\epsilon_0}-1)}{\frac{\epsilon}{\epsilon_0}+2}$$

$$\vec{D}_{in} = \epsilon_0\vec{E}_{out}\left[\frac{3\epsilon/\epsilon_0}{2+\epsilon/\epsilon_0}\right] = \epsilon E_{in}$$

$$\vec{P}_{in} = \vec{D}_{in} - \epsilon_0\vec{E} = \epsilon_0\vec{E}_{out} \times \frac{3\left(\frac{\epsilon}{\epsilon_0}-1\right)}{\left(\frac{\epsilon}{\epsilon_0}+2\right)}$$

Images and dielectrics: point charge in medium 1, above medium 2 – both linear dielectrics



$$\epsilon_1 \nabla\cdot E = \rho \qquad z>0$$

$$\epsilon_2 \nabla\cdot E = 0 \qquad z<0$$

$$\nabla\times E = 0$$

and $D\cdot n$ continuous $\quad \epsilon_1 E_z^{(1)} = \epsilon_2 E_z^{(2)}$ at $z=0$

$E_{\parallel}$ " $\quad E_x^{(1)} = E_x^{(2)}$ at $z=0$

We guess that to solve for the potential in (1), an image at $z=-d$ will do the trick, but maybe the charge is different

$$\Phi^{(1)} = \frac{1}{4\pi\epsilon_1}\left\{\frac{q}{R_1} + \frac{q'}{R_2}\right\}$$

$$R_1^2 = \underbrace{x^2+y^2}_{\rho^2} + (d-z)^2 \qquad R_2^2 = \rho^2 + (z+d)^2$$

There are no charges in region (2), but there will be a surface polarization charge on the interface. Let's guess that Φ in region (2) can be given by a point charge q'' at the location of the real charge

$$\Phi^{(2)} = \frac{1}{4\pi\epsilon_2}\frac{q''}{R_1}$$

Need 2 identities

~~Derivatives~~: (a) $\left.\dfrac{\partial}{\partial z}\dfrac{1}{R_1}\right|_{z=0} = -\left.\dfrac{\partial}{\partial z}\dfrac{1}{R_2}\right|_{z=0} = \dfrac{d}{[\rho^2+d^2]^{\frac{3}{2}}}$

(b) $$\frac{\partial}{\partial \rho}\frac{1}{R_1}\bigg|_{z=0} = \frac{\partial}{\partial \rho}\frac{1}{R_2}\bigg|_{z=0} = \frac{-\rho}{[\rho^2+d^2]^{3/2}}$$

so $D\cdot n$: $\epsilon_1 E_z^{(1)} = \epsilon_2 E_z^{(2)}$

$$\frac{\epsilon_1}{4\pi\epsilon_1}\left\{\frac{\partial}{\partial z}\left[\frac{q}{R_1}+\frac{q'}{R_2}\right]\right\} = \frac{\epsilon_2}{4\pi\epsilon_2}\frac{\partial}{\partial z}\frac{q''}{R_1}$$

$\boxed{q - q' = q''}$ from (a)

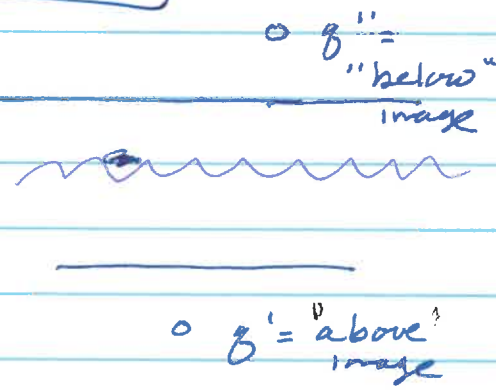
$$E_x^{(1)} = E_x^{(2)} \qquad \frac{1}{4\pi\epsilon_1}\frac{\partial}{\partial \rho}\left[\frac{q}{R_1}+\frac{q'}{R_2}\right] = \frac{1}{4\pi\epsilon_2}\frac{\partial}{\partial \rho}\frac{q''}{R_1}$$

$\boxed{\frac{1}{\epsilon_1}[q+q'] = \frac{1}{\epsilon_2}q'' \quad \text{from } b}$

$$\frac{q}{\epsilon_1}-\frac{q'}{\epsilon_1}=\frac{q''}{\epsilon_1}$$

$$\frac{q}{\epsilon_1}+\frac{q'}{\epsilon_1}=\frac{q''}{\epsilon_2}$$

$$q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\,q$$



$$q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right) q$$

note if $\epsilon_1 = \epsilon_2$, $q' = 0$, $q'' = q$

The ~~[illegible]~~ volume polarization charge density is

$\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ - it is zero (except at the point charge q) - but, at the interface, there's a surface polarization charge $\sigma_p = (\vec{P}_1 - \vec{P}_2)\cdot\hat{n}$ ---

