Cylindrical Geometry (ρ, ϕ, z)

Begin with 2-d problems: ρ, ϕ, no z dependence

$$0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} = 0$$

separation of variables:

$$\Phi = R(\rho) e^{im\varphi}, \quad m \text{ integer}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial R}{\partial \rho} - \frac{m^2}{\rho^2} R = 0$$

$m \neq 0$ solution: guess $R = \rho^b$

$$\frac{b^2 R}{\rho^2} - \frac{m^2}{\rho^2} R = 0 \Rightarrow b = \pm m$$

$m = 0 \quad R = E + F \ln \rho$

$\frac{dR}{d\rho} = \frac{F}{\rho}$, $\rho R' =$ constant so 2nd deriv $= 0$

$$\Phi = E + F \ln \rho + \sum_{m=1}^{\infty}\left(A_n \rho^n + \frac{B_n}{\rho^n}\right)$$

$$\times (c_n \cos n\varphi + d_n \sin n\varphi)$$

These are called "cylindrical harmonics" – they are easy to use, like Legendres. Sometimes you see the last term written as

$$F_n \cos n(\varphi - \varphi_n).$$

2-d Green's function

$$-4\pi\,\delta^2(\vec{x}-\vec{x}') = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]G \qquad (1)$$

use $$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \qquad (2)$$

$$G(x,x') = \frac{1}{2\pi}\sum_m e^{im(\varphi-\varphi')} g_m(\rho,\rho') \qquad (3)$$

so $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g_m}{\partial\rho} - \frac{m^2}{\rho^2}g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho') \qquad (4)$$

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon}\frac{\partial}{\partial\rho}\rho\frac{\partial g_m}{\partial\rho} = -4\pi \qquad (\text{cancel } \rho\text{'s})$$

$$\rho\frac{\partial g_m}{\partial\rho}\Big)_{\rho=\rho'-\epsilon}^{\rho=\rho'+\epsilon} = -4\pi \qquad (5)$$

I'll be a bit slower than Jackson, since there are peculiarities in the answer -- Imagine we have boundaries $a<\rho<b$ (take $a\to 0$, $b\to\infty$ at the end). Treat $m\neq 0$, $m=0$ separately

a) $m\neq 0$. Solution a ~~combination~~ product of ~~terms~~ homogeneous term satisfying b.c.'s

$$g_m = q\left[\rho_<^m - \frac{a^{2m}}{\rho_<^m}\right]\left[\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\right] \equiv g_< g_>$$

5) is $$q\left[\rho\frac{\partial g_>}{\partial\rho}g_< - \rho\frac{\partial g_<}{d\rho}g_>\right]_{\rho=\rho'} = -4\pi$$

$$-4\pi = \xi \Big[\rho\Big(\frac{-m}{\rho^{m+1}} - \frac{m\rho^{m-1}}{b^{2m}}\Big)\Big(\rho^m - \frac{a^{2m}}{\rho^m}\Big)$$

$$- \rho\Big(m\rho^{m-1} + \frac{m a^{2m}}{\rho^{m+1}}\Big)\Big(\frac{1}{\rho^m} - \frac{\rho^m}{b^{2m}}\Big)\Big]$$

$$= m\,\xi \Big[-1 - \Big(\frac{\rho'}{b}\Big)^{2m} + \Big(\frac{a}{\rho'}\Big)^{2m} + \Big(\frac{a}{b}\Big)^{2m}$$

$$-1 + \text{''} - \text{''} + \Big(\frac{a}{b}\Big)^{2m}\Big]$$

$$-4\pi = -2m\,\xi\Big(1 - \Big(\frac{a}{b}\Big)^{2m}\Big)$$

$$\boxed{g_m = \frac{2\pi}{m}\,\frac{1}{\Big[1 - \Big(\frac{a}{b}\Big)^{2m}\Big]}\Big(\rho_<^m - \frac{a^{2m}}{\rho_<^m}\Big)\Big(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\Big)}$$

This is well behaved as $a \to 0$, $b \to \infty$, just $\frac{2\pi}{m}\Big(\frac{\rho_<}{\rho_>}\Big)^m$.

no issues with limits for $m \neq 0$!

