The "more useful" technique is to expand in orthogonal polynomials. You have all seen this used before - a little bit - but now it becomes a powerful tool.

Idea is to find a useful set of functions, write your solution to specific problem as linear ~~solutions~~ combination of these functions.

~~useful~~ in 1-d,

$$f(x) = \sum_n a_n \psi_n(x)$$

n ← some label

where

$$\int_a^b dx\, \psi_n(x)^* \psi_m(x) = \delta_{n,m}$$

~~useful~~

↑ range you desire to know $f(x)$

~~and~~ This is only useful if ~~(useful ...)~~ in some region of x, only a few terms saturate the sum

~~So~~ In principle, many possible choices

In practice, geometry often suggests a "best choice"

"Standard manipulations" inverts the sum

$$a_n = \int_a^b dx'\, \psi_n^*(x')\, f(x')\, dx'$$

use some info about $f(x)$ to find a's.

(this will be boundary conditions)

By the way, notice

$$f(x) = \int_a^b dx' \sum_n \psi_n(x)\, \psi_n^*(x')\, f(x')$$

This implies completeness relation – or definition of δ-fn!

$$\delta(x-x') = \sum_n \psi_n^*(x') \psi_n(x)^*$$

S-1.2

Our version of the QM relation

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$\langle x|x'\rangle = \delta(x-x') = \sum_n \langle x|n\rangle\langle n|x'\rangle = \sum_n \psi_n^*(x) \psi_n(x')^*$$

It's very useful to express δ-functions in terms of orthogonal functions – we can write expressions which know about b.c. – useful for direct solve of

$$\nabla^2 G(x,x') = -4\pi\, \delta^3(x-x')$$

For example, Fourier series, $-a/2 < x < a/2$

even $\quad \psi_0^e(x) = \frac{1}{\sqrt{a}}$

$$\psi_m^e(x) = \sqrt{\frac{2}{a}} \cos\left(2\pi \frac{mx}{a}\right) \qquad m \geq 1$$

odd $\quad \psi_m^o(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi m x}{a} \qquad m \geq 1 \quad \Leftarrow$

$$f(x) = A_0 + \sum_{m\geq 1} \left[A_m \psi_m^e(x) + B_m \psi_m^o(x) \right]$$

$$\begin{matrix} A_m \\ B_m \end{matrix} = \sqrt{\frac{2}{a}} \int_{-a/2}^{a/2} f(x)\, dx \begin{cases} \cos 2\pi m x/a \\ \sin 2\pi m x/a \end{cases}$$

δ-function in space of functions vanishing at $x = \pm a$

$$\delta(x-x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin \frac{2n\pi x'}{a} \sin \frac{\pi n x}{a}$$

QM version

$$\langle x | x' \rangle = \delta(x - x')$$

C ~~continuous~~ eigenstate of Hermetian op w/ continuous spectrum

$$\mathbb{1} = \sum_n |n\rangle\langle n| \quad \text{completeness rel}$$

in terms of eigenstates of operator w/ discrete spectrum

$$\langle x | x' \rangle = \langle x | \mathbb{1} | x' \rangle$$

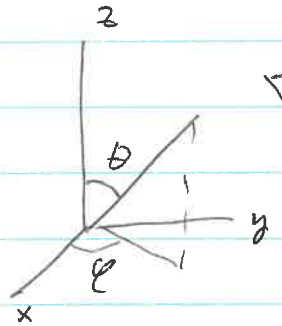
$$\delta(x - x') = \sum_n \langle x | n \rangle \langle n | x' \rangle$$

$$\langle x | n \rangle = \psi_n(x)$$

$$\delta(x - x') = \sum_n \psi_n(x) \psi_n^*(x')$$

Let's focus on problems which are simplest in spherical coordinates and develop all the tools and formalism - then we can treat cylindrical and planar problems very easily "by analogy"

We begin by trying to separate the Laplace eqn in spherical coordinates



$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2}$$

$$\equiv \nabla_r^2 + \frac{1}{r^2}\nabla_{\theta,\varphi}^2$$

We assume a solution $\equiv R(r)\, Y(\theta,\phi) =$

$$\Phi(r,\theta,\varphi) = \left(\frac{u(r)}{r}\right) Y(\theta,\varphi)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\frac{u(r)}{r} = \frac{1}{r}\frac{\partial^2 u}{\partial r^2}$$

$$\frac{\partial}{\partial r}\left\{ r^2\left(\frac{u'}{r} - \frac{u}{r^2}\right)\right\} = \frac{\partial}{\partial r}\left(r u' - u\right)$$

$$= r u'' + u' - u' = r u''$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\frac{u(r)}{r} = \frac{1}{r} u''(r)$$

$$\left\{ \frac{1}{r} u'' Y + \frac{u}{r}\nabla^2 Y = 0 \right\}$$ cancel r $\div$ by Yu

$$\left(\nabla_r^2 + \frac{1}{r^2}\nabla^2_{\theta,\varphi}\right)\Phi(r,\theta,\varphi) = 0$$

Guess $\Phi(r,\theta,\varphi) = R(r)\, Y(\theta,\varphi)$

$$\left[Y \cdot \nabla_r^2 R + \frac{1}{r^2} R \nabla^2_{\theta,\varphi} Y = 0 \right] \frac{r^2}{RY}$$

$$\frac{r^2}{R(r)} \nabla_r^2 R(r) + \frac{1}{Y}\nabla^2_{\theta,\varphi} Y = 0$$ for all r, θ, φ

$\Rightarrow$ each term is a constant,

$$C_r + C_{\theta\varphi} = 0$$

will be useful if b.c. also separate: $\Phi(r=R) = \ldots$

Angular part

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y(\theta,\varphi)$$
$$= C_{\theta,\varphi} Y(\theta,\varphi)$$

You saw this in your quantum mechanics class, for a ~~particle~~ particle in a central potential: nonsingular solutions are the Spherical Harmonics

$$Y_\ell^m(\theta,\varphi), \quad \nabla^2_{\theta,\varphi} Y_\ell^m(\theta,\varphi) = -\ell(\ell+1) Y_\ell^m(\theta,\varphi)$$

with ℓ an integer ≥ 0

$$-\ell \leq m \leq \ell$$

They are defined and normalized so that

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \; Y_{\ell'}^{m'}(\theta,\varphi)^* \, Y_\ell^m(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

and their completeness relation

$$\sum_{\ell m} |\ell m\rangle\langle \ell m| = \mathbb{1} \Leftrightarrow \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\theta',\varphi')\, Y_\ell^m(\theta,\varphi)^*$$

$$= \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

~~$(\langle x|x'\rangle = \delta^3(x-x')$~~
~~$\sum_{\ell m} \langle x|\ell m\rangle\langle \ell m|x'\rangle$~~

Also $\quad Y_\ell^{-m}(\theta,\varphi) = (-)^m\, Y_\ell^m(\theta,\varphi)^*$

(recall $\quad Y_\ell^m(\theta,\varphi) \propto e^{im\varphi}$)

Who has not memorized the first few?

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi} = -\sqrt{\frac{3}{4\pi}}\,\frac{\hat{x}+i\hat{y}}{\sqrt{2}\,r}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\,\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{\hat{z}}{r}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{2i\varphi} = \sqrt{\frac{15}{32\pi}}\left(\frac{\hat{x}+i\hat{y}}{r}\right)^2$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\frac{\hat{z}(\hat{x}+i\hat{y})}{r^2}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}\,\left[3\frac{z^2}{r^2}-1\right]$$

(recall $\frac{\hat{z}}{r} = \cos\theta$, $\frac{\hat{x}}{r} = \sin\theta\cos\varphi$, $\hat{y} = \sin\theta\sin\varphi$)

Radial solution is now

$$\nabla_r^2 R(r) = \frac{\ell(\ell+1)}{r^2} R(r)$$

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} R(r) = \frac{\ell(\ell+1)}{r^2} R$$

Useful aside: Very useful to write $R(r) = \frac{u(r)}{r}$

Why: because $\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} \frac{u(r)}{r} = \frac{1}{r}\frac{d^2u}{dr^2}$

check $\quad \frac{d}{dr} r^2 \left[\frac{u'}{r} - \frac{u}{r^2}\right] = \frac{d}{dr}\left[r u' - u\right]$

$$= u' + r u'' - u' = r \frac{d^2u}{dr^2}$$

$$\nabla_r^2 R = \frac{1}{r} \frac{d^2u}{dr^2}$$

For us, it's easy ... $\frac{1}{r}\frac{d^2u}{dr^2} = \frac{\ell(\ell+1)}{r^2}\frac{u(r)}{r}$

cancel the $\frac{1}{r}$'s

$\Rightarrow$ homogeneous eqn: $u = r^p$ $\quad$ cancel $\frac{1}{r}$'s

$$p(p-1)\, r^{p-2} = \ell(\ell+1)\, r^{p-2}$$

so $\quad p = \ell+1$ or $p = -\ell$

AcN

$$u(r) = A r^{\ell+1} + \frac{B}{r^\ell}$$

$$R(r) = A r^\ell + \frac{B}{r^{\ell+1}}$$

Putting all the pieces together, the general solution to $\nabla^2 \Phi = 0$ in spherical coordinates is

$$\Phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\overset{C}{A_{\ell m}} r^{\ell} + \frac{\overset{D}{B_{\ell m}}}{r^{\ell+1}} \right) Y_\ell^m(\theta,\varphi) \qquad (*)$$

In ~~electrostatics~~ physics, one so-often encounters problems with azimuthal symmetry that an enormous edifice of formalism exists to solve them. If there is no φ-dependence, $m=0$ in $(*)$ and we replace the spherical harmonic by the Legendre polynomial: $Y_\ell^0(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$

(The P_ℓ's are defined so that $P_\ell(1) = 1$.) They are the set of polynomials defined to be orthogonal on the range $[-1, 1]$

$$\int_{-1}^{1} P_\ell(z) P_n(z)\, dz \propto \delta_{\ell n}$$

In actual fact, the "funny" normalization (very useful in practice) of $P_\ell(1) = 1$ makes the RHS

$$\int_{-1}^{1} P_\ell(z) P_n(z)\, dz = \frac{2}{2\ell+1} \delta_{\ell n}$$

Can you recall

$$P_0(z) = 1$$

$$P_1(z) = z$$

$$P_2(z) = \frac{3z^2-1}{2}$$

Note $P_\ell(-z) = (-)^\ell P_\ell(z)$. This is just parity (in z).