$D = \epsilon E = \epsilon_0 E + P$ so $\vec{P} = (\epsilon - \epsilon_0)\vec{E} = \left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)\vec{D}$

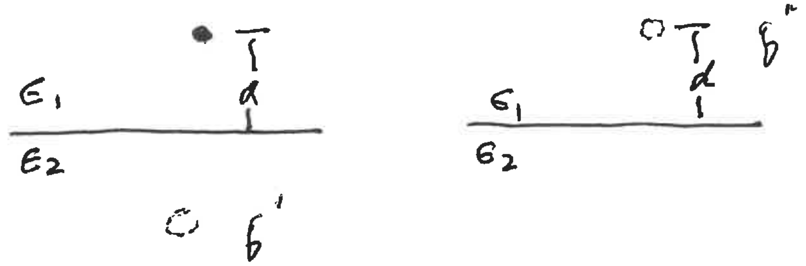
~~$\sigma_p = \left(\frac{\epsilon_1-\epsilon_0}{\epsilon_1}\right) - \frac{\epsilon_2-\epsilon_0}{}$~~

Recap:

$$\Phi^{(1)} = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{R_1} + \frac{q'}{R_2}\right]$$

$$\Phi^{(2)} = \frac{1}{4\pi\epsilon_2}\left[\frac{q''}{R_1}\right]$$

$$R_1^2 = \rho^2 + (z-d)^2 \qquad R_2^2 = \rho^2 + (z+d)^2$$



$\vec{D}\cdot\hat{n}$ continuous: $\epsilon_1 E_z^{(1)} = \epsilon_2 E_z^{(2)}$

$$-\epsilon_1 \left.\frac{\partial \Phi^{(1)}}{\partial z}\right)_{z=0} = -\epsilon_2 \left.\frac{\partial \Phi^{(2)}}{\partial z}\right)_{z=0}$$

$\vec{E}_{tan}$ continuous $\quad -\left.\frac{\partial \Phi^{(1)}}{\partial \rho}\right)_{z=0} = -\left.\frac{\partial \Phi^{(2)}}{\partial \rho}\right)_{z=0}$

2 eqns, 2 unknowns q' + q''

(algebra ---)

$$q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\, q \qquad q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right) q$$

check: if $\epsilon_1 = \epsilon_2$ $\;q' = 0$, $q'' = q$ (no interface)

There could be a volume polarization charge density:

$\rho_{pol} = -\vec{\nabla}\cdot\vec{P}$ but $\vec{P} = \frac{(\epsilon-\epsilon_0)}{\epsilon}\vec{D}$ and $\nabla\cdot D = 0$

except at q. There is a surface charge density

$\sigma_p = (\vec{P}_1 - \vec{P}_2)\cdot n$ due to discontinuity in ϵ.

Electrostatic Energy

If a point charge q_i is brought from infinity to a point x_i, against a potential Φ (which vanishes at infinity) the work done on it is

$$W_i = q_i \Phi(x_i)$$

Suppose Φ comes from $(n-1)$ point charges

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|x - x_j|}$$

The total work done to assemble all the charges is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{q_i q_j}{|x_i - x_j|}$$

(they all fight eachother.) This is a sum over pairs; sum over everybody, with a 1/2:

$$W = \frac{1}{2} \sum_{i} \sum_{\substack{j \\ \{i \neq j\}}} \frac{q_i q_j}{|x_i - x_j|} \frac{1}{4\pi\epsilon_0}$$

pass to continuum be careful about $i \neq j$ (neglect self energy)

$$W = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \int \frac{\rho(x)\rho(x')}{|x - x'|} d^3x\, d^3x'$$

or $\frac{1}{2} \int \rho(x) \Phi(x)\, d^3x$ ----> $\rho = -\epsilon_0 \nabla^2 \Phi$

or $\frac{-\epsilon_0}{2} \int \Phi(x) \nabla^2 \Phi(x)\, d^3x$

or $\frac{\epsilon_0}{2} \int |\vec{\nabla}\Phi|^2 d^3x \overset{\text{parts}}{=} \frac{\epsilon_0}{2} \int E^2 d^3x$

$\therefore$ energy density in electrostatic field $w = \frac{\epsilon_0}{2} E^2$

all very standard...

Energy and Dielectrics

Energy cost to assemble charges all brought in from infinity is

$$W = \frac{1}{2}\int \rho(x)\,\Phi(x)\,d^3x$$

counts all the charge (microscopic, macroscopic) and all the fields.

