7310

Electricity and Magnetism

T. DeGrand - F319 - 2-8602

thomas.degrand@colorado.edu

Grading:

homework	250
midterm	100
final	150
	500

Homework out Wednesday due Friday 10 days later, one set per week @ (return to physics mailbox if you have one)

office hrs - Thurs afternoon, Weds afternoon - plus when you can find me

text: Jackson 3d ed. (of course). You can use an earlier one but
a) problem labels changed b) CGS vs MKS

Prerequisites: an undergrad E+M course - physical concepts should not be new to you - though the math might be

Also need mathematics background to solve problems involving vector calculus, separation of variables, orthogonal polynomials.

Physics 7310

T. DeGrand–office Gamow Tower F-319, tel 492-8602, (though the phone doesn't work)

email `thomas.degrand@colorado.edu`

Class meets MWF 1:25 to 2:15 in Duane G-125

Office hours – Wednesday –3:30-4 (between quantum and the colloquium) and and Thursday–1 to 5, plus when you can find me – NOT right before class or Friday morning, please!

I'll use the regular class web page

- `http://www-hep.colorado.edu/~degrand/p7310.html`

to get information out to you. I'll use Canvas as little as possible. "Secret" things will go there, if necessary. Everything public will be mirrored on the regular web page.

Grade:

- homework 250
- midterm 100
- final 150
- total 500 points

The midterm will be in the evening, 1 1/2 hours long, in mid October. We'll figure out a date later. The final exam is Tuesday 19 December, 430-7 PM in our classroom.

Homework will typically be given out on Wednesdays, due Friday of the next week. The questions will be posted on the class web page. The grader will probably be marking papers over the weekend and I will want to post solutions at some reasonable time after the Friday deadline, so keep to a schedule and negotiate with me IN ADVANCE if you feel you have to turn in something late. Homework solutions will be scanned onto the course web page.

The grader is Andrew Osborne `Andrew.osborne-1@colorado.edu`.

Text: Jackson, "Classical electrodynamics." I will not follow Jackson's order of topics when I think I can do better. I hope to end the first semester with cavities and wave guides.

Books I like, and will try to put on reserve in the Engineering-Math-Physics library include

- Zangwill, "Modern Electrodynamics." A new book, almost orthogonal in its approach to the more traditional Jackson. Check it out; there is some amazing stuff in there.

The book is notorious for the difficulty of its problems, and its tendency to treat non-obvious conclusions as self-evident"

Wikipedia

"~~[illegible]~~"

~~[illegible]~~ Alternate Texts

Zangwill - new, quite interesting - different emphasis

Landau & Lifshitz - Classical theory of Fields
Electrodynamics of continuous media

Also old texts - Panofsky + Phillips, (Smythe, Stratton)

Useful undergrad texts - Griffiths, Reitz Milford Christy
(but not too useful) - ~~[illegible]~~

You'll also want some good math methods books
Morse & Feshbach
Matthews & Walker
Stone & Goldbart
and tables of properties of special functions
Abramowitz & Stegun (tables obsolete, not formulas)
Nist web site

A good integral table ≤ Gradshteyn & Rhyzik

It is very useful to be able to look up properties of special functions in a venue which reveals more information than you seek.

Homework is mostly Jackson. Don't google it.

Work together. Start early!

Exams - open book ≡ clean copy of Jackson
+ integral table
midterm $1\frac{1}{2}$ hr in evening - date TBA
Allowing infinite resources would be too cruel!

Mechanics

Semester I - we will try to get through Ch 1-8 of Jackson

electrostatics (~ $\frac{1}{2}$ semester)

magneto statics

induction (not for ~~much~~ much!)

plane waves

wave guides & cavities - done badly

Semester II Ch 9 -12, 14 - plus other material

radiation (specified ρ, e, radiation, scattering, diffraction ...)

special relativity (reasonably high brow)

relativity + radiation

classical rel. field theory

-plus- very simple quantum theory of E & M field

So ... 1-12, 14, ++

Why are you taking this course?

1) To learn useful (and sometimes deep) things about classical E+M.

- all probes of natural phenomena use classical electromagnetism
- many macroscopic phenomena are dominantly electromagnetic
- Classical E+M is a "gauge theory" - not all DoF are independent - redundancies in variables - gauge theories describe all phys. in our world

2) To learn useful (and sometimes deep) things about systems which resemble classical E+M - this is really a course in "applied mathematical methods"

These systems include (not exclusive list!)

- other classical field theories (acoustics, hydrodynamics) the edge of GR
- Quantum Mechanics esp. wave mechanics
- Quantum Field Theory

Math methods include
Ordinary Differential Eqns
Partial " "

(lots of vector calculus!)

use of special functions (learn & forget!)

use of Green's function techniques: response to pt. source

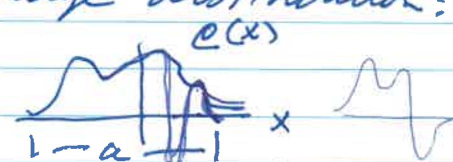
$$(\text{diff ops})\, G(z, z_0) = \delta(z - z_0) + \text{boundary cndn}$$

(A ubiquitous, powerful technique ... this is really the ~~only~~ one new thing you learn)

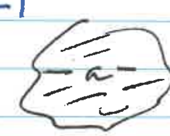
When you open Jackson, this all looks horrible - very high ambient noise level of mathematics. Few of us can deal with this easily. But if you look more closely, you notice - ~~no~~ ~~not always~~ bode is almost never exact solutions (real problems never have exact solutions) - it's really about finding approximations. Good approximations are usually powerful & subtle, not obvious at first. Finding them is an art. Good to learn about them in slightly risky environment

Example: multipole approximation - What is potential from some localized but arbitrary charge distribution?

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')\, d^3x'}{|\vec{x} - \vec{x}'|}$$



do the integral ... or think, first



Go to $x \gg a$. Details of ρ don't matter

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_{TOT}}{R} + \cdots \right]$$

$$\cdots = \frac{(\ )}{R^2} + \frac{(\ )}{R^3} + \cdots$$; each ( ) successively harder to find but successively less & less important at big R $\left(\sim \left(\frac{a}{R}\right)^p \frac{1}{R}\right)$

Finite # of orders is approximate but perhaps "good enough" - and you can compute how accurate it is.
(small expansion parameter a/R)

How to do this intelligently + efficiently is absolutely vital if you are going to do anything in ~~theoretical~~ physics - even if / especially if your interests are primarily experimental.

Best way to learn how to do it - to work hard problems.

and - no way around it, Jackson type problems are hard

However - they ~~do have answers~~ ~~some~~ can be presented as simple questions, and they do have answers which can be found in finite time.