The analog of (*) is

$$\Phi(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

This is quite useful!

Examples: Φ between 2 spheres with boundary conditions on sphere

$$\Phi(r=R_1,\theta) \equiv g(\cos\theta)$$

$$\Phi(r=R_2,\theta) = h(\cos\theta)$$

Call $\quad A_\ell r^\ell + \dfrac{B_\ell}{r^{\ell+1}} \equiv f_\ell(r)$

at $r=R_1 \qquad g(\cos\theta) = \sum_\ell f_\ell(R_1)\, P_\ell(\cos\theta)$

find $f_\ell(R_1)$ using orthogonality

$$G_n \equiv \int_{-1}^{1} g(\cos\theta)\, d\cos\theta \times P_n(\cos\theta)$$

$$= \sum_\ell f_\ell(R_1) \int P_\ell(\cos\theta)\, P_n(\cos\theta$$

$$G_n = \sum_\ell \frac{2}{2\ell+1} f_\ell(R_1)\, \delta_{\ell n} = \frac{2}{2n+1} f_n(R_1)$$

and at R_2

$$H_n = \int_{-1}^{1} h(\cos\theta)\, P_n(\cos\theta)\, d\cos\theta$$

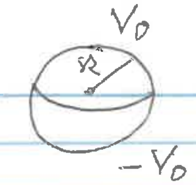
$$= \frac{2}{2n+1} f_n(R_2)$$

$$\left(\frac{2n+1}{2}\right) G_n = A_n R_1^n + B_n / R_1^{n+1}$$

$$\Longrightarrow A_n, B_n$$

$$\left(\frac{2n+1}{2}\right) H_n = A_n R_2^n + B_n / R_2^{n+1}$$

explicit example



$V(R,\theta)$ fixed, Φ everywhere, $\Phi \to 0$ at ∞.

Outside, $A_\ell = 0$ so $\Phi \to 0$ at $r \to \infty$

Inside $B_\ell = 0$ so Φ well behaved at origin

$$\Phi_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \qquad = \sum \frac{B_\ell r^\ell}{R^{2\ell+1}} P_\ell(\cos\theta)$$

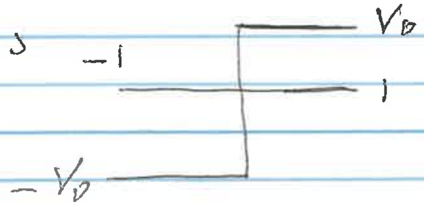
$$\Phi_{out} = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad = \sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Φ is continuous at R so

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}$$

and

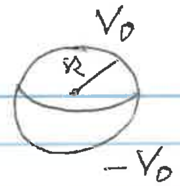
$$\frac{B_\ell}{R^{\ell+1}} \cdot \frac{2}{2\ell+1} = \int_{-1}^{1} V(R,\theta) P_\ell(\cos\theta)\, d\cos\theta$$

For this case,  , $V(\theta)$ is

parity odd - only odd ℓ survives

$$B_1 = \frac{3}{2} R^2 V_0 \left[\int_0^1 \mu\, d\mu - \int_{-1}^{0} \mu\, d\mu \right] = \frac{3}{2} R^2 V_0$$

explicit example

200)


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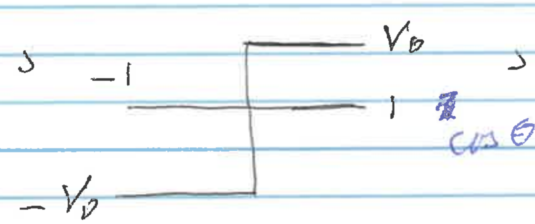
and
$$\frac{B_\ell}{R^{\ell+1}} \cdot \frac{2}{2\ell+1} = \int_{-1}^{1} V(R,\theta) P_\ell(\cos\theta)\, d\cos\theta$$

For this case, $V(\theta)$ is



parity odd - only odd ℓ survives

$$B_1 = \frac{3}{2} R^2 V_0 \left[\int_0^1 \mu\, d\mu - \int_{-1}^{0} \mu\, d\mu \right] = \frac{3}{2} R^2 V_0$$

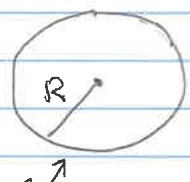
$$r>R \quad \Phi(r,\theta) = V_0 \left\{ \frac{3}{2}\frac{R^2}{r^2} P_1(\cos\theta) - \frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \cdots \right\}$$

$$r<R \quad \Phi(r,\theta) = V_0 \left\{ \frac{3}{2}\frac{r}{R} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \cdots \right\}$$

Note the utility for $r \gg R$, or $r \ll R$

What if ~~the charge~~ the charge density is specified on the surface?

We know $E_\perp$ is discontinuous so



$\sigma(R,\theta)$

$$\left(\vec{E}^{out} - \vec{E}^{in}\right)\cdot\hat{r} = \frac{\sigma(R,\theta)}{\epsilon_0}$$

$$= -\left[\frac{\partial\Phi^{out}}{\partial r} - \frac{\partial\Phi^{in}}{\partial r}\right]_{r=R} = \frac{\sigma(R,\theta)}{\epsilon_0}$$

Φ is continuous so this is a second b.c. which must be satisfied

(Continuous E_{tan} is equivalent to continuous Φ - involves angular derivatives, radial dependence unaffected)

Another problem - more complicated b.c. as $r \to \infty$.

Example: grounded sphere in nonzero electric field. ($E_z = E_0$)

$$\text{as } r\to\infty \quad \Phi(r,\theta) = -E_0 r\cos\theta$$

$$\Phi(R) = 0$$

Solution needs only one partial wave!

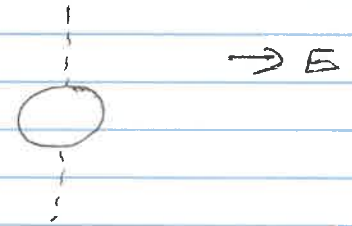
$$\Phi = \left(-E_0 r + B/r^2\right)\cos\theta \qquad r>R$$

$$B = E_0 R^3 \text{ so } \Phi(R) = 0$$

$$\Phi(r) = 0 \text{ if } r<0.$$

$$\Phi = \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta \quad \text{if } r>R$$

Notice that the entire $z=0$ plane is at $V=0$.
This is Jackson 2.10 abs. He wants an image solution - what we have is easier!

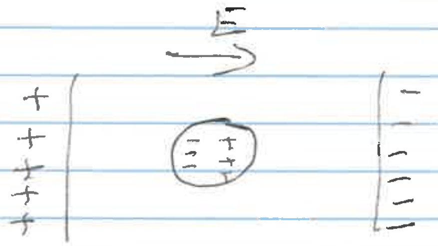


And inside, $\Phi=0$. There is a charge density induced on the sphere,

$$\frac{\sigma}{\epsilon_0} = -\left.\frac{\partial\Phi}{\partial r}\right)_{r=R} = -\left(-E_0 - \frac{2E_0R^3}{R^3}\right)\cos\theta$$

$$= +3E_0\cos\theta$$

Sign?



The total charge on the front hemisphere is

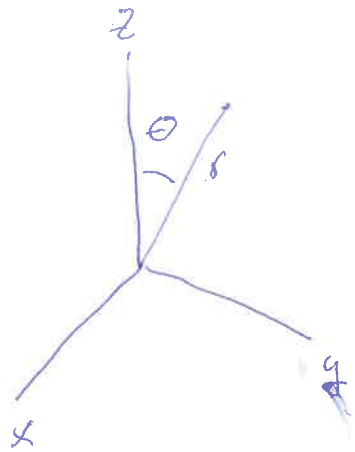
$$Q = R^2\int_0^{2\pi} d\varphi\int_0^1 d\cos\theta\cdot\sigma(\theta,\varphi) = (2\pi R^2)(3E_0)\frac{\epsilon_0}{2}$$

$$= \cancel{+3\pi\epsilon_0 E_0 R^2} \; +3\pi\,\epsilon_0 R^2 E_0$$

Finally, we have the great trick: at $\theta=0$,
$P_\ell(1)=1$ so

$$\Phi(r,\theta=0) = \sum\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)$$

If you can compute Φ at $\theta=0$ - by direct integration, for example, you can expand in a power series in r, and identify the A_ℓ's and B_ℓ's to be used in the general formula, valid at any θ!



Great trick

$z = r\cos\theta$

If azimuthal symmetry

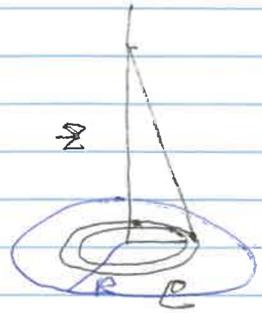
$$\Phi(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

at $\theta \to 0$ $\quad P_l(\cos\theta) = 1$, $\quad r = z$

$$\Phi(z,0) = \sum_l A_l z^l + \frac{B_l}{z^{l+1}}$$

so go backwards --- direct solve for $\Phi(z,0)$, fill in $\Phi(r,\theta)$.