Typically we want to bring together free charges and pre-assembled dielectrics, and get a bill for them, not for the cost of the dielectric.

Start over

Make a change $\delta\rho$ in macroscopic charge density

$$\delta W = \int \delta\rho(x)\,\Phi(x)\,d^3x \qquad \text{(keep this)}$$

Φ is potential due to all charge already present, $\vec{E} = -\vec{\nabla}\Phi$

but $\vec{\nabla}\cdot\vec{D} = \rho$, $\delta\rho = \vec{\nabla}\cdot\delta\vec{D}$

$$\delta W = \int [\nabla\cdot(\delta D)]\,\Phi\, d^3x \xrightarrow[\text{parts}]{} \int \delta\vec{D}\cdot\vec{E}\,d^3x$$

2 − signs $(\vec{E} = -\vec{\nabla}\Phi)$

$$W = \int d^3x \int_0^D \delta\vec{D}\cdot\vec{E}$$

Formal total electrostatic energy, gotten by bringing D from 0 to its final value.

Linear medium only $\quad \vec{E}\cdot\delta\vec{D} = \frac{1}{2}\delta(\vec{E}\cdot\vec{D})$

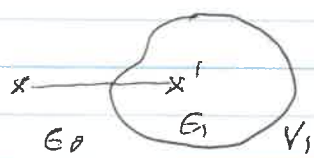
$$W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d^3x$$

A different problem: What is the change in energy when a linear dielectric is placed in an E field, whose sources (charges) are fixed?

Initially $\rho_0(x)$, $\epsilon_0(x)$, $\vec{E}_0(x)$, $\vec{D}_0(x) = \epsilon_0 \vec{E}_0(x)$

$$W_0 = \frac{1}{2}\int \vec{E}_0 \cdot \vec{D}_0 \, d^3x$$

Now drop in a dielectric with volume V_1



$\vec{E}_0 \to \vec{E}$, $\vec{D} = \epsilon \vec{E}$ $\qquad W_1 = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x$

$$W_1 - W_0 = \Delta W = \frac{1}{2}\int d^3x \, (\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)$$

$$= \underbrace{\frac{1}{2}\int [E\cdot D_0 - D\cdot E_0] \, d^3x}_{\text{zero outside } V_1} + \underbrace{\frac{1}{2}\int (E+E_0)\cdot(D-D_0)\, d^3x}_{\text{will be zero}}$$

drop for now $\equiv I_2$

$D = \epsilon_0 E$ outside, $D = \epsilon_1 E$ inside

$D_0 = \epsilon_0 E_0$ everywhere

so outside $E\cdot\epsilon_0 E_0 - \epsilon_0 E\cdot E$

$$\Delta W = \frac{1}{2}\int_{V_1} [\epsilon_0 - \epsilon_1]\, \vec{E}\cdot\vec{E}_0 \, d^3x \qquad (*)$$

~~In free space~~ $\qquad D = \epsilon_0 E + P = \epsilon_1 E$

$(\epsilon_1 - \epsilon_0)\vec{E} = \vec{P}$

$$W = -\frac{1}{2}\int_{V_1} \vec{P}\cdot\vec{E}_0 \, d^3x$$

energy density of linear dielectric placed in field E_0 whose sources are fixed

$$w = -\frac{1}{2}\vec{P}\cdot\vec{E}_0$$

like $W = \cdots - \vec{p}\cdot\vec{E}_0$ as for dipole $\vec{p}$ with $\frac{1}{2}$ because medium is polarizable $(\frac{1}{2}D\cdot E)$

$\therefore$ if $\epsilon_1 > \epsilon_0$, dielectric sucked into region of increasing field E_0

Again, this is for fixed charge, not fixed voltage (no batteries)

The term which is zero!

$$\nabla\times(E+E_0)=0 \Rightarrow E+E_0 = -\nabla\tilde{\Phi}$$

$$I_2 = -\frac{1}{2}\int \vec{\nabla}\tilde{\Phi}\cdot(D-D_0)\,d^3x$$

$$= \frac{1}{2}\int \tilde{\Phi}\,\vec{\nabla}\cdot(\vec{D}-\vec{D}_0)\,d^3x$$

but ρ_{free} is unaffected by the dielectric:

$$\nabla\cdot D = \nabla\cdot D_0$$

$$I_2 = 0$$

Fixed charge: W = energy. If you shift a coordinate $\vec{x} \to \vec{x} + \delta\vec{x}$, the "principle of virtual work" says

$$\delta W = -\vec{F} \cdot \hat{n}\, \delta x$$

$$\therefore \quad \vec{F} = \text{force acting on dielectric} = -\nabla W \Big|_{Q \text{ fixed}}$$

Case of fixed Voltage is different. As ~~[illegible]~~, as dielectric moves, wires carry charge in + out of battery to maintain potential. Energy is supplied by source.