$m=0$ term: to satisfy b.c. of $g=0$ at $\varrho = a, b$

$$g_< = A \ln \frac{\varrho}{a} \qquad \varrho < \varrho' \qquad = A \ln \frac{\varrho_<}{a}$$

$$g_> = B \ln \frac{\varrho}{b} \qquad \varrho > \varrho' = B \ln \frac{\varrho_>}{a}$$

or $$g = C \ln \frac{\varrho_<}{a} \ln \frac{\varrho_>}{b}$$

$$C \left[\varrho \frac{\partial g_>}{\partial \varrho} \cdot g_< - \varrho \frac{d g_<}{d\varrho} g_> \right]_{\varrho=\varrho'} = -4\pi$$

since $\varrho \dfrac{\partial}{\partial \varrho} \ln \dfrac{\varrho}{c} = \varrho \cdot \dfrac{1}{\varrho} = 1$

$$C \left[\ln \frac{\varrho}{a} - \ln \frac{\varrho}{b} \right]_{\varrho=\varrho'} = C \ln \frac{b}{a} = -4\pi$$

$$C = \frac{4\pi \ln \frac{\varrho_>}{b} \ln \frac{\varrho_<}{a}}{\ln a/b}$$

$$G(\varrho, \phi; \varrho', \phi') = \frac{2 \ln \frac{\varrho_>}{b} \ln \frac{\varrho_<}{a}}{\ln \frac{a}{b}}$$

$$+ 2 \sum_{m=0}^{\infty} \frac{\cos(\varphi - \varphi')}{m} \frac{\left[\varrho_<^m - \frac{a^{2m}}{\varrho_<^m} \right] \left[\frac{1}{\varrho_>^m} - \frac{\varrho_>^m}{b^{2m}} \right]}{\left(1 - \left(\frac{a}{b}\right)^{2m} \right)}$$

Clearly vanishes at $\varrho = a$ and $\varrho = b$.

RPm

Limiting forms are more memorable:

$m \neq 0$: easy to take $a \to 0$, $b \to \infty$

$$G_{m\neq 0} = 2 \sum_{m=\phi}^{\infty} \frac{\cos(\varphi-\varphi')}{m} \left(\frac{\varrho_<}{\varrho_>}\right)^m$$

Monopole term trickier ... we can take $a \to 0$, hold b fixed

$$G_{m=0} = \left(2 \ln \frac{\varrho_>}{b}\right)\left(\frac{\ln \varrho_< - \ln a}{\ln a - \ln b}\right) \quad \text{exact}$$

$$\xrightarrow[a \to 0]{} -2 \ln \frac{\varrho_>}{b}$$

The $\frac{1}{b}$ is not important in $\frac{\partial G}{\partial n}$ on a circular surface. For a line charge $\lambda \delta(\varrho') \frac{1}{\varrho'}$ (recall spherical problem)

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int G(x, x')\, d\varrho' \lambda\, \delta(\varrho')$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left(-2 \ln \frac{\varrho}{b}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{\varrho}$$

This vanishes at $\varrho = b$, as b.c. demand.

Alternative formulas have hidden assumptions:

$$\ln \frac{1}{\varrho^2 + \varrho'^2 - 2\varrho\varrho' \cos(\phi-\phi')} = 2 \ln \frac{1}{\varrho_>} + \cdots \quad (3.152)$$

assumes the "1" in the numerator. The log blows up at both ends! Really have to talk about $\underline{\Phi(r_1) - \Phi(r_2)}$, not $\Phi(r_1)$!

$$G_{m\neq 0} = 2\sum \frac{\cos(m(\varphi-\varphi'))}{m}\left(\frac{\rho_<}{\rho_>}\right) \quad \text{for } 0<\rho<\infty$$

$$G_{m=0} = \frac{2\ln\frac{\rho_>}{b}\ln\frac{\rho_<}{a}}{\ln\frac{a}{b}} \quad \text{for } a<\rho<b$$

$$\rightarrow 2\ln\frac{b}{\rho} \quad \text{if } a>\rho>b$$

In $\frac{\partial G}{\partial n}$ the $\frac{1}{b}$ isn't important

Potential of line charge $= \lambda$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int\left[-2\ln\frac{\rho}{b}\right]\rho' d\rho' \frac{\lambda}{\rho'}\delta(\rho')$$

$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{\rho} \quad (\text{vanishes at } \rho=b)$$

~~You have to be careful in 2-d!~~

see 3.152 ... you have to be careful with the log

The full 3-d problem uses the same techniques, just some new functions (cylindrical Bessel fns)