A sample problem: uniform σ on a disc of radius R



$$\Phi(z) = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{\rho\, d\rho}{\sqrt{z^2+\rho^2}}$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0} \sqrt{z^2+\rho^2}\,\Big|_0^R$$

$$\Phi(z) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2} - z\right] \qquad (z>0 \text{ assumed})$$

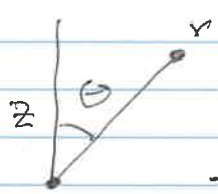
For $z \gg R$, $\sqrt{R^2+z^2} = z + \frac{R^2}{2z} + \frac{1}{8}\frac{R^4}{z^3} + \cdots$

$$\Phi(z) = \frac{\sigma}{2\epsilon_0}\left[\frac{R^2}{2z} + \frac{1}{8}\frac{R^4}{z^3} + \cdots\right]$$

$$= \frac{(\pi R^2\sigma)}{4\pi\epsilon_0 z} + \frac{(\pi R^4\sigma)}{4}\frac{1}{4\pi\epsilon_0}\frac{1}{z^3} + \cdots = \frac{C_1}{z} + \frac{C_3}{z^3}$$

This is

$$\Phi(r, \theta=0) = \frac{C_1}{r} + \frac{C_3}{r^3}$$

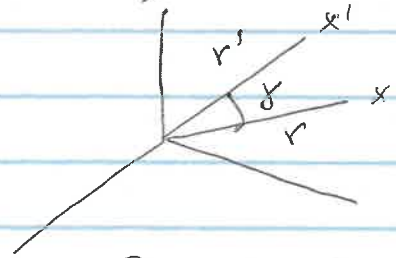


so

$$\Phi(r,\theta) = \frac{C_1}{r}P_0(\cos\theta) + \frac{C_3}{r^3}P_2(\cos\theta) + \cdots$$

$$\frac{1}{4\pi\epsilon_0}\frac{(\pi R^2\sigma)}{r} + \frac{(\pi R^4\sigma)}{4\pi\epsilon_0}\frac{P_2(\cos\theta)}{r^3} + \cdots$$

Potential from a point charge has a simple ~~Before examples~~ and useful Legendre expansion



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

2 ways to see this

$r_<$ - smaller of $|x'|, |x|$
$r_>$ - greater

a) $|x-x'| = (r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}$

suppose $r > r'$; expand as

$$|\vec{x}-\vec{x}'| = r(1+\epsilon)^{1/2} \qquad \epsilon = \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\gamma$$

$$= r\left[1 + \frac{r'^2}{r^2} - \frac{2r'}{r}\cos\gamma\right]^{1/2}$$

$$\frac{1}{|x-x'|} = \frac{1}{r}\left[1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\gamma + \frac{r'^2}{r^2}\left(-\frac{1}{2} + \frac{3}{8}\cdot 4\cos^2\gamma\right) + \cdots\right]$$

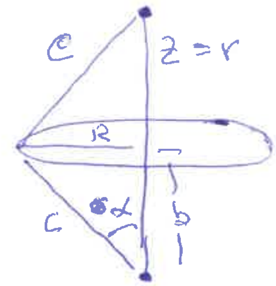
$$= \frac{1}{r} + \frac{r'}{r^2}\cos\gamma + \frac{r'^2}{r^3}\left(\frac{3\cos^2\gamma - 1}{2}\right) + \cdots$$

or use great trick. At $\gamma = 0$, if $r > r'$

$$\frac{1}{|r-r'|} = \frac{1}{r}\sum\left(\frac{r'}{r}\right)^\ell = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}}$$

Insert Legendres by hand!

Jackson example: Φ from a ring



$$\Phi(z=r) = \frac{1}{4\pi\epsilon_0}\frac{q}{c}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{\left[r^2 + c^2 - 2rc\cos\alpha\right]^{1/2}}$$

and $c^2 = b^2 + R^2$

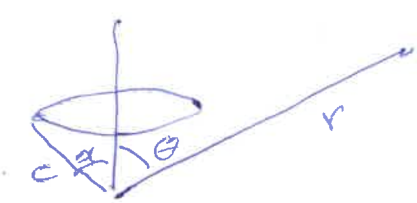
This expands out to

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{c^\ell}{r^{\ell+1}} P_\ell(\cos\alpha) \quad \text{if } r > c$$

$$= \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{r^\ell}{c^{\ell+1}} P_\ell(\cos\alpha) \quad \text{if } r < c$$

All this for points a distance r out the z-axis

Now go to <u>arbitrary</u> r, θ



$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\alpha) P_\ell(\cos\theta)$$

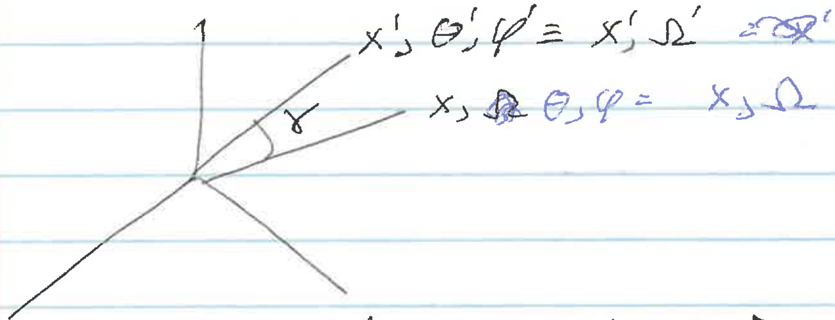
$$r_{\lessgtr} = \begin{matrix}\min\\ \max\end{matrix}(c, r)$$

If we put our coordinate axes at the center of the ring, $c = R$, $\alpha = \frac{\pi}{2}$ $P_\ell(\frac{\pi}{2}) = 0$ if ℓ = odd

Back to $\frac{1}{|x-x'|} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$

This can be awkward – recall

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$



<u>but</u> there is a Magic Formula

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_m Y_\ell^m(\theta',\varphi')^* Y_\ell^m(\theta,\varphi)$$

so

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell m} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_\ell^m(\theta',\varphi')^* Y_\ell^m(\theta,\varphi)$$

Extremely useful – because it's factorized.

<u>Proof</u> needs another magic result from QM: A rotation of coordinates transforms a state $|jm\rangle$ into a superposition of states $|jm'\rangle$ – with same j

$$Y_\ell^m(\bar{\Omega}) = \sum_{m'=-\ell}^{\ell} Y_\ell^{m'}(\Omega)\, C^\ell_{mm'} \qquad *$$

~~Recap~~ A Little check

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega')^{*}\, Y_\ell^m(\Omega)$$

$$\cos\gamma = \cos\theta\cos\theta' + \mathrm{sen}\,\theta\,\mathrm{sen}\,\theta'\cos(\varphi-\varphi')$$

Check for $\ell = 1$ $\quad P_\ell(\cos\gamma) = \cos\gamma$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\; e^{i\varphi}\,\mathrm{sen}\,\theta \qquad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\; e^{-i\varphi}\,\mathrm{sen}\,\theta$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\,\cos\theta$$

$$\cos\gamma = \frac{4\pi}{3}\left\{\frac{e^{-i\varphi'}\mathrm{sen}\,\theta'\, e^{i\varphi}\mathrm{sen}\,\theta}{2} + \frac{e^{+i\varphi'}\mathrm{sen}\,\theta'\, e^{-i\varphi}\mathrm{sen}\,\theta}{2} + \cos\theta\cdot\cos\theta'\right\}\left(\sqrt{\frac{3}{4\pi}}\right)^2$$

$$\cos\gamma = \cos\theta\cos\theta' + \mathrm{sen}\,\theta\,\mathrm{sen}\,\theta'\cos(\varphi-\varphi')$$

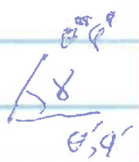
---

by the way, $P_\ell(1) = 1$

$$\frac{2\ell+1}{4\pi} = \sum_m \left|Y_\ell^m(\theta,\varphi)\right|^2$$

"closed shells are round"

$$P_\ell(\cos\gamma) \overset{?}{=} \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega) \qquad \text{S-10}$$



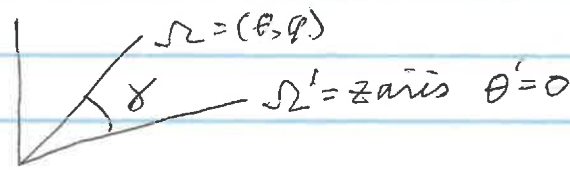
Tricky: $\delta^2(\Omega-\Omega') = \sum_{\ell m} Y_\ell^m(\Omega')^* \, Y_\ell^m(\Omega)$ <u>completeness</u>

$\delta^2(\Omega-\Omega')$ is a function of γ - expand it as

$$\delta^2(\Omega-\Omega') = \sum_\ell B_\ell P_\ell(\cos\gamma)$$

$$B_\ell = \frac{2\ell+1}{2}\int d\cos\gamma \; \delta^2(\Omega-\Omega') P_\ell(\cos\gamma)$$

Evaluate this with coordinates along Ω'



and then $\gamma=\theta$, $d\cos\gamma = d\cos\theta = d\cos\theta \int \frac{d\varphi}{2\pi} = \frac{d\Omega}{2\pi}$

$$B_\ell = \frac{2\ell+1}{4\pi}\int d\Omega \; \delta^2(\Omega-\Omega') P_\ell(\cos\gamma)$$