Start with

$$W = \frac{1}{2}\int \rho \Phi\, d^3x \quad \text{counts all charges, as assembled.}$$

Change $\rho \to \rho + \delta\rho$, $\Phi \to \Phi + \delta\Phi$, imagine independent.

$$\delta W = \frac{1}{2}\int d^3x \left[\rho\, \delta\Phi + \Phi\, \delta\rho\right] \qquad (*)$$

point to $***$ on ED-1

If dielectric properties weren't changed this would be

$\delta W = \int \delta\rho \cdot \Phi\, d^3x$, so the 2 terms in $(*)$ would be equal. But if dielectric properties change, $\epsilon(x) \to \epsilon(x) + \delta\epsilon(x)$, polarization charge changes, $\rho = \rho_{free} + \rho_{pol}$, we only want to deal with ρ_{free}.

How to proceed:

1) Disconnect the battery, move the dielectric. Free Charges are fixed. We did this already

$$\delta W_1 = \frac{1}{2}\int \rho\, \delta\Phi_1\, d^3x = -\frac{1}{2}\int [\epsilon_1 - \epsilon_0]\, \vec{E} \cdot \vec{E}_0\, d^3x$$

in this process $\delta\Phi_1 \neq 0$ – potentials changed

2) Reconnect the battery. $\delta\Phi_2 = -\delta\Phi_1$ to restore all the potentials – but there is also a $\delta\rho_2$

$$\delta W_2 = \frac{1}{2}\int [\rho\,\delta\Phi_2 + \Phi\,\delta\rho_2]\,d^3x$$

but – dielectric properties are not changed – this is

$$\delta W = \int \delta\rho \cdot \Phi$$

so $\quad \rho\,\delta\Phi = \Phi\,\delta\rho$

And $\quad \delta\Phi = \delta\Phi_2 = -\delta\Phi_1$

$$\delta W_2 = -2\,\delta W_1$$

so $\quad \delta W = (1-2)\,\frac{1}{2}\int \rho\,\delta\Phi_1\,d^3x' = -\delta W_1$

the sign flips!

$$F = +\nabla W\Big)_{\text{fixed } V}$$

Horrible business – best "physical" description is in Griffiths – see p-188

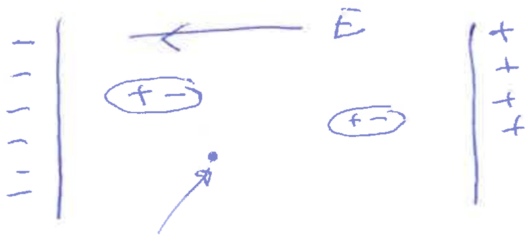
Move dielectric w/ $\epsilon > 1$ into region of greater $\vec{E}$ at fixed voltage, energy increases.

Macroscopic and microscopic quantities

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$\vec{P}$ = polarization
$\vec{p}$ = dipole moment

χ_e is a macroscopic quantity. How is it related to microscopic properties of the material? Interesting, somewhat complicated story if done right - see Zangwill, Ch 6. What follows is a story based on (semi) classical modelling of material as little dipoles



$\vec{E}$ Field at a point = $\vec{E}_{TOT}$ = macro field + local variation

$$\vec{E}_{tot} = \vec{E} + \vec{E}_i$$

$\vec{E}_i = \vec{E}_{near}$ (from a real calculation) $- \vec{E}_P$

(E_P = field from smooth background of polarization - this is a macroscopic quantity, already included in $\vec{E}$)

- $E_{TOT} = \vec{E} + \vec{E}_{near} - \vec{E}_P$.

Now recall the story about the dipole moment
(δ-fn in dipole field)

$$\int_{r<R} \vec{E}(x)\, d^3x = -\frac{\vec{p}}{3\epsilon_0}$$

$\vec{P}$ = dipole moment / unit vol $\Rightarrow \vec{p} = \frac{4\pi R^3}{3} \vec{P}$

$$\vec{E}_P = \text{average field} = \frac{1}{\left[\frac{4\pi R^3}{3}\right]} \int_{r<R} E\, d^3x = -\frac{\vec{P}}{3\epsilon_0}$$

so $\vec{E}_i = \frac{\vec{P}}{3\epsilon_0} + \vec{E}_{near}$ is correction to macro $\vec{E}$ in the dielectric