This will not be true of the problems you will encounter in your research. (being presented with a simple question, which has a simple answer, is such a special occasion you can't let it go by)

Finally - being a physicist means that you have the ability - and the people you work with expect you to have the ability - to analyze physical systems and describe their behavior. You are not supposed to fear technical difficulties, and you are supposed to be able to do anything. ~~Specialization is for insects~~!

Classical E + M is a good place to build your skill set.

The equations of electrodynamics

Charge conservation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Maxwell's equations in MKS

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0} \qquad \mu_0 = 4\pi \times 10^{-7} \text{ (exact)} \rightarrow \frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$$

$$c = 299\ 792\ 458 \text{ m/s (exact)}$$

Lorentz force law

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The "right way to begin" is to start with a Lagrangian, derive these as eqns of motion --- next semester!

Units

Jackson has an appendix – conversion factors between conventions. I put some notes on line with a slightly better table (see below). For a lecture, better to step back and ask

~~They are human constructs~~!

Where do units come from? Why use one set vs another?

They are human constructs!

3 (at least) purposes for units

1) Produce order unity numbers in ~~calculations~~ typical calculations (helps physical intuition)

2) Build in physics constraints which are known to be true

3) Keep formalism simple

MKS is an example of (1)'s for human sized engineering, kg, sec, m ... Ampere. Also nice for freshman E&M: electrostatics, magnetostatics – ϵ_0, μ_0, eventually discover $c^2 = \frac{1}{\mu_0 \epsilon_0}$.

But that is 1860's physics – can't we move on?

And we do: $\frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$, c's in eqns.

You never need to know ϵ_0.

And now we are at item (2). What do you want to build in?

a) Always charge conservation $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

b) special relativity

(Automatic in Maxwell's eqns if coefficients chosen properly)

Different unit systems differ in

- Relation of charge/current to force
- Relative dimensionality of $\vec{E}$ + $\vec{B}$, $\vec{A}$ + Φ.

3 common systems for E + M

$$\vec{F} = q\vec{E} \quad \text{for all}$$

$$\vec{E} = k_1 \frac{q}{r^2}\hat{r}, \quad V = k_1 \frac{q_1 q_2}{r} \quad \Phi = k_1 \frac{q}{r}$$

MKS: $k_1 = \frac{1}{4\pi\epsilon_0}$: electric forces $\neq$ ordinary forces; charge its own unit

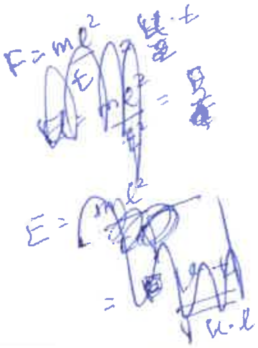
CGS $k_1 = 1$: $[q^2]$ = energy × length

Lorentz Heaviside $k_1 = \frac{1}{4\pi}$ $\quad \Phi = \frac{q}{4\pi r}$

used in QFT. Why? FT of $\frac{1}{4\pi r} = \frac{1}{k^2}$,
in perturbative QFT calc's, often work in k-space.

MKS $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \Rightarrow [B] = \frac{1}{v}[E]$

CGS $\quad = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}] \quad [B] = [E]$



These choices populate the individual Maxwell eqns, enforcing

- special rel
- $I = \frac{q}{t}$

More useful things: electron, proton have charge e

CGS $\quad \dfrac{e^2}{\hbar c} = \dfrac{\text{energy} \times \text{length}}{(\text{energy} \times \text{time}) \times \left(\dfrac{\text{length}}{\text{time}}\right)} = \text{pure } \#$

$$= \frac{1}{137}$$

L-H $\quad \dfrac{1}{4\pi} \dfrac{e^2}{\hbar c} = \dfrac{1}{137}$

MKS $= \dfrac{1}{4\pi\epsilon_0} \dfrac{e^2}{\hbar c} = \dfrac{1}{137}$

You rarely need to know 1.6×10^{-19} C

instead $\quad \hbar c = 1970$ eV-Å or 197 MeV-fm

$c = 3 \times 10^{8}$ m/s $= 3 \times 10^{16}$ Å/sec

Example of use of atomic units.

Why do batteries come in volts?

$$U \approx \frac{e^2}{r} \quad \text{in atom,} \quad r \sim a_0 \sim 1\,\text{Å}$$

$$U = \frac{e^2}{\hbar c}\,\frac{\hbar c}{a_0} = \frac{1}{137} \times \frac{1970\ \text{eV-Å}}{1\,\text{Å}}$$

$$= 10\ \text{eV}$$

$$= e \times 10\ \text{volts}$$

Jackson eds 1 & 2 are pure CGS

blue Jackson is MKS ~~to~~ ch 1-12
CGS after

Why? { build in physics constants known
{ keep formalism simple

Special relativity!

Under S.R. transformations

E + B mix

A & Φ mix

bundle them together: $(\Phi, \vec{A})$ = 4 vector

E & $B \longrightarrow F_{\mu\nu}$ (rank 2 tensor)

in fact re-label $A^\mu = (A^0, \vec{A})$ notation!

very annoying to have different ~~dimension~~ dimensionality for components!

Electrostatics

In the beginning there is Coulomb's law

<u>Static</u> point charge q_0 at $\vec{x}_0$ feels a force due to
<u>Static</u> point charge q_2 at x_2

$$\vec{F}(x_0) = k_1 q_0 q_2 \frac{(\vec{x}_0 - \vec{x}_2)}{|\vec{x}_0 - \vec{x}_2|^3}$$

$k_1 \equiv \frac{1}{4\pi\epsilon_0}$ in MKS

$\vec{F}(x_0) \equiv q_0 \vec{E}(x_0)$ defines electric field $\vec{E}$

Electrodynamics is <u>linear</u> - field of many point charges is sum of individual fields

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$

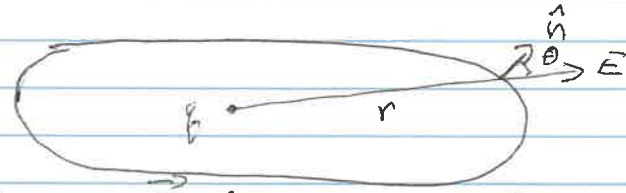
Take the limit of a continuously varying charge distribution

$$\Delta q(x') = \rho(x')\, d^3x'$$

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0} \int \rho(x') \frac{[\vec{x} - \vec{x}']}{|\vec{x} - \vec{x}'|^3}\, d^3x'$$

Only technical details remain! -- ~~alternative~~ differential alternative might be more useful.