When $\Omega=\Omega'$, $\cos\gamma = 1$, $P_\ell(\cos\gamma)=1$

$$B_\ell = \frac{2\ell+1}{4\pi} P_\ell(1) = \frac{2\ell+1}{4\pi}$$

<u>So</u> ~~Also~~ $\delta^2(\Omega-\Omega') = \sum_{\ell m} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega) = \sum_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma)$

But $P_\ell(\cos\gamma) \propto Y_\ell^0$ - * says ~~this~~ the RH equality expression is true term by term

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_m Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)}$$

$$G(x,x') = \frac{1}{|x-x'|} = \sum \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^m(\theta',\varphi')^* Y_\ell^m(\theta,\varphi)$$

Notice the extreme utility of this expression

$$\frac{1}{|x-x'|} = G(x,x') \text{ is the Green's function for } \nabla^2 \text{ with free b.c.}$$

Thus we have a representation for the Greens function which is factorized into separate functions of x and x'. For example, suppose we wanted to solve

$$\Phi(x) = \Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')\, d^3x'}{|x-x'|}$$

and $\rho(x')$ could be written simply in spherical coordinates –

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\int d\Omega'\, r'^2 dr'\, \rho(r',\Omega')$$

$$\times \sum_{\ell m} \frac{4\pi}{2\ell+1} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega) \frac{r_<^\ell}{r_>^{\ell+1}}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{\ell m} \frac{4\pi}{2\ell+1} Y_\ell^m(\Omega)\Big\{\int d\Omega'\, Y_\ell^m(\Omega')$$

$$\times \left[\frac{1}{r^{\ell+1}}\int_0^r r'^2 dr'\, \rho(r',\Omega')(r')^\ell \right.$$

$$\left. + r^\ell \int_r^\infty r'^2 dr'\, \rho(r',\Omega') \frac{1}{(r')^{\ell+1}}\right]\Big\}$$

For all problems solved by images, where

$$G(x,x') = \sum_j \frac{f_j}{|\vec{X}-\vec{X}_j|},$$ we can expand the sums and combine, too.

Example - electronic repulsion in Helium

$$H = -\frac{\hbar^2}{2m}\left(\nabla_1^2 + \nabla_2^2\right) - 2e^2\left[\frac{1}{r_1} + \frac{1}{r_2}\right] + \frac{e^2}{r_{12}}$$

neglect last term, assume

~~assume~~ $\psi(r_1, r_2) = \psi_{1s}(r_1)\,\psi_{1s}(r_2) = \frac{Z^3}{\pi a_0^3} e^{-\frac{Z}{a_0}(r_1+r_2)} \equiv f(r_1, r_2)$

i.e. product of hydrogenic WFs

~~Compute~~ $\cancel{\langle \psi | \frac{e^2}{r_{12}} | \psi \rangle}$ ~~as perturbation in operator~~

~~Group~~ Compute correction in perturbation theory ~~to~~ to ground state energy

$$\langle V \rangle = \langle \psi | \frac{e^2}{r_{12}} | \psi \rangle = \int d^3r_1\, d^3r_2\, \psi(r_1, r_2)^* \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} \psi(r_1, r_2)$$

To do this quickly and easily, write

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^m(\Omega_1)^* Y_\ell^m(\Omega_2)$$

$$\langle V \rangle = 4\pi e^2 \cancel{\left[\frac{Z^3}{\pi a_0^3}\right]^2} \sum_{\ell m} \frac{1}{2\ell+1} \int d\Omega_1\, Y_\ell^m(\Omega_1)^* \int d\Omega_2\, Y_\ell^m(\Omega_2)$$

$$\times \int r_1^2 dr_1 \int r_2^2 dr_2 \; \underbrace{\cancel{e^{-\frac{2Z}{a_0}(r_1+r_2)}}}_{f(r_1, r_2)} \; \frac{r_<^\ell}{r_>^{\ell+1}}$$

Now $\int d\Omega\, Y_\ell^m(\Omega)^* = \sqrt{4\pi} \int d\Omega\, Y_\ell^m(\Omega)^* \frac{1}{\sqrt{4\pi}}$

$= \sqrt{4\pi} \int d\Omega\, Y_\ell^m(\Omega)^* Y_0^0(\Omega) = 0$ unless $\ell = 0, m = 0$!
$= \delta_{\ell 0}\delta_{m 0} \cdot \sqrt{4\pi}$

The sum collapses to one term: $\frac{1}{r_>}$

$$\langle V \rangle = (4\pi)^2 \cancel{\left[\frac{Z^3}{\pi a_0^3}\right]^2} e^2 \left\{ \int_0^\infty r_1^2 dr_1 \left[\int_0^{r_1} r_2^2 dr_2\, f(r_1, r_2) \cdot \frac{1}{r_1} + \int_{r_1}^\infty r_2^2 dr_2\, f(r_1, r_2) \cdot \frac{1}{r_2} \right] \right.$$

= 2 1-d integrals!

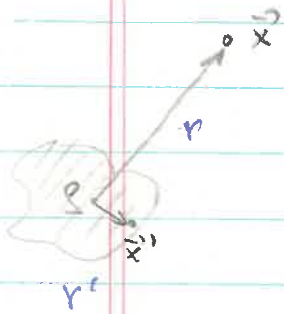
There are a couple of directions we can push this: we can continue to study Green's functions and potentials in free space, or we can introduce boundaries. The "free space" path takes us to a very useful formalism, very clean

Electric Multipole Expansion

In general,

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(x')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_l \sum_m \left\{ \int d^3x' \left(\frac{r_<^l}{r_>^{l+1}}\right) Y_l^{m*}(\Omega')\, \rho(\vec{x}') \right\} \times Y_l^m(\Omega) \times \frac{4\pi}{2l+1}$$



If $\rho(x') \to 0$ ~~...~~ at big r', and if $r_>$ is always r $(=|x|)$, we can write

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_l \sum_m \frac{4\pi}{2l+1} q_{lm} \frac{Y_l^m(\Omega)}{r^{l+1}}$$

with

$$q_{lm} = \int Y_l^m(\theta',\varphi')^* \, r'^l \rho(\vec{x}')\, d^3x'$$

are called "multipole ~~coefficients~~ moments", or "spherical multipole moments." Let's look at a few of them, rewriting the Y_l^m in terms of Cartesian variables

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$

$$q_{00} = \frac{1}{\sqrt{4\pi}}\int \rho(x')\, d^3x' = \frac{1}{\sqrt{4\pi}} q$$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}$, $\sin\theta e^{i\varphi} = \left(\frac{x+iy}{r}\right)$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\int (x'-iy')\rho(x')\, d^3x' \equiv -\sqrt{\frac{3}{8\pi}}(p_x - ip_y)$$

$$Y_1^0 = \sqrt{\tfrac{3}{4\pi}}\cos\theta = \sqrt{\tfrac{3}{4\pi}}\tfrac{z}{r}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\int z'\rho(x')\,d^3x' = \sqrt{\frac{3}{4\pi}}\,p_z$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\int (x'-iy')^2\rho\,d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{11}-2iQ_{12}-Q_{22}\right)$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}}\int z'(x'-iy')\rho = -\sqrt{\frac{1}{3}}\sqrt{\frac{15}{8\pi}}\left(Q_{13}-iQ_{23}\right)$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\int (3z'^2-r'^2)\rho = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,Q_{33}$$

Because ρ is real, the reflection properties of the Y_ℓ^m's gives $q_{\ell m} = (-)^m q^*_{\ell,-m}$ (defn of Y_ℓ^m) ; $2\ell+1$ $q_{\ell m}$'s!

Most of the formulas you'll see, in the literature are in terms of the Cartesian multipole moments

$$\cancel{Q_{ijk} = \int d^3x\,\rho(x)\,x^i y^j z^k}$$

although the $q_{\ell m}$'s are much more useful ~~to~~ ~~who~~ if you know your angular momentum group theory. Nevertheless...

the Dipole moment $\vec{p} = \int \rho(x)\,\vec{x}\,d^3x$

and Quadrupole moment tensor (symmetric, traceless)

$$Q_{ij} = \int \left(3x_ix_j - \delta_{ij}x^2\right)\rho\,d^3x$$

Finally, $\Phi(x) = \dfrac{1}{4\pi\epsilon_0}\left[\dfrac{q}{r} + \dfrac{\vec{p}\cdot\vec{r}}{r^3} + \dfrac{1}{2}\sum_{ij} Q_{ij}\dfrac{r_ir_j}{r^5} + \dots\right]$

(gets messy fast ...)

[recall conducting sphere in external $\vec{E}$ field:

$$\Phi = -E_0 r + \frac{E_0R^3\cos\theta}{r^2} \Rightarrow \frac{\vec{p}}{4\pi\epsilon_0} = \vec{E}R^3 \quad]$$

Multipole summary

<u>spherical basis</u> $\Phi(x) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_\ell^m(\Omega)}{r^{\ell+1}}$

$$q_{\ell m} = \int Y_\ell^{m}(\theta,\varphi)\, r^\ell\, \rho(r)\, d^3r$$

$2\ell+1$ $q_{\ell m}$'s per ℓ

useful if angular momentum is important: for example $\rho(r) \sim Y_L^M(\Omega) f(r)$

<u>Cartesian</u> basis - very messy beyond quadrupole

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{r_i r_j}{r^5} + \cdots\right]$$

$$q = \int \rho(x)\, d^3x \quad , \quad p_i = \int \rho(x)\, x_i\, d^3r$$

$$Q_{ij} = \int \left[3x_i x_j - \delta_{ij} x^2\right] \rho(x)\, d^3x$$

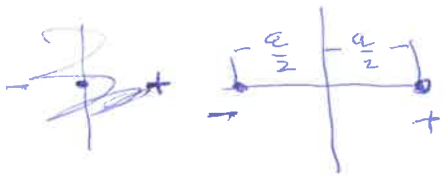
i.e. $Q_{zz} = \int [3z^2 - r^2]\,\rho(x)\, d^3x \qquad r^2 = x^2+y^2+z^2$

Q_{ij} symmetric $(Q_{ij} = Q_{ji})$, traceless $\left(\sum_i Q_{ii} = 0\right)$

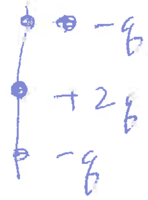
$$\sum_i Q_{ii} = \int \left[3(x^2+y^2+z^2) - 3r^2\right]\rho\, d^3x = 0$$

$9 \to 6 \to 5$ independent ones.