Size? $\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$

$$\chi_e = \frac{N\gamma_m}{1 - \frac{1}{3}N\gamma_m} \sim N\gamma_m \text{ if } N\gamma_m \text{ is small}$$

Back of the envelope in CGS...

gas: $N \sim \frac{6 \cdot 10^{23}}{22.4 \times 10^3 \text{cm}^3} \times \left(\frac{1 \text{ cm}}{10^8 \text{Å}}\right)^3 \sim 10^{-5} \frac{1}{\text{Å}^3}$

$$N\frac{e^2}{m\omega^2} = N \frac{e^2}{\hbar c} \frac{1}{mc^2} \frac{1}{(\hbar\omega)^2} (\hbar c)^3$$

$$= \frac{10^{-5}}{\text{Å}^3} \times \frac{1}{137} \times \frac{1}{5 \times 10^5 \text{eV}} \times \frac{1}{(\text{eV})^2} \times (2000 \text{ eV-Å})^3$$

$$= \frac{1}{137} \times 10^{-5-5+9} \sim 10^{-3}$$

$\frac{e^2}{\epsilon_0} = 4\pi\left(\frac{e^2}{4\pi\epsilon_0}\right)$

$\frac{\epsilon}{\epsilon_0} = 1.005$ for air

N is $\sim 10^5$ times greater for solids -

$$\frac{\epsilon}{\epsilon_0} \sim 1 + 1$$

$\cancel{E_{tot} = E_0 + E_{near} + \ldots}$

$\vec{E}_{near}$ involves a better calculation. Sometimes (believe it or not) it is zero! (amorphous materials, gases, Lorentz model - see Jackson). If that is so we

$$\vec{\underline{P}} = N \langle \vec{P}_{mol} \rangle$$

N = density
$\langle P_{mol} \rangle \equiv$ average molecular dipole moment.

Define molecular polarizability γ_m via

$$\langle \vec{P}_{mol} \rangle = \epsilon_0 \gamma_m \vec{E}_{TOT}$$

$$\vec{\underline{P}} = \epsilon_0 N \gamma_m E_{TOT} = \epsilon_0 N \gamma_m (\vec{E} + \vec{E}_L)$$

$$= \epsilon_0 N \gamma_m \left(\vec{E} + \frac{1}{3\epsilon_0} \underline{P} \right)$$

solve for $\underline{P}$:

$$\underline{P} = \epsilon_0 \left[\frac{N \gamma_m}{1 - \frac{1}{3} N \gamma_m} \right] \vec{E}$$

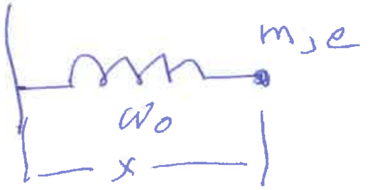
$$\chi_E = \frac{N \gamma_m}{1 - \frac{1}{3} N \gamma_m} \; ; \quad \frac{\epsilon}{\epsilon_0} = 1 + \chi_E$$

Inverse is called the Clausius Mossotti formula (1850, 1879)

$$\gamma_m = \frac{3}{N} \left[\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right] \qquad [\;] \text{ like } \sigma_p$$

Kind of odd - micro γ_m in terms of macro ϵ - this is from the ~~day~~ early days of atoms - but they knew macro facts, trying to get at micro facts

An oscillator model for χ_m $(\vec{p} = \epsilon_0 \chi_m \vec{E})$



electron on a spring!

$$e\vec{E} = m\omega_0^2 \vec{x}$$

$$\vec{p} = e\vec{x} = \frac{e^2}{m\omega_0^2}\vec{E}$$

$$\chi_m = \frac{1}{\epsilon_0}\frac{e^2}{m\omega_0^2} \longrightarrow \frac{1}{\epsilon_0}\sum_i \frac{e_i^2}{m_i(\omega_0^i)^2}$$

mks: drops for cgs $\left(\frac{1}{\epsilon_0} \leftrightarrow 4\pi\right)$

Or – at finite temperature

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 - eEz$$

$$\langle p_z = e\cdot z\rangle_{T=0} = \frac{\int d^3p \int d^3x \, e^{-H/kT}\cdot ez}{\int d^3p \int d^3x \, e^{-H/kT}}$$

$$= \frac{\int dz \cdot ez \cdot e^{-\frac{1}{2}m\omega_0^2 z^2 - eEz}}{\int dz \, e^{-\frac{1}{2}m\omega_0^2 z^2 - eEz}}$$

$$\langle\vec{p}\rangle = \frac{e^2}{m\omega_0^2}\vec{E} \text{ again}$$

We will use generalizations of this model again & again!