Gauss' law



on the surface $\vec{E} \cdot \hat{n}\, dA = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \cos\theta\, dA$

note $\hat{n}$ = unit normal. Jackson writes $\frac{\partial \Phi}{\partial n} = \hat{n} \cdot \vec{\nabla} \Phi$

$\cos\theta \, dA = r^2 d\Omega$ - area of an equivalent spherical patch

$$\vec{E} \cdot \hat{n} \, dA = \frac{q}{4\pi\epsilon_0} d\Omega \frac{\cancel{r^2}}{\cancel{r^2}}$$

Integrate over the whole surface

$$\int_S E \cdot n \, dA = 4\pi \frac{q}{4\pi\epsilon_0} = \frac{q}{\epsilon_0}$$

q = total charge inside sphere

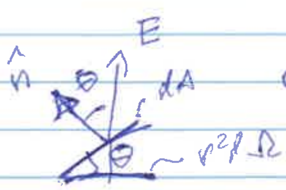
$\Longrightarrow$ Gauss' law

$$\int_{S'} \vec{E} \cdot \hat{n} \, dA = \frac{1}{\epsilon_0} \sum_i q_i = \frac{1}{\epsilon_0} \int_V \rho(x') \, d^3x'$$

Gauss' theorem $\int_{S'} \vec{E} \cdot \hat{n} \, dA = \int_V \vec{\nabla} \cdot \vec{E} \, d^3x'$

$$\int d^3x \left[\vec{\nabla} \cdot \vec{E} - \frac{1}{\epsilon_0} \rho(x) \right] = 0 \quad \text{for any volume}$$

or $\vec{\nabla} \cdot \vec{E}(x) = \frac{\rho(x)}{\epsilon_0}$



$\cos\theta \, dA = r^2 d\Omega$

$$E = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')\,(\vec{x}-\vec{x}')}{|x-x'|^3}\, d^3x' \qquad \nabla\cdot E = \frac{\rho}{\epsilon_0} \qquad E3$$

Electrostatic potential

use $\quad \dfrac{\vec{x}-\vec{x}'}{|x-x'|^3} = -\vec{\nabla}_x \dfrac{1}{|\vec{x}-\vec{x}'|}$

to write

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0}\cancel{\int \rho \ldots} \;\; \vec{E}(x) = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}_x \int \frac{\rho(x')\, d^3x'}{|x-x'|}$$

$$= -\vec{\nabla}\Phi(x)$$

Φ is the electrostatic ~~a~~ scalar potential.

$$\vec{\nabla}\times\vec{E} = 0 \quad \text{because} \quad \vec{\nabla}\times\vec{\nabla}\Phi = 0$$

$$E = -\nabla\Phi, \quad \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

Φ and work: transport a test charge from A to B

Work done moving the charge

$$W = -\int_A^B F\cdot dl = -\int_A^B q\,\vec{E}\cdot d\vec{l}$$

$$= \int_A^B q\,\nabla\Phi\cdot dl = q\,[\Phi(B) - \Phi(A)]$$

or $\quad \displaystyle\int_A^B \vec{E}\cdot d\vec{l} = -[\Phi(B) - \Phi(A)]$

$$\oint_{\text{closed path}} E\cdot dl = 0 \quad (B=A) = \int dA\,(\underbrace{\nabla\times E}_{0})\cdot n$$

$\Rightarrow$ one Φ, 3 (constrained) $\vec{E}$'s !

~~Second order derivatives of eqns~~

Second order equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad + \quad E = -\nabla \Phi$$

$$\left.\begin{array}{ll} \nabla^2 \Phi = -\dfrac{\rho}{\epsilon_0} & \text{Poisson's eqn} \\ \text{if } \rho = 0 \quad \nabla^2 \Phi = 0 & \text{Laplace's eqn} \end{array}\right\} \ast\ast\ast$$

$\ast\ast\ast$ "Electrostatics" is either of these,

plus boundary conditions - i.e.

Φ ⟹ Given region V with charge density ρ, boundary $\Phi(S)$ fixed, or $\hat{n}\cdot\vec{E}$

Let's push into the math before pulling back to $\boxed{\Phi = ?}$

physics - there is a general answer to Φ

It uses "Green's theorem," "Green's functions" - auxillary quantity

Derivation is roundabout

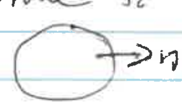
Start w/ Gauss law

$$\int_V \vec{\nabla} \cdot \vec{E} \, d^3x = \int_S \vec{E} \cdot \hat{n} \, dA$$

Call $\vec{E} = \phi \vec{\nabla} \psi$

LHS $\quad \nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla} \psi$

RHS $\quad \phi \vec{\nabla} \psi \cdot \hat{n} = \phi \dfrac{\partial \psi}{\partial n}$ derivative with direction of outer normal to V

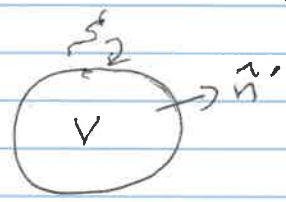


$$\int_V \left(\phi \nabla^2 \psi + \nabla\phi \cdot \nabla\psi\right) d^3x = \int_S dA\, \phi \frac{\partial \psi}{\partial n}$$

exchange ϕ & ψ

$$\int_V \left[\psi \nabla^2 \phi + \vec{\nabla}\psi \cdot \vec{\nabla}\phi\right] d^3x = \int dA\, \psi \frac{\partial \phi}{\partial n}$$

subtract, prime x & n



$$\int_V \left[\phi(x') \nabla'^2 \psi(x') - \psi(x') \nabla'^2 \phi(x')\right] d^3x'$$

$$= \int_{S'} dA' \left(\phi(x') \frac{\partial \psi}{\partial n'} - \psi(x') \frac{\partial \phi}{\partial n'} \right)$$

True for arbitrary ϕ, ψ — this is Green's th. [1.35]

Use in electrostatics — specific choice for ψ, ϕ

set $\phi(x') = \Phi(x')$, $\nabla^2 \Phi(x') = -\frac{\rho(x')}{\epsilon_0}$

$\psi(x') \equiv G(x, x')$, $\nabla^2 G(x, x') = -4\pi \delta^3(x - x')$

[pause — why 4π? What is $\delta^3(x - x')$? what is G? We will revisit] —

plug in —

$$\int_V d^3x' \left[\Phi(x') \left[-4\pi \delta^3(x - x')\right] - G(x, x') \frac{\rho(x')}{\epsilon_0} \right]$$

$$= \int_{S'} dA' \left[\Phi(x') \frac{\partial G(x, x')}{\partial n'} - G(x, x') \frac{\partial \Phi}{\partial n'} \right]$$

or

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho(x')\, G(x,x')\, d^3x' \qquad (1\text{-}36)$$

$$+ \frac{1}{4\pi}\int_{S'} dS' \left[G(x,x')\frac{\partial\Phi}{\partial n'} - \Phi(x')\frac{\partial G(x,x')}{\partial n'} \right]$$

Very general - but very cryptic! Let's look at special cases.

a) $\rho(x') = 0$: $\Phi(x = \text{interior})$ depends on Φ on bndy + $\partial\Phi/\partial n$
"boundary value problem"

b) suppose $\Phi \to 0$ as $x \to \infty$, only volume term is left and we recover

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \rho(x')\, G(x-x')\, d^3x'$$

where $\nabla^2 G(\vec{x}-\vec{x}') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

"obviously" -in this case- $G(\vec{x}-\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} \equiv \frac{1}{R}$ in Jackson

(we must show this)

For Jackson 1.10, just plug $\frac{1}{R}$ into 1.36 and proceed.