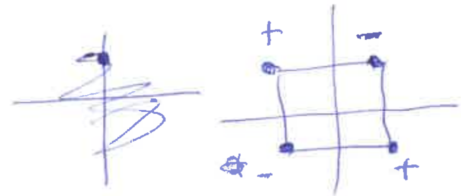
Visualization



= dipole



or



= quadrupole

"Induced dipole moment", recall conducting sphere in external $\vec{E}$ field, potential was

$$\Phi = -E_0 r + \frac{E_0 R^3}{r^2}\cos\theta$$

read off

2nd term is $\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$, $\frac{\vec{p}}{4\pi\epsilon_0} = R^3\vec{E}$ ($\longrightarrow E$)

annoyance: all multipole moments (except q) depend on location of origin – math statement (not useful, but true) – lowest non vanishing term is independent of choice of origin – see 4.4

Electric field for dipole easiest from

$$\vec{E} = -\vec{\nabla}\Phi = -\vec{\nabla}\,\frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

$$= \frac{3\vec{r}(\vec{p}\cdot\vec{r}) - \vec{p}\,r^2}{r^5}\cdot\frac{1}{4\pi\epsilon_0}$$

almost!

~~Let's return to this...~~

Issue: there are exact results for dipoles. Suppose we have a localized $\rho(x)$ creating an $\vec{E}(x)$. Integrate $\vec{E}(x)$ over a sphere of radius R. If we chose coordinates ~~with~~ so center of sphere is at $\vec{x}=0$ then

a) If no charge inside R,

$$\vec{E}(0) = \frac{3}{4\pi R^3}\int_{r<R} d^3r\;\vec{E}(r)$$

b) if all charge is inside R

$$-\frac{4\pi}{3}\,\frac{\vec{p}}{4\pi\epsilon_0} = \int_{r<R} d^3r\;\vec{E}(r) = -\frac{\vec{p}}{3\epsilon_0}$$

$\vec{p}$ = dipole moment measured with respect to center of sphere proof is in Jackson – not particularly illuminating. (sec. 4.1)

here's the issue – dipole at $\vec{x}_0$

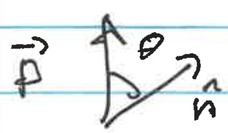
$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3}\right]$$

Note trouble at $x_0 \to x$.

$$\int d^3x \, \vec{E}(x) \stackrel{?}{=}$$

If we naively integrate $\vec{E}$ over angles, not worrying about the singularity, we get zero

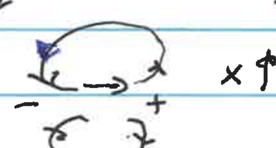
$\vec{p}(n) = 3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}$: pick $\vec{p}$ along z axis



$$\vec{z}\cdot\vec{p}(n) = [3\underbrace{(\cos\theta)}_{n\cdot z}\underbrace{\cos\theta}_{p\cdot n} - 1]\, p_0$$

$$\int E_z \, d^3x = \int_{-1}^{1} d\cos\theta \, [3\cos^2\theta - 1] = 0$$

$\int E_x \, d^3x = 0$ also



But what if we integrate radially?

$$\int \frac{r^2 dr}{r^3} = ?$$

Multipoles and energy of a localized charge distribution in an <u>external</u> potential.

Start with $W = \int \rho(x)\,\Phi(x)\,d^3x$ (*)

Taylor expand Φ about the origin

$$\Phi(x) = \Phi(0) + \vec{x}\cdot\vec{\nabla}\,\Phi(0) + \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial^2 \Phi(0)}{\partial x_i \partial x_j} + \cdots$$

$\vec{E} = -\vec{\nabla}\Phi$ and $\vec{\nabla}\cdot\vec{E} = 0$ (potential due to <u>external</u> charges)

$$\Phi(x) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial}{\partial x_j} E_i(0) + \cdots$$

Add zero $= \frac{1}{6} r^2\, \nabla\cdot E$

$$\Phi(x) = \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_i}{\partial x_j}$$

put this in * & recall definitions of moments ($\vec{p} = \int \rho(x)\,\vec{x}\,d^3x$ etc.)

$$W = q\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_i(0)}{\partial x_j} + \cdots$$

Notice how different multipoles talk to the external field. 2nd term is the usual dipole energy formula. Last term - quadrupoles talk to gradients of fields

Jackson: external field is from electrons in molecules, or crystal lattices. Multipole is for nucleus. Different $|JM\rangle$ states (m states for fixed J) have different Q_{ij}'s, energy is different, levels split - Like the Stark effect (which is for dipoles). Use to measure Q's of nuclei - that's the physics of the Jackson problems.

patch – what you see in books

$$* \quad \vec{E}(x) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta^3(\vec{x}-\vec{x}_0)\right\}$$

with convention

$\langle$ first term $\rangle = 0$ always (don't think about singularity)

$$\int d^3x \ \text{2nd term} = -\frac{4\pi}{4\pi\epsilon_0}\frac{\vec{p}}{3} = -\frac{\vec{p}}{3\epsilon_0}$$

$\int d^3x\, \vec{E}(x) =$

expression * for the field of a point dipole satisfies an exact result.

There's a similar magnetic dipole formula – seen much more often in QM, for hyperfine splitting

This is never discussed (except in Jackson)

the Alternative is horrible:
dipole's extension must be monitored
you can't think about an idealized microscopic dipole as a point
in any $\int \vec{E}\, d^3x$
as in usual dipole-plot it!

Green's functions via spherical harmonics

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$$

Recall the "external to the sphere problem" w/ b.c. $\Phi(r=R)=0$, solved with an image

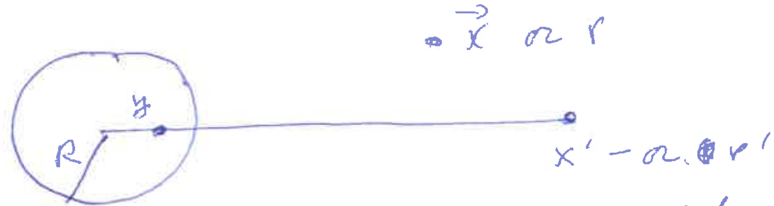


Image term $-\frac{R}{r'} \cdot \frac{1}{|\vec{x}-\vec{y}|}$; $(y) = \frac{R^2}{r'}$ (rescaled q)

Image is inside the sphere, want solution outside the sphere so $|y|$ is always less than r

$$\frac{r_<^\ell}{r_>^{\ell+1}} = \left(\frac{R^2}{r'}\right)^\ell \frac{1}{r^{\ell+1}}$$; there's another $\frac{R}{r'}$

$$\boxed{R^{2\ell}\cdot R = \frac{(R^2)^{\ell+1}}{R}}$$

$$(\text{image}) = -\sum \frac{4\pi}{2\ell+1}\left(\frac{R^2}{rr'}\right)^{\ell+1}\frac{1}{R} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$$

i.e. $$G(\vec{x}, \vec{x}') = \sum_{\ell m} \frac{4\pi}{2\ell+1} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{R^2}{rr'}\right)^{\ell+1}\right\}$$

if $r<r'$; $\{\} = \left(\frac{1}{r'}\right)^{\ell+1}\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)$

if $r>r'$ $\{\} = \left(\frac{1}{r}\right)^{\ell+1}\left(r'^\ell - \frac{R^{2\ell+1}}{(r')^{\ell+1}}\right)$

Notice that this a) vanishes if $r=R$ or $r'=R$ (one b.c.)

b) is symmetric in $r \leftrightarrow r'$

c) vanishes as $r \to \infty$ or $r' \to \infty$, the other b.c.

d) It is 2 linear combinations of Laplace's eqn -

$$\alpha r^{l} + \beta / r^{l+1}$$

Hold that thought, start over - direct attack

$$\nabla_x^2 G(x, x') = -4\pi \delta^3(x-x')$$

$$G = 0 \text{ if } x \in S' \text{ or } x' \in S' \text{ (surfaces)}$$

Let's write the δ-fn in spherical coordinates as

$$\delta^3(x-x') = \frac{1}{r^2}\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\Omega')^* Y_\ell^m(\Omega) \qquad (*)$$

and ~~try to~~ guess a solution

$$G(\vec{x},\vec{x}') = \sum_{\ell m} g_\ell(r,r')\, Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$$

$$\nabla^2 G(x,x') = \sum_{\ell m}\nabla^2\left[g_\ell(r,r')\, Y_\ell^m(\Omega)\right] Y_\ell^m(\Omega')^*$$

$$= \sum_{\ell m}\left\{\left(\frac{1}{r}\frac{d^2}{dr^2} r g_\ell\right) Y_\ell^m(\Omega) - g_\ell\frac{\ell(\ell+1)}{r^2} Y_\ell^m(\Omega)\right\}\cdot Y_\ell^m(\Omega')^*$$

and this is equal to $-4\pi\delta^3(\vec{r}-\vec{r}')$

$$= -\frac{4\pi}{r^2}\delta(r-r')\,\delta(\cos\theta\cos\theta')\,\delta(\varphi-\varphi')$$

$= (*)$ — so: Mode by mode

$$\boxed{\frac{1}{r}\frac{d^2}{dr^2} r g_\ell(r,r') - \frac{\ell(\ell+1)}{r^2} g_\ell(r,r') = -\frac{4\pi}{r^2}\delta(r-r')}$$

Note: for $r\neq r'$, $g_\ell(r,r')$ is a solution to the Homogeneous ODE. What happens at $r=r'$?