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial \varphi^2} + \nabla_\varrho^2$$

and ~~...~~ the only new lesson is how to deal w/ new functions

A particular solution to $\nabla^2 \Phi = 0$ is

$$\Phi = \varphi_m(\varphi)\, Z(z)\, R_m(k\varrho)$$

$$\varphi_m(\varphi) = A_m \sin m\varphi + B_m \cos m\varphi \quad , \quad m = \text{integer}$$

$$Z(z) = C e^{kz} + D e^{-kz}$$

$R_m(k\varrho)$ a solution to

from $e^{\pm kz}$

$$\frac{1}{\varrho}\frac{\partial}{\partial \varrho}\varrho\frac{\partial R}{\partial \varrho} \leftarrow \left(\frac{d^2 R_m}{d\varrho^2} + \frac{1}{\varrho}\frac{dR_m}{d\varrho}\right) + \left(+k^2 - \frac{m^2}{\varrho^2}\right) R_m(k,\varrho) = 0 \qquad (1)$$

This is Bessel's eqn. The general Φ will involve superpositions of the particular solutions.

Before going on, construct a possible Greens fn. (for all φ, z values, possible ϱ surfaces)

Let's arbitrarily pick

$$\delta(\varphi - \varphi') = \frac{1}{2\pi}\sum_m e^{im(\varphi-\varphi')}$$

$$\delta(z - z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk \cos k(z-z')$$

$$G(x,x') = \frac{1}{2\pi^2}\int_0^{\infty} dk \sum_m e^{im(\varphi-\varphi')} \cos k(z-z')\, g_m(k,\varrho,\varrho')$$

note e^{ikz} not $e^{\pm kz}$ so radial eqn is

2) $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g}{\partial\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)g = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')$$

Solutions to (1) and homogeneous part of (2) are tabulated - Bessel functions.

1) $J_m(k\rho) \equiv$ Bessel fn

$N_m(k\rho) \equiv$ Neumann fn

$H_m^{(1,2)} = J_m(k\rho) \pm i N_m(k\rho)$
$\equiv$ Hankel function

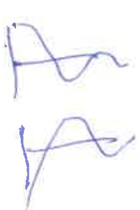
Think "sines, cosines, $e^{\pm ik\rho}$"

2) opposite sign of k^2 is like sinh, cosh

$I_\nu(x) = (i)^{-\nu} J_\nu(ix) \sim \sinh x$

$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \sim e^{-x}$

Everything you need to know about Bessel fns!



1) at small x, $J_\nu(x) \sim x^\nu$; $N_\nu(x) \sim \frac{1}{x^\nu}$, $N_0 \sim \log x$

2) at big x $J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$ (wave in 2d;

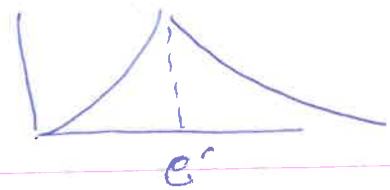
$N_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$ $|A|^2 \sim \frac{1}{r}$)

Zeros of $J_\nu(x_{\nu n}) = 0$, $N_\nu(x_{\nu n}) = 0$ tabulated

3) $I_\nu(x)$ well behaved at origin, bad at $x \to \infty \sim \frac{e^x}{\sqrt{2\pi x}}$

4) $K_\nu(x)$ diverges at origin, dies at $x \to \infty \sim \sqrt{\frac{\pi}{2x}}\exp(-x)$

Easiest first to discuss free space Green's fn.

Want $g_m(k,\rho,\rho') \sim$ 

so $g_m = A_m I_m(k\rho_<) K_m(k\rho_>)$

Matching condition $\left.\frac{dg_m}{d\rho}\right)_+ - \left.\frac{dg_m}{d\rho}\right)_- = -\frac{4\pi}{\rho'}$

With this g_m, this is $A_m\left(I_m K_m' - K_m I_m'\right)$

can show (this is called the Wronskian) - the product is $\propto \frac{1}{\rho}$ for any value of ρ. So evaluate in asymptotic region - for example $\rho \gg 1$.

Answer:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\int_0^\infty dk \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \cos k(z-z')$$

$$\times I_m(k\rho_<) K_m(k\rho_>)$$

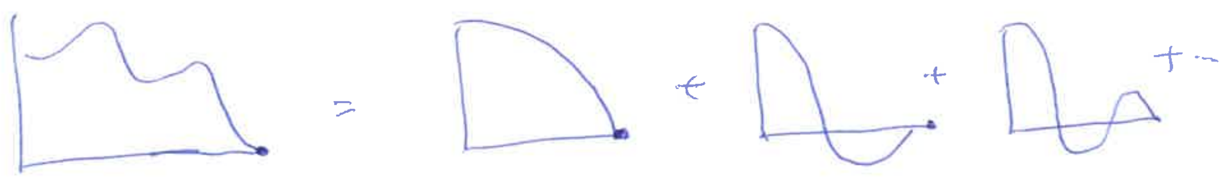
With boundaries, separation constants get quantized.
J's are orthogonal in too many ways to remember (get a book!)