Summarize

- ~~general formulation for solution of~~

Potential problem: Solve $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$

subject to specified b.c.'s

Solution uses auxillary function $G(x, x')$

solution to $\nabla^2 G(x, x') = -4\pi \delta^3(\vec{x} - \vec{x}')$

with another set of specified b.c.'s

$G(x, x') \equiv$ "Green's function" for electrostatics

Math issues

1) I claim $G(x, x') = \frac{1}{|\vec{x} - \vec{x}'|} + F(x, x')$

where $\nabla^2 F = 0$. Is it?

2) What is $\delta^3(x - x')$?

3) What are allowed b.c.'s & useful b.c.'s?

Let's take a break from the math, do some physics - look at energy - come back

Electrostatic energy

If $\Phi(x) \to 0$ at infinity then a point charge q_i brought in from infinity to location x_i costs an amount of work

$$W_i = q_i \Phi(x_i)$$

Suppose Φ comes from $n-1$ point charges

$$\Phi(x_i) = \sum_{j=1}^{n-1} \frac{q_j}{4\pi\epsilon_0 |\vec{x}_i - \vec{x}_j|}$$

Total work to assemble all the charges one at a time

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{q_i q_j}{|x_i - x_j|}$$

each one sees the ones already there.

This is a sum over pairs

$$= W = \frac{1}{2} \sum_{\substack{\text{all } i \\ i \neq j}} \sum_{\text{all } j} \frac{q_i q_j}{|x_i - x_j|} \frac{1}{4\pi\epsilon_0}$$

Pass to continuum, be casual about the $i \neq j$

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x)\rho(x')}{|x - x'|} d^3x\, d^3x'$$

$$= \frac{1}{2} \int d^3x\, \rho(x) \Phi(x)$$

$$\rho = -\epsilon_0 \nabla^2 \Phi$$

$$W = -\frac{\epsilon_0}{2} \int \Phi \nabla^2 \Phi\, d^3x = \frac{\epsilon_0}{2} \int [\vec{\nabla}\Phi]^2 d^3x$$

(parts)

$$= \frac{\epsilon_0}{2} \int \vec{E}(x)^2 d^3x$$

$\Rightarrow$ interpret electrostatic energy as residing in E-field: energy density

$$w(x) = \frac{\epsilon_0}{2} \left| \vec{E}(x) \right|^2$$

$w(x) > 0$!

(note funny business w/ self energy $\rightarrow w(x) > 0$ — see Jackson example)

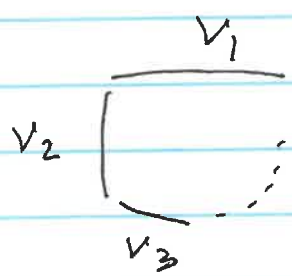
Capacitance, coeffs of capacitance –

$\sim \Phi = V_1$ ("equipotential surface")

$\sim \Phi = 0 \quad \Rightarrow \quad \vec{E}(x) \propto V_1$

$$W \propto V_1^2$$

$$\equiv \frac{1}{2} C V_1^2 \qquad C \equiv \text{capacitance}$$



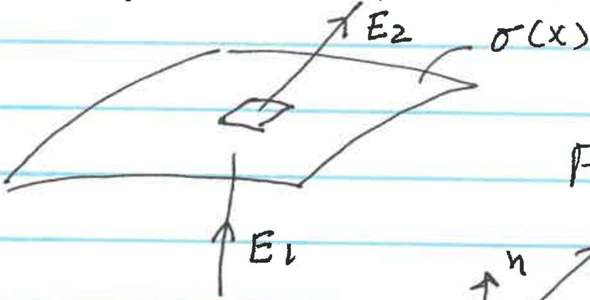
$$W = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j$$

($E = \sum$ terms for each)

$C_{ij} \equiv$ coefficients of capacitance

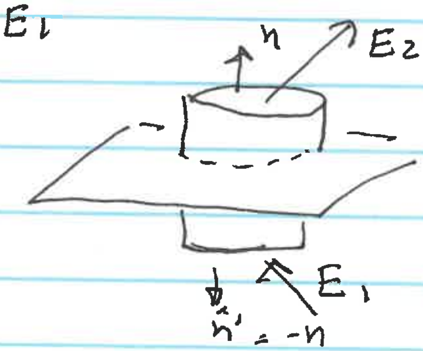
Surface charge & ~~Boundary Conditions~~ Continuity

only case you ever use is very simple - surface charge density. How does E change?



Familiar Gauss's law story:

1)



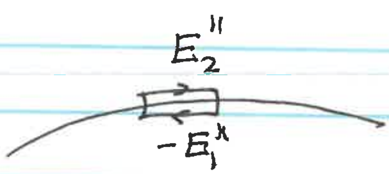
Gaussian pillbox

$$\int \vec{E}\cdot\hat{n}\, dA = (\vec{E}_2 - \vec{E}_1)\cdot\hat{n}\cdot A$$

$$= \frac{q}{\epsilon_0} = \sigma(x)\cdot\frac{A}{\epsilon_0}$$

$$\therefore \left[\vec{E}_2(x) - \vec{E}_1(x)\right]\cdot\hat{n} = \frac{\sigma(x)}{\epsilon_0}$$

2)



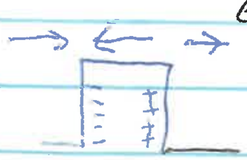
$$\nabla\times E = 0 \Rightarrow \oint E\cdot dl = 0$$

$$= E_2^{\parallel} - E_1^{\parallel}$$

$$\Rightarrow \begin{cases} \vec{E}_t & \text{continuous everywhere} \\ \vec{E}_n & \text{continuous everywhere that } \sigma = 0. \end{cases}$$

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(x')\, dA'}{|x - x'|}$$ is continuous everywhere

$$\lim_{\epsilon\to 0}\int_{x-\epsilon}^{x+\epsilon} E\cdot dl = 0$$ if E bounded $\Rightarrow \Phi(x-\epsilon) = \Phi(x+\epsilon)$



Not true for dipole layer but you never (!) encounter them

Back to Green's functions

Recall (if $\Phi \to 0$ as $x \to \infty$)

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x'$$

Check consistency w/ Poisson - is $\nabla^2 \Phi(x) = \frac{\rho(x)}{\epsilon_0}$?

