SET 8 – due 2 December

1. [30 points] Calculate the differential and total cross sections for e^+e^- annihilation into two massive scalar particles $\phi^+\phi^-$. You will have to fill in several intermediate steps: (a) [5] ϕ is a complex field. If you had two real fields, the kinetic part of the Lagrangian would be

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2. \tag{1}$$

Defining $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, convince yourself (and me) that the appropriately normalized kinetic part of the $\mathcal{L} = (\partial_\mu \phi)^*(\partial_\mu \phi)$. (b) [10] Next, you must figure out what is the conserved current. By considering a global gauge transformation $\phi \to (1 + i\alpha)\phi$ show that the Noether current is $J_\mu = -ie[\phi^+\partial_\mu\phi - \phi\partial_\mu\phi^+)$. As a final hint, make sure that your vertex for $\gamma \to \phi(k_1) + \phi(k_2)$ is proportional to $(k_1 - k_2)_\mu$. (c) [15] After completing the calculation, contrast the angular distribution, threshold behavior, and asymptotic energy behavior to that of $e^+e^- \to \mu^+\mu^-$.

2. [20 points] Srednicki (around p. 570) says that the effective Lagrangian for the decay of the π^0 meson to a pair of photons is

$$\mathcal{L} = \frac{e^2}{32\pi^2 f_\pi}\pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{2}$$

where f_π is a dimensionful constant, about 95 MeV. You can understand most of this when I tell you that the π^0 is a pseudoscalar field, and to make $\mathcal{L}$ a scalar, it couples to the electromagnetic field through $\vec{E} \cdot \vec{B}$, which is the tensor expression in Eq. 2. The $1/f_\pi$ just makes the dimensions correct. Find the decay rate Γ for $\pi^0 \to \gamma\gamma$. Srednicki says that the decay amplitude is

$$T = \frac{e^2}{4\pi^2 f_\pi}\epsilon^{\mu\nu\rho\sigma} k_{1\mu}\epsilon_{1\nu} k_{2\rho}\epsilon_{2\sigma} \tag{3}$$

(I think there is a missing i...) and

$$\Gamma = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2}. \tag{4}$$

To get this, you'll need

- The formula for $d\Gamma$ from Set 6

- A factor of 1/2 from two identical photons in the final state (plus potentially other 2's...)
- The magic identity

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} = -2[\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}] \tag{5}$$

- and the Feynman gauge polarization sum trick

$$\sum_{\lambda} \epsilon_{\mu}(\lambda)\epsilon^{*}_{\nu}(\lambda) = -g_{\mu\nu} \tag{6}$$

1) $e^+e^- \to \phi^+\phi^-$

To begin: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 + \cdots$

so if $\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}$, $\partial_\mu \varphi = \frac{1}{\sqrt{2}}(\partial_\mu \varphi_1 + i\partial_\mu \varphi_2)$

and $(\partial_\mu \varphi)^\dagger(\partial^\mu \varphi) = \frac{1}{2}\left[(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2\right]$.

The Noether current arises from the (lack of) variation of $\mathcal{L}$ under $\varphi \to \varphi' = e^{i\alpha}\phi$ or $\delta\varphi = i\alpha\varphi$. Equivalently

$$(\varphi_1' + i\varphi_2') = (1 + i\alpha)(\varphi_1 + i\varphi_2)$$

$$\delta\varphi_1 = -\alpha\varphi_2$$

$$\delta\varphi_2 = \alpha\varphi_1$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\delta(\partial_\mu\varphi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\delta(\partial_\mu\varphi_2)$$

$$= (\partial^\mu\varphi_1)(-\alpha\partial_\mu\varphi_2) + (\partial^\mu\varphi_2)(\alpha\partial_\mu\varphi_1) = 0$$

The equation of motion is $\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)} = \frac{\partial\mathcal{L}}{\partial\varphi_i}$ so