Integrate both sides of the DE from $r = r' - \epsilon$ to $r = r' + \epsilon$, take $\epsilon \to 0$

$$\int_{r'-\epsilon}^{r'+\epsilon} dr \left[\frac{d^2}{dr^2} r g_\ell(r,r') - \frac{\ell(\ell+1)}{r^2} r g_\ell(r,r') \right] = \int_{r'-\epsilon}^{r'+\epsilon} dr \left[-\frac{4\pi}{r} \delta(r-r') \right]$$

or 1) $$\frac{d}{dr}\left[r g_\ell(r,r') \right] \Big|_{r'-\epsilon}^{r'+\epsilon} = -\frac{4\pi}{r'}$$

Derivative is discontinuous!

Integrate once more $\int_{r'-\epsilon}^{r'+\epsilon} dr$

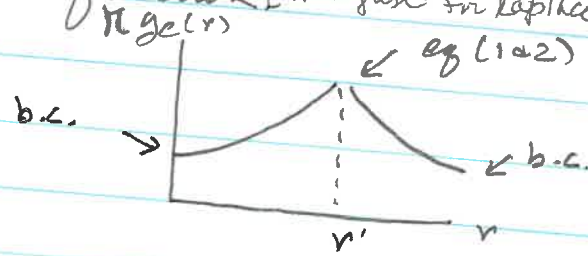
2) $$g(r,r') \Big|_{r=r'+\epsilon} - g(r,r') \Big|_{r=r'-\epsilon} = 0$$

g is continuous

So we can find $g_\ell(r,r')$ by the following set of steps:

- Solve the homogeneous equation in the regions $r<r'$ and $r>r'$, satisfying any b.c. there
- Match the 2 solutions at $r=r'$ (eq (2))
- Derivatives are discontinuous at $r=r'$ (eq (1))

This method will work for almost any kind of Dirichlet Green's function (not just for Laplace eqn)



Let's look at some examples:

a) Free-space Dirichlet b.c.

$$0 < r, r' < \infty$$

$$g \to 0 \quad \text{at } r \to \infty$$

$$g \to 0 \quad \text{at } r \to 0$$

For $r<r'$ $\quad g_\ell(r,r') = A_\ell r^\ell$

$r>r'$ $\quad g_\ell(r,r') = \dfrac{B_\ell}{r^{\ell+1}}$

Recall the story

$$G(x,x') = \sum_{\ell m} g_\ell(r,r') Y_\ell^{m*}(\Omega') Y_\ell^m(\Omega)$$

$$g_\ell(r,r')\Big|_{r=r'+\epsilon} = g_\ell(r,r')\Big|_{r=r'-\epsilon} \qquad (4)$$

$$\frac{d}{dr}\left[r g_\ell(r,r')\right]\Bigg|_{r=r'-\epsilon}^{r=r'+\epsilon} = -\frac{4\pi}{r'} \qquad (3)$$

1) $g_\ell(r=r_{min}) = 0$ $\qquad$ $g_\ell(r=r_{max}) = 0$ 2)

$$\frac{1}{r}\frac{d^2}{dr^2} r g_\ell(r,r') - \frac{\ell(\ell+1)}{r^2} g_\ell(r,r') = 0$$

if $r \neq r'$

p. 518a (1) is $\frac{d}{dr}\left(\frac{B_\ell}{r^\ell}\right) - \frac{d}{dr}\left(A_\ell r^{\ell+1}\right)\Big|_{r=r'} = -\frac{4\pi}{r'}$

$$-\frac{\ell B_\ell}{(r')^{\ell+1}} - (\ell+1)A_\ell (r')^\ell = -\frac{4\pi}{r'}$$

$$-\ell B_\ell - (\ell+1)A_\ell (r')^{2\ell+1} = -4\pi (r')^\ell$$

(2) is $\frac{B_\ell}{(r')^{\ell+1}} = A_\ell (r')^\ell$ or $B_\ell = A_\ell (r')^{2\ell+1}$

$$\therefore B' = \frac{4\pi}{2\ell+1} r'^\ell \qquad A_\ell = \frac{4\pi}{2\ell+1}\frac{1}{(r')^{\ell+1}}$$

$$G(x,x') = 4\pi \sum_{\ell m} \frac{Y_\ell^m(\Omega)\, Y_\ell^m(\Omega')^*}{2\ell+1} \cdot \begin{cases} \frac{r^\ell}{(r')^{\ell+1}} & r<r' \\ \frac{(r')^\ell}{r^{\ell+1}} & r>r' \end{cases}$$

That was re-assuring. Lets consider another case, a shell bounded by $a<r<b$ w/ $G(a,r') = G(b,r') = 0$

For $r<r'$ $\quad g_\ell = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \equiv g_\ell^<$

1) $0 = A_\ell a^\ell + B_\ell / a^{\ell+1}$

For $r>r'$ $\quad g_\ell = C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}} \equiv g_\ell^>$

2) $0 = C_\ell b^\ell + \frac{D_\ell}{b^{\ell+1}}$

and 3) $-\frac{4\pi}{r'} = \frac{d}{dr}(r g^>)\Big|_{r=r'} - \frac{d}{dr}(r g^<)\Big|_{r=r'}$

4) $g^>(r', r') = g^<(r', r')$

4 equations, 4 unknowns. looks bad!

Recap: $\cancel{G(x,x')}$

$$G(x,x') = \sum_{\ell m} g_\ell(r,r')\, Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)$$

$$\frac{1}{r}\frac{d^2}{dr^2} r\, g_\ell(r,r') - \frac{\ell(\ell+1)}{r^2} g_\ell(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

1) $g_\ell(r_{min} \equiv a) = 0$

2) $g_\ell(r_{max} = b) = 0$

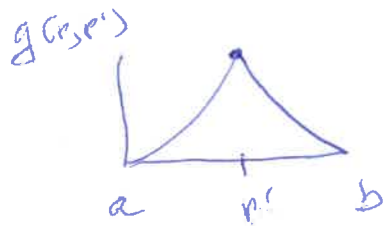
4) $g_\ell(r,r')\Big|_{r=r'-\epsilon} = g_\ell(r,r')_{r=r'+\epsilon}$

3) $\frac{d}{dr}\left[r g_\ell(r,r')\right]\Big|_{r=r'-\epsilon}^{r=r'+\epsilon} = -\frac{4\pi}{r'}$

and 5) if $r \neq r'$ $g_\ell(r)$ a solution of homogeneous eqn

$$g_\ell(r,r') = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}$$

and 6) $G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x})$ for Dirichlet b.c.



A tricky way to solve this problem is to realize that $g(r,r')$ is symmetric under exchange of r & r'. Solve bc's by inspection (at $r = a$ & b)

$$g_\ell(r,r') = \begin{cases} C\left[r^\ell - \dfrac{a^{2\ell+1}}{r^{\ell+1}}\right]\left[\dfrac{1}{r'^{\ell+1}} - \dfrac{r'^\ell}{b^{2\ell+1}}\right] & \text{if } r<r' \\ \text{or} \\ C\left[\dfrac{1}{r^{\ell+1}} - \dfrac{r^\ell}{b^{2\ell+1}}\right]\left[r'^\ell - \dfrac{a^{2\ell+1}}{r'^{\ell+1}}\right] & \text{if } r>r' \end{cases}$$

This solves (1), (2), and (4). Derivative expression is

$$\frac{d}{dr}\left[r g_\ell\right]_{r=r'+\epsilon} = C\left[\frac{-\ell}{(r')^{\ell+1}} - \frac{(\ell+1)(r')^\ell}{b^{2\ell+1}}\right]\left[r'^\ell - \frac{a^{2\ell+1}}{(r')^{\ell+1}}\right]$$

$$\frac{d}{dr}\left[r g_\ell\right]_{r=r'-\epsilon} = C\left[\frac{1}{(r')^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right]\left[(\ell+1)r'^\ell + \frac{\ell a^{2\ell+1}}{(r')^{\ell+1}}\right]$$

multiply and combine – one eqn for C

$$-\frac{4\pi}{r'} = C\left\{-\frac{(2\ell+1)}{r'} - 0 - 0 - \frac{2\ell+1}{r'}\left(\frac{a}{b}\right)^{2\ell+1}\right\}$$

$$G(\vec{x},\vec{x}') = 4\pi \sum_{\ell m} \frac{Y_\ell^m(\Omega')^* \, Y_\ell^m(\Omega)}{(2\ell+1)\left[1+\left(\frac{a}{b}\right)^{2\ell+1}\right]} \times \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

Most of this is obvious? (take $a \to 0$, $b \to \infty$)

Examples using $\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x\, \rho(x')\, G(x,x')$

$$- \frac{1}{4\pi}\int_S dA\, \Phi(x') \frac{\partial G}{\partial n'}$$

and suppose we have an interior problem ($a = 0$, $G = 0$ at $r = b$)

$$G(x,x') = 4\pi \sum_{\ell m} \frac{Y_\ell^{m}(\Omega')^* Y_\ell^m(\Omega)}{2\ell+1}\, r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right]$$