Call $\vec{x} - \vec{x}' \equiv \vec{r}$

$$\nabla^2 \frac{1}{r} = \vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{r} \right) = -\vec{\nabla} \cdot \left[\frac{\vec{r}}{r^3} \right]$$

$$= -\left(\frac{\vec{\nabla} \cdot \vec{r}}{r^3} + \vec{r} \cdot \vec{\nabla} \frac{1}{r^3} \right)$$

$$= -\left(\frac{3}{r^3} - \vec{r} \cdot \frac{3\vec{r}}{r^5} \right) = 0 \quad \text{if } r \neq 0$$

$\nabla^2 \frac{1}{r} = 0$ if $r \neq 0$.

$$\nabla^2 \Phi(x) = \frac{1}{4\pi\epsilon_0} \int \rho(x') d^3x' \, \nabla^2 \frac{1}{|x-x'|}$$

Expect problems as $x - x' \to 0$ so split the integral into 2 parts

$$\int_{|x-x'| \text{ tiny}} d^3x' + \int_{\text{everything else}} d^3x'$$

In 2nd term $\nabla^2 \frac{1}{r} = 0$

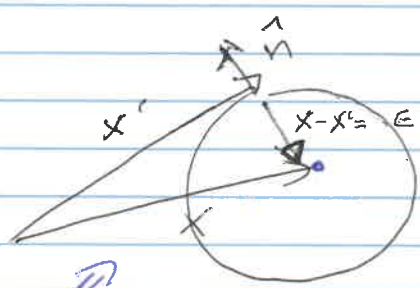
In first term write $\rho(x') = \rho(x) + \vec{r} \cdot \vec{\nabla} \rho + \cdots$
keep only first term

and $\int d^3x' = \int d^3(x'-x)$

Next, $\nabla^2 \frac{1}{|x-x'|} = \nabla'^2 \frac{1}{|x-x'|} = \nabla' \cdot \frac{(\vec{x}-\vec{x}')}{|x-x'|^3}$ 2 minus signs

$$\nabla^2 \Phi = \frac{1}{4\pi\epsilon_0}\, \rho(x) \int_{|x-x'|\ \text{small}} d^3(x'-x)\; \nabla' \cdot \frac{(\vec{x}-\vec{x}')}{|x-x'|^3}$$

$$= (\text{Gauss theorem}) = \frac{\rho(x)}{4\pi\epsilon_0} \int_{\substack{\text{little sphere} \\ \text{centered on } x}} dA' \, \frac{(\vec{x}-\vec{x}')\cdot\hat{n}}{|x-x'|^3}$$



$$\frac{(\vec{x}-\vec{x}')\cdot\hat{n}}{|x-x'|^3} = - \quad = (-)\left(\frac{\epsilon}{\epsilon^3}\right) = -\frac{1}{\epsilon^2}$$

$$dA' = \epsilon^2 \, d\Omega$$

$\vec{x}' + \vec{x} - \vec{x}' = \vec{x}$

$$\nabla^2 \Phi(x) = \frac{\rho(x)}{4\pi\epsilon_0}\left[(-1)\cdot\frac{4\pi\epsilon^2}{\epsilon^2}\right] = -\frac{\rho(x)}{\epsilon_0}$$

Yes, Poisson checks

By the way,

$$f(r) \equiv \nabla^2 \frac{1}{r} = 0 \quad \text{if } r \neq 0$$

$$\int \left(\nabla^2 \frac{1}{r}\right) d^3r = -4\pi$$

$$\therefore \quad \nabla^2 \frac{1}{r} = -4\pi\, \delta^3(\vec{r})$$

or $$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\, \delta^3(\vec{x}-\vec{x}')$$

Quick summary of properties of Dirac δ-function

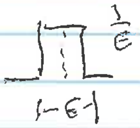
Ref: Lighthill, "Fourier analysis and generalized functions"

in 1-dimension $\delta(x-a) = 0$ if $x \neq a$

$$\int \delta(x-a) f(x)\, dx = f(a)$$

$\delta(x-a)$ – this spiky thing – is not a function, it is a "~~[illegible]~~ "distribution" – usually only has meaning ~~[illegible]~~ as a limiting form of some function (if you want to derive properties, you must use these limiting forms)

Examples of δ-functions



a) $$\delta(x-a) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\Theta\left(x-\left(a+\tfrac{\epsilon}{2}\right)\right) - \Theta\left(x-\left(a-\tfrac{\epsilon}{2}\right)\right) \right]$$

Not useful – cant differentiate it.

b) $$\lim_{\Lambda \to \infty} \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} e^{ik(x-a)}\, dk$$ ~~[illegible]~~ see QM class

c) $$\lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi}\,\epsilon} \exp\left[-\left(\frac{x-a}{\epsilon} \right)^2 \right]$$

Much more useful defs anice!

Many ~~identities~~ satisfying identities – the most useful is

$$\int f(x)\, \delta(g(x))\, dx = \sum_i \frac{f(x_i)}{\left| \frac{dg}{dx} \right|_{x_i}}$$

where $g(x_i) = 0$, note absolute values

~~Proof: use (c), change variables to $y = g(x)$ ...~~

~~Absolute value: $\delta(-x) = \delta(x)$~~

~~[crossed out]~~ check this w/ Gaussian

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}\,\epsilon} \exp\left[-\left(\frac{g(x)}{\epsilon}\right)^2\right] f(x)\,dx$$

At small ϵ exp very peaked at place where $g(x) = 0$

$$g(x) \simeq g'(x_i)\cdot(x - x_i)$$

$$f(x) \simeq f(x_i) + \cdots$$

Gaussian $\int$ gives $\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}\,\epsilon} \exp\left[-\frac{g'(x_i)^2(x-x_i)^2}{\epsilon^2}\right] \times \left(f(x_i) + \cdots\right)$

$$= \frac{f(x_i)}{|g'(x_i)|}$$

special case: $\int f(x)\,\delta(bx - c)\,dx = \frac{1}{b} f\left(\frac{c}{b}\right)$

Sometimes need $\int f(x)\,\delta'(x-a)\,dx = -f'(a)$

(do a parts $\int$.)

3-d δ-function $\delta^3(\vec{x} - \vec{a})$ defined so

$$\int d^3x\, f(\vec{x})\,\delta^3(\vec{x} - \vec{a}) = f(\vec{a})$$

In rectangular coordinates $\delta^3(\vec{x} - \vec{a}) = \delta(x - a_x)\,\delta(y - a_y) \times \delta(z - a_z)$

Watch out for Jacobians in other coordinate systems!