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta(\partial_\mu\varphi_i) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i$$

$$= \sum_i \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\delta\varphi_i\right]$$

and the Noether current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\delta\varphi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\delta\varphi_2,$$

or $J^\mu = -(\partial^\mu \varphi_1)\varphi_2 + (\partial^\mu \varphi_2)\varphi_1$ up to an overall factor. In terms of φ,

$$\left.\begin{aligned} \varphi &= \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \\ \varphi^* &= \frac{\varphi_1 - i\varphi_2}{\sqrt{2}} \end{aligned}\right\} \qquad \begin{aligned} \varphi_1 &= \frac{\varphi + \varphi^*}{\sqrt{2}} \\ \varphi_2 &= \frac{\varphi - \varphi^*}{i\sqrt{2}} \end{aligned}$$

$$\begin{aligned} J^\mu &= \frac{1}{2i}\Big\{ -(\partial^\mu \varphi + \partial^\mu \varphi^*)(\varphi - \varphi^*) \\ &\qquad + (\partial^\mu \varphi - \partial^\mu \varphi^*)(\varphi + \varphi^*) \Big\} \\ &= \frac{1}{i}\left[\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^* \right]. \end{aligned}$$

The interaction term is $J^\mu A_\mu = A_\mu \left[\varphi^*(k_1) k_2^\mu \varphi(k_2) - \varphi^*(k_1) k_1^\mu \varphi(k_2) \right]$

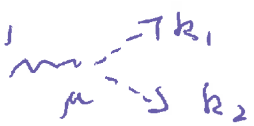
associating k_1 with φ^*, k_2 with φ.

We can get this from

$$(D^\mu \varphi)^* (D_\mu \varphi) = (\partial^\mu \varphi^* + ieA^\mu \varphi^*)(\partial_\mu \varphi - ieA_\mu \varphi)$$

$$\to \left[ik_1^\mu \varphi^*(k_1) + ieA^\mu \varphi(k_1)^* \right] \times \left[ik_{2\mu} \varphi(k_2) - ieA_\mu \varphi(k_2) \right]$$

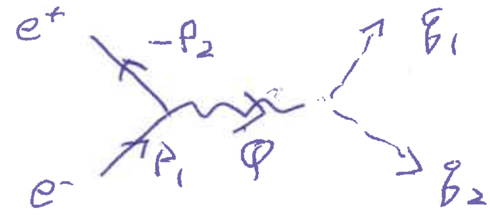
so $\mathcal{L}^{int} = eA_\mu \varphi^*(k_1) \varphi(k_2) [k_2 - k_1]^\mu$

The vertex is  $= -ie[k_2 - k_1]^\mu$

($-ie$ from Dyson series.) There is also a vertex

$e^2 A^2 \varphi^2$:  as in the Schrodinger eqn.

$e^+e^- \to \phi^+\phi^-$ is



$$\mathcal{M} = [-ie]^2\, \bar{v}(p_2)\gamma_\mu u(p_1)\left[\frac{-i g^{\mu\nu}}{Q^2}\right](q_1-q_2)_\nu$$

Define $k = q_1 - q_2$

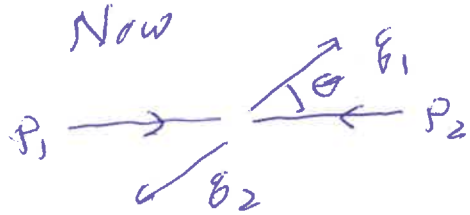
$$\mathcal{M} = \frac{(-ie)^2}{Q^2}\, \bar{v}(p_2)\not{k}\, u(p_1)$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4Q^4}\,\mathrm{Tr}\, \not{p}_2 \not{k} \not{p}_1 \not{k} \qquad (\text{if } m_e \approx 0)$$

$$= \frac{e^4}{4Q^4}\cdot 4 \cdot \left\{ 2\, p_1\cdot k\; p_2\cdot k - p_1\cdot p_2\, k^2 \right\}$$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\cdot \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

Now



$$p_1 = \frac{Q}{2}(1, 1, 0, 0)$$
$$p_2 = \frac{Q}{2}(1, -1, 0, 0) \qquad \therefore p\cdot q = \frac{Q^2}{2}$$

$$q_1 = \frac{Q}{2}(1, v\cos\theta, v\sin\theta, 0)$$
$$q_2 = \frac{Q}{2}(1, -v\cos\theta, -v\sin\theta, 0)$$