One useful one = Fourier Bessel series

$$\int_0^a \rho \, d\rho \, J_\nu\left(x_{\nu m}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) = \underline{\delta_{nm}} \, \frac{a^2}{2} J_{\nu+1}(x_{\nu n})^2$$

where $x_{\nu m}$ = mth zero of $J_\nu(x)$

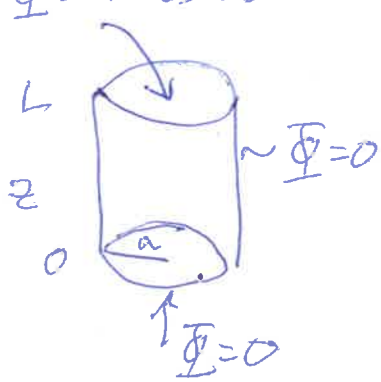
use: expand $f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$



Example: start w/ "standard form"

$$\Phi = \sum_{m,n} \left(A_{mn} \sin m\varphi + B_{mn} \cos m\varphi\right) \times \left[C_{mn} e^{k_{mn} z} + D_{mn} e^{-k_{mn} z}\right] \times \left[E_{mn} J_m(k_{mn}\rho) + F_{mn} N_m(k_{mn}\rho)\right]$$

$\Phi = V(\rho, \varphi)$



$\Longrightarrow$

- $\Phi(z=0) = 0 \Rightarrow \sinh k_{mn} z$
- Interior soln <u>can't</u> have N's
- $\Phi(\rho = a) = 0$ quantizes k_{mn}'s

so

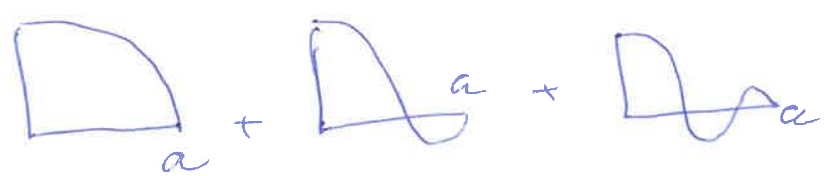
$$\Phi(\rho, z, \varphi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh k_{mn} z$$
$$\times \left[A_{mn} \sin m\varphi + B_{mn} \cos m\varphi \right]$$

and $J_m(k_{mn}a) = 0$. fixes k_{mn}

Orthogonality ~~gre~~ wrt φ, ρ gives

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \left[\frac{1}{\sinh k_{mn} L} \right] \left[\frac{2}{\pi a^2 J_{m+1}(k_{mn} a)} \right]$$
$$\times \int_0^{2\pi} d\varphi \int_0^a \rho \, d\rho \, V(\rho, \varphi) J_m(k_{mn}\rho) \cdot \begin{cases} \sin m\varphi \\ \cos m\varphi \end{cases}$$

Recall $J_0 \sim$



$J_1 \sim$

If the ρ-range is infinite the Fourier-Bessel series becomes an integral. For example, imagine specifying $V(\rho,\varphi)$ at $z=0$, and we want $\Phi \to 0$ as $z \to \infty$. The z dependence is only $\exp(-kz)$, if we want a well behaved solution at $z=0$, only J_m, no N_m

$$\Phi(\rho,\phi,z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(k\rho) e^{-kz} \cdot \left[A_m(k) \sin m\varphi + B_m(k) \cos m\varphi \right]$$

Invert wrt ϕ at $z=0$

$$(*) \qquad \frac{1}{\pi} \int_0^{2\pi} V(\rho,\phi) \cdot \begin{Bmatrix} \sin m\varphi \\ \cos m\varphi \end{Bmatrix} d\varphi = \int_0^{\infty} J_m(k\rho) \begin{Bmatrix} A_m(k) \\ B_m(k) \end{Bmatrix} dk$$

Another magic fact

$$\frac{1}{k} \delta(k-k') = \int_0^{\infty} \rho\, d\rho\, J_m(k\rho) J_m(k'\rho)$$

Multiply both sides of * by $\int_0^{\infty} \rho\, d\rho\, J_m(k'\rho)$

$$\begin{Bmatrix} A_m(k') \\ B_m(k') \end{Bmatrix} = \frac{k}{\pi} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\, V(\rho,\phi) \times J_m(k'\rho) \times \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

- Doing the integral could still be a challenge – but one of the reasons you take a class like this is so the answer doesn't look surprising!