Back to solutions of Poisson eqn

Many possibilities for boundary conditions -

a) Dirichlet b.c. $\Leftrightarrow$ Φ specified on boundary
b) Neumann b.c. $\partial\Phi/\partial n$ " " "
c) mixed ; D ~~some~~ someplace, N someplace else
d) both (Cauchy b.c.)
e) periodic, antiperiodic (~~compact~~ compact spaces)
。。。

Dirichlet & Neumann specify Φ uniquely.
Proof: suppose 2 ... Φ_1 & Φ_2 exist.
~~Greens the~~ ... ρ/ϵ_0
Define ... Clearly $\nabla^2 U = 0$,
on S, $U = 0$ ... $) = 0$ ~~and~~

SKIP

Greens th. ... $\phi = \psi = U$ is 0

$$\int d^3x' \left[U \cancel{\nabla^2 U} + \nabla U \cdot \nabla U \right] = \int U \frac{\partial U}{\partial n} \, dS' = 0$$

$$\int d^3x' \left[\vec{\nabla} U\right]^2 = 0 \quad \text{but} \quad \left[\vec{\nabla} U\right]^2 \geq 0$$

so $\vec{\nabla} U = 0$ everywhere so $U =$ constant.
D: $U = 0$ on surface $\rightarrow U = 0$ everywhere
N: Φ unique up to an overall additive constant.

No Cauchy b.c. for $\nabla^2 \Phi = \rho/\epsilon_0$: D + N solutions are just different

If fine (don't ~~really~~ often encounter Neumann b.c.'s)

Neumann b.c. - might try to impose

$$\frac{\partial G_N}{\partial n}(x, x') = 0 \quad \text{for } x' \in S$$

but Gauss' law says

$$\int dV\, \nabla^2 G = -4\pi = \int_S dA \frac{\partial G}{\partial n}$$

is inconsistent with this desire - simplest thing to try is

$$\frac{\partial G_N(x, x')}{\partial n'} = -\frac{4\pi}{S}$$

[S = surface area]

Then

$$\Phi(x) = \underbrace{\frac{1}{S}\int dS' \, \Phi(x')}_{\text{constant}} + \frac{1}{4\pi\epsilon_0}\int \rho G \, dV$$

$$+ \frac{1}{4\pi}\int \frac{\partial \Phi}{\partial n} \cdot G \, dS'$$

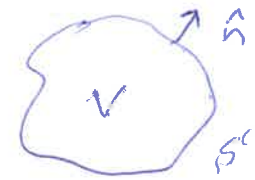
Not ~~so~~ automatic that $G_N(x, x') = G_N(x', x)$, can often impose this as a separate requirement

Return to magic formula

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V d^3x'\, G(x,x')\, \rho(x') + \frac{1}{4\pi} \int_{S'} dS' \left[G(x,x') \frac{\partial \Phi}{\partial n'} - \Phi(x') \frac{\partial G(x,x')}{\partial n'} \right]$$

n' = outward normal from inside V

$$\nabla^2 G(x) = -4\pi\, \delta^3(\vec{x} - \vec{x}')$$



Suppose your boundary value problem is Dirichlet - you want to specify Φ on S'. You can make a formal solution if you kill the first term: i.e. define $G_D(x,x') = 0$ for $x' = S'$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V d^3x'\, \rho(x')\, G_D(x,x') + \frac{1}{4\pi} \int_{S'} dS' \frac{\partial G_D(x,x')}{\partial n'} \Phi(x')$$

Very useful! Find G_D once ($\nabla^2 G = -4\pi\delta(x-x')$, $G_D = 0$ on S') $\leftarrow$ {$G = \Phi$ of pt charge at x', all boundaries grounded} solve poisson for Any ρ, Any $\Phi(S)$!

and G is mathematically unique - any way you solve it is fine!

Also, $G_D(x,x') = G_D(x',x)$: exchange source + sink

Finally, we are ready to start electrostatics!

Q: given $\rho(x)$ + boundary conditions on Φ, $\Phi(x)$

A1: direct solve of $\nabla^2\Phi = \frac{-\rho}{\epsilon_0}$ (separation of variables) best for $\nabla^2\Phi=0$

A2: find $G(x,x') \to \Phi$

2 approaches:

1) Exact solutions - essentially only method of images. Main example: Φ or G inside or outside spheres. All in Ch 2

This is ultimately a tragic story. We can't finish any useful calculation. Only do it to check later results

2) Approximate solutions presented as a series (expansion for short/long distances) - A mix of Ch 2, 3, 4. I will differ from Jackson, do everything first for problems w/ spherical geometry. (Pick variables so boundary conditions naturally separate - become 1-d constraints)

a) Spherical geometry

i) and azimuthal symmetry

ii) More general forms for G

iii) Multipoles

b) repeat, cylindrical geometry

c) repeat, cartesian geometry

Why? typically, geometry constrains solutions, as we go a) → b) → c) constraints weaken - more & more blind alleys open up.

2nd order ODE's (what you get after separation of variables) have 2 linearly independent soln's, ~~typically have~~ nice to have a good reason to discard one from the start (like, a solution is singular someplace

$P_\ell(z)$ & $Q_\ell(z)$ (legendre & associated legendre)

vs sin & cos, sinh & cosh ...

(Reasonably) Exact Solutions in 3-d

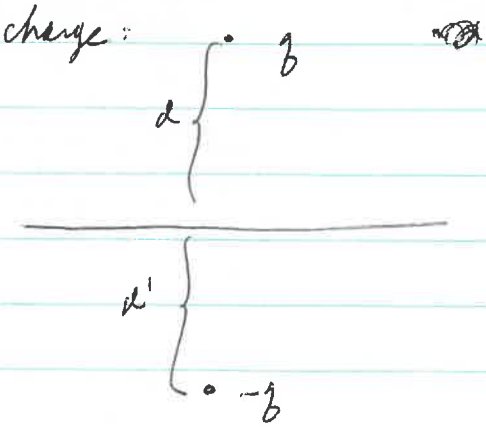
... begin with $G(x,x') = \dfrac{1}{|\vec{x}-\vec{x}'|}$ (1)

if the b.c. is $\Phi \to 0$ at $|x| \to \infty$. Other b.c. make this harder.

Solutions for
A number of b.c. can be ~~stated~~ found using (1) plus cleverness. The idea is to mock up the b.c. by adding unphysical point charges in unphysical regions. This is called the "method of images"

A pure art form!