$\rho = 0$, $\Phi(b)$ specified $= V(\theta', \varphi')$

$$\frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} \quad \text{; here } r' = r_>$$

$$\frac{\partial}{\partial r'}\left[\frac{1}{(r')^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right]_{r'=b} = \frac{-(\ell+1)}{b^{\ell+2}} - \frac{\ell\, b^{\ell-1}}{b^{2\ell+1}} = \frac{-(2\ell+1)}{b^{\ell+2}}$$

$$\boxed{\frac{\partial G}{\partial n'} = -\frac{4\pi}{b^2}\sum_{\ell m} Y_\ell^m\, Y_\ell^m \left(\frac{r}{b}\right)^\ell} \; ; \quad dA' = b^2 d\Omega'$$

$$\Phi(x) = \sum_{\ell m}\left[\int b^2 d\Omega'\, \frac{V(\theta'\varphi')\, Y_\ell^m(\theta'\varphi')}{b^2}\right] \left(\frac{r}{b}\right)^\ell Y_\ell^m(\Omega)$$

exactly! $\Phi(x) = \sum_{\ell m} c_{\ell m}\, r^\ell\, Y_\ell^m(\Omega)$

$c_{\ell m}$ from b.c. at $r = b$

Examples using $\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x\, \rho(x')\, G(x,x')$

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$\rho = 0$, $\Phi(b)$ specified $= V(\theta', \varphi')$

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'}\Big)_{r'=b} \quad ; \text{ here } r' = r_>$$

$$\frac{\partial}{\partial r'}\left[\frac{1}{(r')^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right]_{r'=b} = -\frac{(\ell+1)}{b^{\ell+2}} - \ell\frac{b^{\ell-1}}{b^{2\ell+1}} = -\frac{(2\ell+1)}{b^{\ell+2}}$$

$$\boxed{\frac{\partial G}{\partial n'} = -\frac{4\pi}{b^2}\sum_{\ell m} Y_\ell^{m} Y_\ell^{m}\left(\frac{r}{b}\right)^\ell} \quad ; \quad dA' = b^2 d\Omega'$$

$$\Phi(x) = \sum_{\ell m}\left[\int b^2 d\Omega'\, \frac{V(\theta'\varphi')\, Y_\ell^m(\theta'\varphi')}{b^2}\right] \left(\frac{r}{b}\right)^\ell Y_\ell^m(\Omega)$$

exactly! $\Phi(x) = \sum_{\ell m} c_{\ell m}\, r^\ell\, Y_\ell^m(\Omega)$

$c_{\ell m}$ from b.c. at $r = b$

Examples using $\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x \, \rho(x')\, G(x,x')$

$$- \frac{1}{4\pi}\int_S dA \, \Phi(x') \frac{\partial G}{\partial n'}$$

and suppose we have an interior problem ($a=0$, $G=0$ at $r=b$)

$$G(x,x') = 4\pi \sum_{\ell m} \frac{Y_\ell^{m}(\Omega')^{*} Y_\ell^m(\Omega)}{2\ell+1} \, r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right]$$

$\rho = 0$, $\Phi(b)$ specified $= V(\theta', \varphi')$

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'}\Big|_{r'=b}$$; here $r' = r_>$

$$\frac{\partial}{\partial r'}\left[\frac{1}{(r')^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}} \right]_{r'=b} = -\frac{(\ell+1)}{b^{\ell+2}} - \frac{\ell \, b^{\ell-1}}{b^{2\ell+1}} = -\frac{(2\ell+1)}{b^{\ell+2}}$$

$$\boxed{\frac{\partial G}{\partial n'} = -\frac{4\pi}{b^2} \sum_{\ell m} Y_\ell^{m} Y_\ell^{m} \left(\frac{r}{b}\right)^\ell} \; ; \quad dA' = b^2 d\Omega'$$

$$\Phi(x) = \sum_{\ell m} \left[\int b^2 d\Omega' \, \frac{V(\theta'\varphi') Y_\ell^m(\theta'\varphi')}{b^2} \right] \left(\frac{r}{b}\right)^\ell Y_\ell^m(\Omega)$$

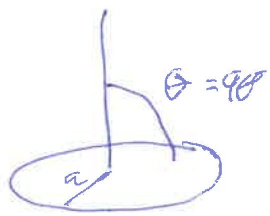
exactly! $\Phi(x) = \sum_{\ell m} c_{\ell m} \, r^\ell \, Y_\ell^m(\Omega)$

$c_{\ell m}$ from b.c. at $r=b$

Now for some horrible Jackson problems

— Charged ring of radius "a" inside grounded shell of radius "b"

According to Jackson



$$\rho(x') = \underbrace{\frac{q}{2\pi a^2}}_{?} \delta(r'-a)\,\delta(\cos\theta)$$

$$q = \int d^3x'\,\rho(x') = \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \int_0^\infty r'^2 dr'$$

$$\times \frac{q}{2\pi a^2}\delta(r'-a)\,\delta(\cos\theta')$$

$$= 2\pi \times 1 \times a^2 \times \frac{q}{2\pi a^2} \quad \checkmark$$

$$\Phi(x) = \int d^3x' \frac{\rho(x')}{4\pi\epsilon_0}\left[\sum_{\ell m} \frac{4\pi}{2\ell+1} Y_\ell^{m*}(\Omega') Y_\ell^m(\Omega) \times r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)\right.$$

φ integral $\rightarrow 2\pi$ and forces $m=0$

θ integral $\rightarrow \theta' = 90°$ ($\cos\theta' = 0$)

r' integral $\rightarrow r' = a$

$$\Phi(x) = \frac{4\pi}{4\pi} \frac{q}{\epsilon_0} \sum_{\ell} \frac{Y_\ell^0(\cos\theta' = 0)\, Y_\ell^0(\Omega)}{2\ell+1}$$

$$\times\; r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right)$$

$r_<$ is $\min(r, a)$

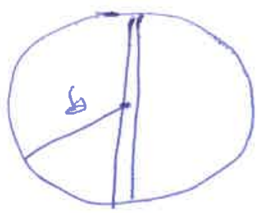
$r_>$ is $\max(r, a)$

$$Y_\ell^0(\theta') Y_\ell^0(\theta) = \frac{P_\ell(0) P_\ell(\cos\theta)}{4\pi}$$

We had all of this before, just an extra $\frac{-r^\ell}{b^{2\ell+1}}$ from the boundary.

A harder one (cont. last spherical one)

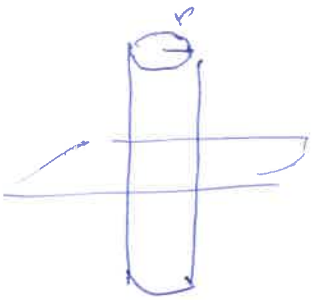
Charged rod, λ = charge/length, embedded in grounded spherical shell



$$\Phi = \frac{1}{4\pi\epsilon_0}\int \rho(x')\, G(x,x')\, d^3x'$$

G as before

Problem: what is $\rho(x)$???



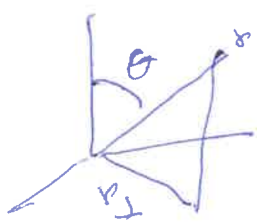
In cylindrical coordinates, with cylindrical radius r

$$\rho(\vec{x}) = \lambda\, \delta(r) \cdot c$$

c chosen so $\int dA\, \rho(\vec{x}) = \lambda$

$$\lambda = \int 2\pi r\, dr\, c\, \delta(r) = \lambda \Rightarrow c = \frac{1}{2\pi r} \quad (!)$$

In spherical coordinates this is $r_\perp = r\sin\theta$



$$\frac{1}{2\pi r_\perp}\delta(r_\perp) = \frac{1}{2\pi r\sin\theta}\delta(r\sin\theta)$$

Problem - want to write in terms of $\cos\theta$

$$\lambda\, \delta(f(\cos\theta)) = \lambda \sum_j \frac{\delta(\cos\theta - \cos\theta_j)}{\left|\frac{df}{d\cos\theta}\right|_{\theta=\theta_j}}$$

$$f(\cos\theta) = r\sin\theta = r\sqrt{1-\cos^2\theta}$$

roots are $\cos\theta_j = \pm 1$, $\left|\frac{df}{d\cos\theta}\right| = r\left|\frac{\cos\theta}{\sin\theta}\right|$

$$\rho(x) = \frac{\lambda}{2\pi r \sin\theta}\left[\frac{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)}{r\,\frac{|\cos\theta|}{|\sin\theta|}}\right]$$

$\theta \sim \epsilon$ n $\pi - \epsilon$: $\sin\theta > 0$, $|\cos\theta| = 1$

$$\rho(x) = \frac{\lambda}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]$$

$$\rho(x) = \frac{\lambda}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \rho(x')\, G(x, x')\, d^3x'$$

$$G = 4\pi \sum_{\ell m} \frac{Y_\ell^m(\Omega')^* Y_\ell^m(\Omega)}{2\ell+1}\, r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

Angular $\int$: no φ dependence in $\rho \Rightarrow m = 0$

$$\cos\theta = \pm 1$$

$$Y_\ell^{m=0}(\Omega') = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta' = \pm 1)$$

$$\Phi(x) = \sum_\ell \left[P_\ell(1) + P_\ell(-1)\right] \frac{\lambda}{2\pi}\, \frac{2\pi}{4\pi\epsilon_0}\, P_\ell(\cos\theta) \quad *$$

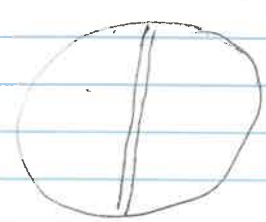
$$\cdot \int_0^b r'^2 dr' \underbrace{\frac{1}{r'^2}}_{\rho(r')\ \text{'s} = \frac{\lambda}{2\pi r^2}\{\delta(\ ) \cdots\}} \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)(r_<^\ell)$$

etc.