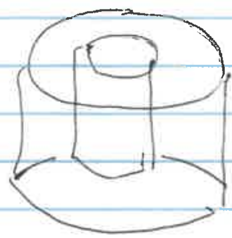
More odds & ends. Return to cylinder with interior shell

$$\Phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\{a_{nm} J_m(k_{mn}\rho) + b_{nm} N_m(k_{mn}\rho)\right\}$$

roots of Bessel's

$$\times \left\{\alpha_{nm}\, e^{k_{nm}z} + \beta_{nm}\, e^{-k_{nm}z}\right\}$$

$$\times \left\{A_{nm}\sin m\varphi + B_{nm}\cos m\varphi\right\}$$



Suppose there is no z dependence in b.c.'s. Then it must be that $k_{nm} \Rightarrow 0$.

That means that the argument of the Bessel function goes to zero. Call $x = k\rho$ for

$$\lim_{x\to 0} J_0(x) = \text{constant}$$

$$\lim_{x\to 0} N_0(x) = \log x = \log\rho + \text{constant}$$

$$\lim_{x\to 0} J_m(x) = x^m \to \rho^m$$ (absorb k into α) $\sum_n$ redundant

$$N_m(x) = \frac{1}{x^m} \to \frac{1}{\rho^{-m}}$$

$$\Phi(\rho,\varphi) = \underbrace{E + F\log\rho}_{m=0} + \sum_{m=1}^{\infty}\left\{a_m\rho^m + b_m\rho^{-m}\right\} \times \left[A_m\sin m\varphi + B_m\cos m\varphi\right]$$

the harmonic series.

Eigenfunction expansions for Green's functions

This is a widely-used technique - you often see it in QM. (I've used it.)

It uses 2 related equations

a) $\nabla_x^2 G(x,x') + [f(x)+\lambda]\, G(x,x') = -4\pi\delta^3(x-x')$

b) $\nabla_x^2 \psi(x) + [f(x)+\lambda]\,\psi(x) = 0$ (∇_x to be removed)

Assume identical boundary conditions for G & ψ.

Typically, b) only has good solutions for special values of λ, $\lambda=\lambda_n$ and $\psi(x)=\psi_n$

$$\nabla^2\psi_n + [f(x)+\lambda_n]\,\psi_n = 0$$

ψ_n's form a complete and orthonormal basis.

Use these to find G

$$G(x,x') = \sum_n a_n(x')\,\psi_n(x)$$

plus $\delta^3(x-x') = \sum_n \psi_n(x')^*\,\psi_n(x)$

$$(\nabla^2+f+\lambda)\,G = \sum_n a_n(x')\,[-\lambda_n+\lambda]\,\psi_n(x) = -4\pi\sum_n \psi_n(x')^*\,\psi_n(x)$$

or

$$a_n(x') = \frac{4\pi\,\psi_n(x')^*}{\lambda_n-\lambda}$$

$$G(x,x') = 4\pi\sum_n \frac{\psi_n(x')^*\,\psi_n(x)}{\lambda_n-\lambda}$$

If n is continuous, sum $\to$ integral, might be singular as $\lambda_n\to\lambda$ - we'll come back to this

Use for Poisson $\lambda=0$, $f(x)=0$. Solve $(\nabla^2+k^2)\,\psi_k(x)=0$

i.e. $\lambda_n \to -k^2$; $\psi_k(x) = \dfrac{e^{ik\cdot x}}{(2\pi)^{3/2}}$; $\int \psi_k(x)^*\,\psi_k(x')\,d^3k = \delta^3(x-x')$

Use for Poisson: $\lambda = 0$, $f(x) = 0$. What's a possible ψ? try $(\nabla^2 + k^2)\psi_k(x) = 0$ — "no b.c."

i.e. $\lambda_n = -k^2$ ①

② $\psi_k(x) = \dfrac{e^{ik\cdot x}}{(2\pi)^{3/2}}$ free particle in QM.