The simplest example of an image problem is, of course, the point charge above a grounded conducting plane: the image is a charge of opposite strength under the true charge:



$\vec{x}' = (0,0,d)$

$x'' = (0,0,-d)$

want soln for $z>0$ only

$$G(x,x') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|}$$

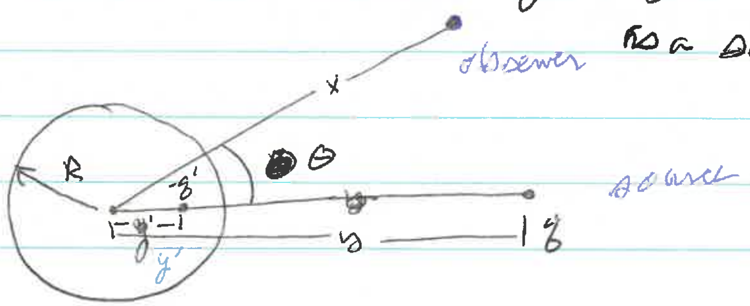
It's a standard undergrad problem to find the surface charge density on the plane, etc.

Notice - you can also find Φ in the upper half plane when $\Phi(x,y,z=0)$ is specified (and $\Phi \to 0$ as $r \to \infty$)

That is, put plane at $z=0$

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int dx'dy'\, \Phi(x',y',0)\left[-\frac{\partial}{\partial z'}G(\vec{r},r')\right|_{r'=(x',y',z'=0)}$$

A somewhat harder problem is a point charge ~~outside~~ a grounded conducting ~~plane~~ sphere of radius R (or inside it!) ~~The~~ Finding Φ will give us an exact Green's function for ~~either~~ Dirichlet ~~spherical~~ b.c. on a spherical surface (to be fair, other methods of solution will be more useful, we will use this soln as a sanity check)



Lets put the image $-q'$ inside R on a line connecting the center of the sphere to q.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} - \frac{q'}{|\vec{x}-\vec{y}'|}\right)$$

Defining θ as shown, and $|\vec{x}| = x$, $|\vec{y}| = y$
~~Putting the z axis along y~~

$$|\vec{x}-\vec{y}| = [x^2+y^2-2xy\cos\theta]^{1/2}$$
$$|\vec{x}-\vec{y}'| = [x^2+y'^2-2xy'\cos\theta]^{1/2}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2-2xy\cos\theta}} - \frac{q'}{\sqrt{x^2+y'^2-2xy'\cos\theta}}\right)$$

We want $\Phi(|x|=R)=0$: this is ~~(q'/q < 0 so flip~~ sign

$$\Phi(R,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left[\frac{R^2+y^2}{q^2} - \frac{2Ry}{q^2}\cos\theta\right]^{1/2}} - \frac{1}{\left[\frac{R^2+y'^2}{q'^2} - \frac{2Ry'}{q'^2}\cos\theta\right]^{1/2}}\right)$$

so we better have — cosine term

$$\frac{y}{q^2} = \frac{y'}{q'^2} \quad (a) \qquad \text{and} \qquad \frac{R^2+y^2}{q^2} = \frac{R^2+y'^2}{q'^2}$$

solve for q' + y'

A little ~~R~~-fiddling yields

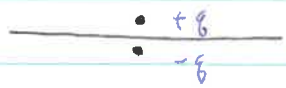
$$q' = \frac{R}{y} q \qquad y'y = R^2$$

check

$$\frac{q'^2}{q^2} = \frac{R^2}{y^2} = \frac{y'}{y} \quad (a) \quad ; \quad \frac{R^2+y'^2}{q'^2} = \frac{R^2 + R^4/y^2}{q^2 \frac{R^2}{y^2}} = \frac{y^2+R^2}{q^2}$$

Notice that as y approaches R, $q' = q$, $y' = y = R$ – planar geometry again (if $y = R+\epsilon$, $y' = R-\epsilon$)

sanity check!



Some interesting physics (for a change): what's the surface charge density?

$$\frac{\sigma}{\epsilon_0} = -\frac{\partial \Phi}{\partial n} = -\frac{\partial \Phi}{\partial x}\bigg|_{x=R}$$

($\hat{n}$ out of sphere — for $\vec{E}$, ~~points~~ in)

$$= \frac{q}{4\pi\epsilon_0}\left\{ \frac{x - y\cos\theta}{[\quad]^{3/2}} - \frac{R}{y}\cdot\frac{(x-y'\cos\theta)}{[\quad]^{3/2}} \right\}_{x=R}$$

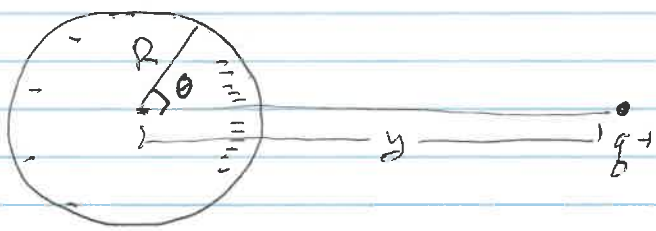
$$= \frac{q}{4\pi\epsilon_0}\left\{ \frac{R - y\cos\theta}{(R^2+y^2-2Ry\cos\theta)^{3/2}} - \frac{R}{y}\cdot\frac{(R - \frac{R^2}{y}\cos\theta)}{\left[R^2 + \frac{R^4}{y^2} - 2R\frac{R^2}{y}\cos\theta\right]^{3/2}} \right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{ \frac{R - y\cos\theta}{(R^2+y^2-2Ry\cos\theta)^{3/2}} - \frac{y^3}{R^3}\frac{R}{y}\frac{[R - \frac{R^2}{y}\cos\theta]}{(R^2+y^2-2Ry\cos\theta)^{3/2}}\right.$$

I don't do this on the board!

$$\sigma = \frac{q R \left[1 - \frac{y^2}{R^2} \right]}{\left[R^2 + y^2 - 2Ry\cos\theta \right]^{3/2}}$$

negative! $\quad y > R \quad$ bigger where $\cos\theta$ is +



~~note: this is~~ $\cancel{\frac{\partial G(x,x')}{\partial n}}$

It had better be true that

$$\int \sigma \, dA = -q' = -q \frac{R}{y}$$

Can check this by direct integration

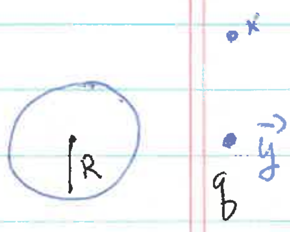
The Point charge is attracted to the sphere because of the opposite charge on the front face. The easiest way to find the force is from the image

$$F = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q q'}{(y-y')^2} \right. = \frac{-q^2}{4\pi\epsilon_0} \frac{R}{y} \frac{1}{\left[y - \frac{R^2}{y} \right]^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{R^2} \left(\frac{R}{y} \right)^3 \frac{1}{\left(1 - \frac{R^2}{y^2} \right)^2} \simeq \left[\begin{array}{ll} \frac{1}{y^3} & \text{at big } y \\ \frac{1}{(y^2-R^2)^2} \sim \frac{1}{[2R(y-R)]^2} & \text{at } y \text{ close in} \end{array} \right.$$

The potential for a point charge inside a grounded sphere is very similar - ~~It will be a homework problem~~. put the image charge outside.
There are a number of variations on this theme!