2 Physics questions

a) Why only even ℓ? (no dipole moment)

b) There is a line charge: shouldn't there be a $\Phi = \ln r$ term?



Yes!

$\Phi_{\ell=0}: \ 2 \cdot \frac{\lambda}{2\pi} \cdot \frac{1}{2\epsilon_0} \cdot 1 \times \int$

Set $\ell=0$ at the start in *

$$\Phi_{\ell=0} = \frac{\lambda}{2\pi\epsilon_0} \int_0^b dr' \left[\frac{1}{r_>} - \frac{1}{b} \right]$$

$$= \frac{\lambda}{2\pi\epsilon_0} \left\{ \left(\frac{1}{r} - \frac{1}{b} \right) \int_0^r dr' + \int_r^b dr' \left(\frac{1}{r'} - \frac{1}{b} \right) \right\}$$

$$\left\{ \left(\frac{1}{r} - \frac{1}{b} \right) r + \ln \frac{b}{r} - \frac{1}{b}(b-r) \right\}$$

the log

$$1 - \frac{r}{b} - 1 + \frac{r}{b} + \ln \frac{b}{r}$$

Messy results ...

you know $\Phi = 0$ at $r=b$

$$\Phi(r,\theta,\varphi) \simeq \frac{2\lambda}{4\pi\epsilon_0} \left[\ln \frac{b}{r} + \sum_{\ell=2,4 \ldots} \left[1 - \left(\frac{r}{b} \right)^\ell \right] \right.$$

$\times$ factors

$\times P_\ell(\cos\theta)$

n~~ote~~

An annoying derivation: there are exact results for dipoles. Suppose we have a localized $\rho(x)$ creating an $\vec{E}(x)$. Integrate $\vec{E}(x)$ over a sphere of radius R, ~~If all charge is outside~~ choose coord so center is at zero

a) If no charge inside R, $\frac{4\pi R^3}{3}\vec{E}(0) = \int_{r<R} d^3r\ \vec{E}(r)$

volume average $\vec{E} = \vec{E}$ at center

b) if all charge inside R $\quad -\frac{4\pi}{3}\frac{\vec{P}}{4\pi\epsilon_0} = \int_{r<R} d^3r\ \vec{E}(r)$

$\vec{p} =$ dipole moment measured with respect of center of sphere.

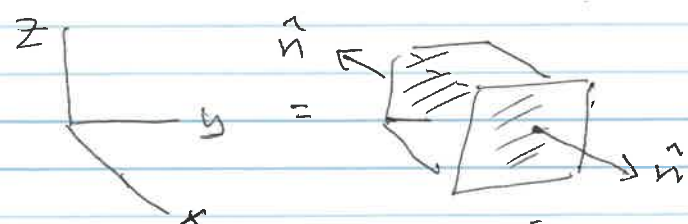
Proof begins with strange vector identity

$$\int \vec{E}\, d^3x = -\int \vec{\nabla}\Phi\, d^3x = -\int_{surface} (\Phi\,\hat{n})\, dA$$

~~Proof~~ Prove for a box

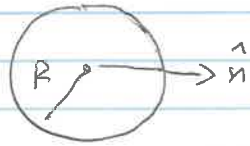
$$\int_{x_{min}}^{x_{max}} dx \int dy \int dz \left[\hat{i}\frac{\partial \Phi}{\partial x} + \hat{j}\frac{\partial \Phi}{\partial y} + \hat{k}\frac{\partial \Phi}{\partial z}\right]$$

$$= \hat{i}\int dy\, dz\, [\Phi(x_{max}, y, z) - \Phi(x_{min}, y, z)] + \cdots$$



look at it $\sim \int$ over 2 surfaces

On a sphere, $dA = R^2 d\Omega$, $\hat{n} = \frac{\vec{x}}{R}$



$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(x')}{|x-x'|}$$

$$\int \vec{E}\, d^3x = -\frac{R^2}{4\pi\epsilon_0}\underbrace{\int d^3x' \rho(x')}_{\equiv \Phi}\int_{\text{sphere}} d\Omega \frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$

$$= -\frac{R^2}{4\pi\epsilon_0}\int d^3x' \rho(x')\, I(x')$$

$$I(x') = \int_{r=R} d\Omega \frac{1}{|\vec{x}-\vec{x}'|}\left\{\hat{i}\frac{x}{R} + \hat{j}\frac{y}{R} + \hat{k}\frac{z}{R}\right\}$$

Do integral exploiting Y_ℓ^m's

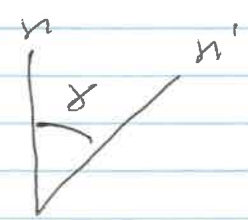
1) $\frac{x}{R} \sim Y_1^m(\Omega)$ $\left(Y_1^m \propto \frac{x+iy}{R}\right)$

2) $$\frac{1}{|x-x'|} = \sum_{\ell m}\frac{4\pi}{2\ell+1} Y_\ell^m(\Omega') Y_\ell^m(\Omega)^* \frac{r_<^\ell}{r_>^{\ell+1}}$$

3) Notice we are going to integrate over Ω, integrand is $Y_1^{m'}(\Omega) Y_\ell^m(\Omega)^* \to$ only $\ell = 1$ contributes!

4) For $\ell = 1$, easiest to use

$$\frac{1}{|x-x'|} = \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$



$$P_1(\cos\gamma) = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$I = \frac{r_<}{r_>^2} \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')\right]$$

$$\times \left[\hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta\right]$$

3 $\int$'s - I'll do the $\hat{k}$ one, write the answer

$$I_k = \frac{r_<}{r_>^2} \cdot 2\pi\cos\theta' \underbrace{\int_{-1}^{1} d\cos\theta \cos^2\theta}_{2/3}$$

$$I = \frac{4\pi}{3}\frac{r_<}{r_>^2}\left[\hat{i}\sin\theta'\cos\varphi' + \hat{j}\sin\theta'\cos\varphi' + \hat{k}\cos\theta\right]$$

$$= \frac{4\pi}{3}\frac{r_<}{r_>^2}\hat{n}' \quad (!)$$

$$\int_{r<R} \vec{E}\, d^3x = \frac{1}{4\pi\epsilon_0}\left(-\frac{4\pi}{3}R^2\right)\int d^3x\, \rho(x')\, \hat{n}'\frac{r_<}{r_>^2}$$

2 cases to consider. First, if all the charge is inside R then $r_< = r'$, $\hat{n}' r' = \vec{r}'$, $r_> = R$

$$\int \vec{E}\, d^3x = -\frac{R^2}{3\epsilon_0}\frac{1}{R^2}\int d^3x'\, \rho(x')\, \vec{r}'$$

$$= -\frac{\vec{P}}{3\epsilon_0}$$

2nd case – all charge is <u>outside</u> R so

$r_< = R$, $r_> = r'$

$$\int \vec{E}\, d^3x = \frac{4\pi R^3}{3}\left[-\frac{1}{4\pi\epsilon_0}\int d^3x'\, \rho(x')\,\frac{\hat{n}'}{r'^2}\right]$$

Recall $\vec{E}(x) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3x'\, \rho(x')\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$

so $$\int \vec{E}\, d^3x = \frac{4\pi R^3}{3}\,\vec{E}(x=0)$$

Now, why did I drag you through this?

1) these are exact formulas

2) our dipole $\vec{E} = -\vec{\nabla}\left(\dfrac{p\cdot r}{4\pi\epsilon_0 r^2}\right)$

~~does~~ does <u>not</u> obey them.

But this was an approximate formula anyway – "$r > r'$ ..."

We want to patch the dipole formula so that its integral agrees with the exact result.

exact: $\int \vec E \, d^3x = -\frac{p}{3\epsilon_0}$ or $\frac{4\pi R^3}{3}\vec E(x=0)$

$$* \quad \vec E(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat n(\vec p\cdot\hat n)-\vec p}{|\vec x-\vec x_0|^3} - \underbrace{\frac{4\pi}{3}\vec p\,\delta^3(\vec x-\vec x_0)}_{\text{patch}}\right]$$

$x_0 \equiv$ location of dipole

Note trouble as $x \to x_0$

If we naively integrate the first term over angles, not worrying about the singularity, we get zero

$\vec p(n) = 3\hat n(\vec p\cdot\hat n)-\vec p$ — pick $\vec p$ along z axis

$\to \hat z\cdot\vec p(n) = \Big[3\underbrace{(\cos\theta)}_{n\cdot z}\underbrace{(\cos\theta)}_{p\cdot n}-1\Big]p_0$

$\int E_z d^3x = \int_{-1}^{1} d\cos\theta\,[3\cos^2\theta-1] = 0$ and other $\int$'s $= 0$ also

Convention $\begin{cases} \langle\text{first term}\rangle = 0 \text{ always, don't think about singularity} \\ \int d^3x \text{ 2nd term} = -\frac{4\pi}{4\pi\epsilon_0}\frac{1}{3}\vec p = -\frac{\vec p}{3\epsilon_0} \end{cases}$

The * formula, for the field of a point dipole, satisfies our exact result

Similar magnetic dipole formula

This is never discussed (except in Jackson)

Alternative – dipole's extension must be monitored, can't think about microscopic dipole in $\int \vec E d^3x$