③ $\sum_n \to \int d^3k$

④ $\int \psi_{k'}(x)^* \psi_k(x)\, d^3x = \int \dfrac{d^3x}{(2\pi)^3} e^{i(k-k')\cdot x} = \delta^3(x-x')$

$$G(x) = \frac{1}{|x-x'|} = 4\pi \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot(x-x')}}{k^2}$$

If $\rho(x)$ has a simple Fourier transform this can be useful

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho(x') G(x,x')$$

$$= \frac{4\pi}{4\pi\epsilon_0} \int d^3x' \rho(x') \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2}$$

$$= \frac{4\pi}{(2\pi)^3 4\pi\epsilon_0} \int d^3k \frac{e^{ik\cdot x}}{k^2} \underbrace{\int d^3x' \rho(x') e^{-ik\cdot x'}}_{\rho(k)}$$

Another exaple, with b.c.: Dirichlet ~~Boo~~ Green's function in a rectangular box

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < c.$$

Try particle-in-a-box ~~wo~~ ψ's

$$\left[\nabla^2 + k^2_{n\ell m}\right] \psi_{n\ell m}(x) = 0$$

$$\psi_{n\ell m} = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\sqrt{\frac{2}{c}} \sin\frac{\ell\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}$$

$$k^2_{n\ell m} = \frac{\pi^2 \ell^2}{a^2} + \frac{\pi^2 m^2}{b^2} + \frac{\pi^2 n^2}{c^2}$$

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{n\ell m} \frac{\psi^*_{n\ell m}(x', y', z')\, \psi_{n\ell m}(x, y, z)}{k^2_{n\ell m}}$$

Quick + easy – might be expensive (3-fold sum)

The method we know might be cheaper...

$$\delta(x - x') = \frac{2}{a} \sum_{\ell} \sin\frac{\ell\pi x'}{a} \sin\frac{\ell\pi x}{a}$$

$$\delta(y - y') = \frac{2}{b} \sum_{m} \sin\frac{m\pi y'}{a} \sin\frac{m\pi y}{a}$$

$$G(\vec{x}, \vec{x}') = \frac{4}{ab} \sum_{\ell m} \sin\frac{\ell\pi x'}{a} \sin\frac{\ell\pi x}{a} \sin\frac{m\pi y'}{b} \sin\frac{m\pi y'}{b} \times g_{\ell m}(x, x')$$

two fold sum

Let's complete the calculation, for practice

$$\nabla^2 G = \frac{4}{ab}\sum_{lm}\left\{\left(\frac{d^2}{dz^2}+\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right) g_{lm}(z,z') \times \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\right\}$$
$$\times \sin\frac{l\pi x'}{a}\sin\frac{m\pi y'}{a}$$

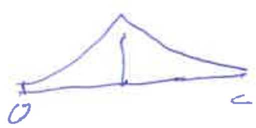
$$= \frac{4}{ab}\sum_{lm}\left\{\frac{d^2 g}{dz^2} - \left(\frac{l\pi}{a}\right)^2 g - \left(\frac{m\pi}{b}\right)^2 g\right\} \times$$
$$\times \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{l\pi x'}{a}\sin\frac{m\pi y'}{b}$$

$$= -4\pi\delta(z-z')\frac{4}{ab}\sum_{lm}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{a}$$
$$\times \sin\frac{l\pi x'}{a}\sin\frac{m\pi y'}{a}$$

<u>OR</u>

$$\left(\frac{d^2}{dz^2} - k_{lm}^2\right) g(z,z') = -4\pi\delta(z-z')$$
$$k_{lm}^2 = \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2.$$

Recall the game:



$g(z,z') = 0$ at $z = 0$, $z = c$

solution of DE is $g = c_1 e^{+k_{lm}z} + c_2 e^{-k_{lm}z}$

near $z=0$ $\quad g_< = \sinh kz_<$
$$g_> = \sinh k(c - z_>)$$

$$g_{lm}(z,z') = D\sinh k_{lm}z_< \sinh k_{lm}(c-z_>)$$

Summary of analytic techniques (ignoring physical insight)

$$\nabla^2 \Phi = 0 \qquad \nabla^2 G(x,x') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$$

1) $\nabla^2 \Phi = 0$: separation of variables in coord system where b.c. are simple

$$\Phi = F_1(x_1)\,F_2(x_2)\,F_3(x_3)$$

$$\frac{1}{F_1}\nabla^2 F_1 = c_1$$

$$\frac{1}{F_2}\nabla^2 F_2 = c_2$$

$$\frac{1}{F_3}\nabla^2 F_3 = c_3$$

$$c_1 + c_2 + c_3 = 0$$

general solution

$$F_1(x_1) = A_{n_1} f^1_{n_1}(x_1) + B_{n_1} g^1_{n_1}(x_1)$$

(2 independent soln's of 2nd order ODE

$$A_n r^n + \frac{B_n}{r^{n+1}}$$

Exception – Legendre polynomials – $A_n r^n \to \infty$ as $r \to \infty$, $B_n / r^{n+1} \to 0$ as $r \to 0$ – have to throw out one or the other. Legendre's eqn has $P_n(z)$, $Q_n(z)$, Q_n ~~stayed~~ has singularities inside $-1 < z < 1$