1) ~~potential~~ conducting, insulated sphere with total charge Q:
in the image problem, the sphere had a total charge $q' = -\frac{qR}{y}$. To give it a total charge Q, just put an image of charge $q'' = Q - q'$ at the center (conductor is an equipotential)



$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} - \frac{Rq}{y}\frac{1}{\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|} + \frac{Q + \frac{R}{y}q}{|\vec{x}|} \right)$$

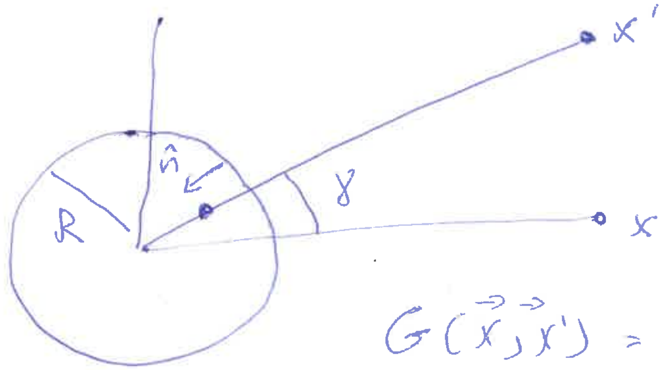
2) Conducting sphere at fixed potential V_0: this time put $V_0 = \frac{q''}{4\pi\epsilon_0 R}$ i.e. image q'' at center of sphere $q'' = 4\pi\epsilon_0 R V_0$

$$\Phi(\vec{x}) = (\text{first 2 terms}) + \frac{V_0 R}{|\vec{x}|}$$

3) For some reason textbooks like to solve the problem of a conducting sphere in a uniform E field, using images. This is done with 2 images, but the much more useful solution is in terms of Legendre polynomials!

Before going on — recall Green's fn equation & rewrite figure

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \rho(x')G(x,x')\,d^3x' + \frac{1}{4\pi}\int_S \Phi(x')\frac{\partial G(x,x')}{\partial n}\,ds' \qquad (1)$$

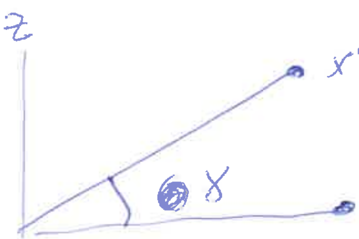


location of the source, before

q'/q

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{R}{x'}\frac{1}{\left|\vec{x} - \frac{R^2}{x'^2}\vec{x}'\right|}$$

This is where we hit the wall. Everything we have done for Φ picked a coordinate system with a preferred z-axis (in terms of θ before, γ now). In (1), we are integrating over x'. Let's ~~continue to~~ fill in the formula holding x fixed



$x' = (x'\sin\theta'\cos\varphi',\ x'\sin\theta'\sin\varphi',\ x'\cos\theta')$

$x = (x\sin\theta\cos\varphi,\ x\sin\theta\sin\varphi,\ x\cos\theta)$

$$x\cdot x' = xx'\left[\sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'\right]$$

$$\equiv xx'\cos\gamma$$

$$G = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x^2x'^2}{R^2}+R^2-2xx'\cos\gamma}}$$

$$\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial x'}\Big)_{x'=R}$$ is almost $\sigma_{/G_0}$ so easy

→ out of region where x lives

$$= \frac{-(x^2-R^2)}{R^2(x^2+R^2-2xR\cos\gamma)^{3/2}} \qquad (!)$$

Consider problem $\nabla^2\Phi = 0$, Φ specified on surface of sphere, want Φ outside sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int \Phi(R,\theta',\varphi')\frac{(x^2-R^2)}{R\left[x^2+R^2-2xR\cos\gamma\right]^{3/2}} R^2\, d\Omega'$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi').$$

$$d\Omega' = d\varphi'\, d\cos\theta'.$$

series of examples --

1) $\Phi = V_0$: should be trivial. Pick $\vec{x}$ along $\hat{z}$

then $\cos\gamma = \cos\theta'$ $(\theta=0)$

$$\Phi(x) = \frac{V_0}{4\pi}\cdot 2\pi \int d\cos\gamma \cdot \frac{(x^2-R^2)R}{\left[x^2+R^2-2xR\cos\gamma\right]^{3/2}}$$

$$= \frac{V_0}{2}(x^2-R^2)R\frac{2}{2xR}\left[\frac{1}{\left[x^2+R^2-2xR\cos\theta\right]^{1/2}}\right]_{-1}^{1}$$

$$= \frac{V_0}{2x}(x^2-R^2)\left[\frac{1}{|x-R|} - \frac{1}{x+R}\right] = \frac{V_0 R}{x} \text{ - OK}$$

$$\frac{x+R-x+R}{x^2-R^2}$$

$\Phi = \cancel{\Phi} =$ 

$$\Phi(x,\theta,\varphi) = \frac{V_0}{4\pi}\int_0^{2\pi} d\varphi' \left[\int_0^1 d\cos\theta' - \int_{-1}^{0} d\cos\theta'\right] \cdot \frac{R(x^2-R^2)}{\left[R^2+x^2-2xR\cos\gamma\right]^{3/2}}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

Change variables in 2nd $\int$ $\theta' \Rightarrow \pi-\theta'$

$\phi' \to \phi' + \pi$

$\cos\theta' \Rightarrow -\cos\theta'$

$\sin\theta' \to \sin\theta'$

$\cos(\varphi-\varphi') \to \cos(\phi-\phi')$

i.e. $\cos\gamma \to -\cos\gamma$

$$\Phi = \frac{V_0}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \times \left\{\frac{1}{\left[R^2+x^2-2Rx\cos\gamma\right]^{3/2}} - \frac{1}{\left[R^2+x^2+2Rx\cos\gamma\right]^{3/2}}\right\}$$

— ---??? stuck!

Last try: special case, $\theta = 0$ - directly "above"

N pole

again $\theta' = \gamma$

Integral is do-able though messy

$$\int_0^1 d\cos\gamma \, \frac{1}{[R^2+x^2-2Rx\cos\gamma]^{3/2}}$$

$$\vdots$$

$$\Phi(x) = V_0\left[1 - \frac{(x^2-R^2)}{x\sqrt{x^2+R^2}}\right]$$

check: at $x = R$ $\Phi = V_0$ - ok

At big x, Taylor expand,

$$\Phi = \frac{3}{2} V \frac{R^2}{x^2} + \cdots$$ much more useful!

Time to step back!