$$\Phi(x_1,x_2,x_3) = \sum_{n_1}\sum_{n_2}\left[A_{n_1} f^1_{n_1}(x_1) + B_{n_1} g^1_{n_1}(x_1)\right]$$

$$\times \left[A_{n_2} f^2_{n_2}(x_2) + B_{n_2} g^2_{n_2}(x_2)\right]$$

$$\times \left[A_{n_3} f^3_{n_3}(x_3) + B_{n_3} g^3_{n_3}(x_3)\right]$$

Typically some relation between n_1, n_2, n_3 from $\Sigma c_i = 0$

2) Green's functions

Again several possibilities for analytic sol'n
- exact solve (for us, just images)
- eigenfunction expansions (p. e_1-e_4)

- convert PDE to ODE using separation of variables & completeness

$$\delta(x_2 - x_2') = \sum_n F_n^*(x_2') F_n(x_2)$$

to get

$$\nabla_1^2 F_1 - [C_2 + C_3] F_1 = -4\pi \delta(x_1 - x_1')$$

solve by $F_1(x_1, x_1') = A \Psi_1(x_<) \Psi_2(x_>)$

Ψ_1 & Ψ_2 chosen to satisfy b.c., then Integrate across δ-fn

$$[\Psi_1 \Psi_2' - \Psi_1' \Psi_2]_{x_1 = x_1'} = -4\pi$$

Comment: $[\ ] \equiv$ "Wronskian" for

Sturm-Liouville problems

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + g(x)y = 0$$

can show Wronskian is a constant ~~$\propto \frac{1}{p(x)}$~~ so (2) becomes normalization

condition on $\Psi_1 \Psi_2$

Ex-1

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta^3(\vec{x} - \vec{x}')$$

rectangular box b.c.'s:

$0 < x < a$,

$0 < y < b$

$0 < z < c$

and pick $\delta(x-x') = \frac{2}{a} \sum_{l} \sin\frac{l\pi x}{a} \sin\frac{l\pi x'}{a}$

$\delta(y-y') = \frac{2}{b} \sum_{m} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$

$$G(x,x') = \frac{4}{ab} \sum_{lm} \sin\frac{l\pi x}{a} \sin\frac{l\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$$

$$\times\, g_{lm}(z-z')$$

(note ~~many~~ at least 3 slightly different looking formulas for $G(\vec{x}, \vec{x}')$!)

$$\nabla^2 G = -4\pi \delta^3(x-x')$$

$$\nabla^2 G = \frac{4}{ab} \sum_{lm} \left\{ \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) g_{lm}(z,z') \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \right\}$$

$$\times \sin\frac{l\pi x'}{a} \sin\frac{m\pi y'}{b}$$

$$= -4\pi \delta(z-z') \sum_{lm} \left\{ \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{l\pi x'}{a} \sin\frac{m\pi y'}{b} \right\}$$

~~orthogonalize with~~

$$= \frac{4}{ab} \sum_{lm} \left\{ \left(\frac{d^2}{dz^2} - \left(\frac{l\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 \right) g_{lm}(z,z') \right\}$$

$$\times \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{a} \sin\frac{l\pi x'}{a} \sin\frac{m\pi y'}{b}$$

orthogonalize wrt x & y --

$$\left[\frac{d^2}{dz^2} \underbrace{- \left(\frac{\ell\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}_{\text{call this } k_{\ell m}^2}\right] g_{\ell m}(z,z') = -4\pi\,\delta(z-z')$$

$$\left(\frac{d^2}{dz^2} - k_{\ell m}^2\right) g = -4\pi\,\delta(z-z')$$

Recall the game: $g(z,z') = 0$ at $z=0,\ z=c$

Solution of DE is $g = c_1 e^{+k_{\ell m} z} + c_2 e^{-k_{\ell m} z}$

Near $z=0$ $\quad z = z_<$ & $g_< = \sinh k z_<$

Near $z=c$ $\quad z = z_>$ $\quad g_> = \sinh k(c - z_>)$

~~From~~ $\quad g(z,z') = g(z',z)$ and

$$g_<(z',z') = g_>(z',z')$$

$\Rightarrow$ $g_{\ell m}(z,z') = D \sinh k_{\ell m} z \, \sinh k_{\ell m}(c - z_>)$

$$g = D \sinh(k z_<) \sinh k(c - z_>)$$

$$\int \frac{d^2 g}{dz^2}\, dz = \left.\frac{dg}{dz}\right)_{z'+\epsilon} - \left.\frac{dg}{dz}\right)_{z'-\epsilon} = -4\pi \int \delta(z-z')\, dz$$

$$D\Big[-k \cosh k(c-z) \sinh kz - k \cosh kz \sinh k(c-z) \Big] = -4\pi$$

$$-kD\Big[(\cosh kc \cosh kz' - \sinh kc \sinh kz') \sinh kz' + (\sinh kc \cosh kz' - \cosh kc \sinh kz') \cosh kz' \Big]$$

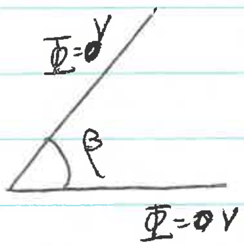
$$= -kD \sinh kc \left[\cosh^2 kz' - \sinh^2 kz' \right] = -4\pi$$

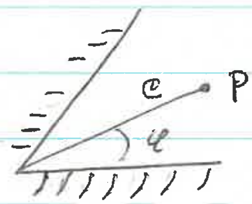
$$kD \sinh kc = 4\pi$$

$$G = 4\pi \cdot \frac{4}{ab} \sum_{lm} \sin\frac{l\pi x}{a} \sin\frac{l\pi x'}{a} \sin\frac{m\pi y}{b} \sin\frac{m\pi y'}{b}$$

$$\times \frac{\sinh k_{lm} z_< \sinh k_{lm}(c - z_>)}{k_{lm} \sinh k_{lm} c}$$

"Wedge" problems for conductors





Obvious to try to work in cylindrical coordinates.

Write $\Phi = V + V_1(\rho, \varphi)$ with $V_1(\rho, 0) = 0$, $V_1(\rho, \beta) = 0$

then $V_1 = [a_\nu \rho^{\nu} + b_\nu \rho^{-\nu}] \sin \nu\varphi$ from $\nabla^2 V_1 = 0$

with $\nu\beta = m\pi \qquad m = 1, 2, 3, \ldots$

if regularity as $\rho \to 0$,
$$\Phi(\rho, \varphi) = V + \sum_{m=1}^{\infty} a_m \rho^{\left(\frac{m\pi}{\beta}\right)} \sin \frac{m\pi}{\beta} \varphi$$

As $\rho \to 0$
$$\Phi = V + a_1 \rho^{\pi/\beta} \sin \frac{\pi\varphi}{\beta}$$
(+ powers of ρ only)

Then $\vec{E} = -\vec{\nabla}\Phi$ says

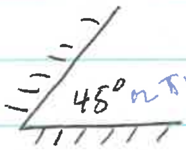
$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\frac{\pi}{\beta} a_1 \left[\rho^{\left(\frac{\pi}{\beta} - 1\right)}\right] \sin \frac{\pi\varphi}{\beta}$$

$$E_\varphi = -\frac{1}{\rho}\frac{\partial \Phi}{\partial \varphi} = -\frac{\pi}{\beta} a_1 \left[\rho^{\pi/\beta - 1}\right] \cos \frac{\pi\varphi}{\beta}$$

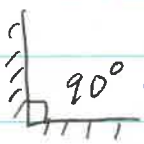
$$\frac{\sigma}{\epsilon_0} = \cancel{\frac{1}{4\pi}}\, \vec{E}\cdot\hat{n} = \left.\frac{-\pi a_1}{\beta}\left[\rho^{\pi/\beta - 1}\right] \cos \frac{\pi\varphi}{\beta}\right|_{\substack{\varphi = 0 \\ \varphi = \beta}}$$

$= E_\varphi$

β small $\Rightarrow \rho^{\pi/\beta - 1} \sim \rho^{\text{large}}$ – $\vec{E}$ + σ die fast in ~~corner~~ corner



45° or $\pi/4$ $\quad E \sim \rho^3$



90° or $\frac{\pi}{2}$ $\quad E \sim \rho$

180° $\beta = \pi$ $\quad E \sim \rho^0$ – of course! (why?)

$\beta > \pi$ $\quad \frac{\pi}{\beta} - 1 \ (\equiv \gamma) < 1$; E becomes weakly singular at $\rho \to 0$: charge piles up on "roof top"

$\beta = \frac{3\pi}{2}$ $\quad \frac{\pi}{\frac{3}{2}\pi} - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$

$\beta = 2\pi$ $\quad E, \sigma \sim \rho^{-1/2}$; high field, high charge on the edge.

$\frac{\pi}{2\pi} - 1 = -\frac{1}{2}$

3-d (needle) similar behavior but needs special functions