$q_1^2 = \frac{Q^2}{4}(1-v^2) = m^2$ so $1-v^2 = \frac{4m^2}{Q^2}$, $v^2 = 1-\frac{4m^2}{Q^2}$

$k = Q(0, v\cos\theta, v\sin\theta, 0)$ so $k^2 = -Q^2 v^2$

$t = (p_1-q_1)^2 = -2\frac{Q^2}{4}(1 - v\cos\theta) + m^2$, so $dt = \frac{Q^2}{2} v\, d\cos\theta$,

$p_1\cdot k = -p_2\cdot k = \frac{Q^2}{2} v\cos\theta$

so putting all the pieces together,

$$d\sigma = \frac{1}{16\pi Q^4}\left[\frac{Q^2 v}{2}\, d\cos\theta\right]\frac{1}{4}\sum |m|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{v}{32\pi Q^2}\frac{(4\pi\alpha)^2}{Q^4}\left\{2P_1\cdot k\, P_2\cdot k - (P_1\cdot P_2)k^2\right\}$$

$$= \frac{\pi\alpha^2 v}{2Q^6}\left\{-2\frac{Q^4}{4}v^2\cos^2\theta + \frac{Q^4 v^2}{2}\right\}$$

$$= \frac{\pi\alpha^2}{4Q^2}v^3\left[1-\cos^2\theta\right]$$

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4Q^2}v^3\sin^2\theta}$$

Note the differences with $e^+e^- \to f^+f^-$

- $\frac{d\sigma}{d\cos\theta} \sim \sin^2\theta$, not $1+\cos^2\theta$
- $\sigma \sim v^3$ at threshold, not v

$$\int_{-1}^{1} d\cos\theta\left[1-\cos^2\theta\right] = 2-\frac{2}{3} = \frac{4}{3}$$

$$\sigma = \frac{\pi\alpha^2}{4Q^2}\cdot\frac{4}{3} = \frac{\pi\alpha^2}{3Q^2} = \frac{1}{4}\sigma(e^+e^-\to f^+f^-)$$

this is at $Q \gg m$ of course. A naive explanation for the $\frac{1}{4}$ is that here there is only one spin state, not $2\times 2 = 4$ for the two fermions

2) $\pi^0 \to \gamma\gamma$: $\mathcal{L} = \dfrac{e^2}{32\pi^2 f_\pi} \, \pi^0 \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$

$\mathcal{L}_I \sim$  $= - \mathcal{H}_I$

$$T = \frac{-ie^2}{32\pi^2 f_\pi} \langle 0 | a_{k_1\epsilon_1} a_{k_2\epsilon_2} \, \mathcal{L}_I \, b^\dagger_{\pi^0} | 0 \rangle$$

$$F_{\mu\nu} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \Big[(ik_\mu \epsilon_\nu - ik_\nu \epsilon_\mu) a_{k\epsilon} e^{ik\cdot x} + h.c.$$

(this is just $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$).

Our first factor of two comes from the field contractions

$$\langle 0 | a_{k_1\epsilon_1} a_{k_2\epsilon_2} \cdots a^\dagger_{q_a \epsilon_a} a^\dagger_{f_b \epsilon_b} \cdots | 0 \rangle$$

is one, and there is the other order. This gives

$$T = \frac{-ie^2}{32\pi^2 f_\pi} \cdot 2 \cdot (i)^2 \, \epsilon^{\mu\nu\rho\sigma} [k_{1\mu}\epsilon_{1\nu} - k_{1\nu}\epsilon_{1\mu}] \times [k_{2\rho}\epsilon_{2\sigma} - k_{2\sigma}\epsilon_{2\rho}]$$

$$= \frac{ie^2}{16\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} k_{1\mu}\epsilon_{1\nu} k_{2\rho}\epsilon_{2\rho} \times 4$$

using the antisymmetry properties of the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$.

$$T = \frac{ie^2}{4\pi^2 f_\pi} \times \cdots .$$

Next, we clean up the phase space integral

$$d\Gamma = \frac{|T|^2}{2M} \frac{d^4k_1}{(2\pi)^3} \frac{d^4k_2}{(2\pi)^3} (2\pi)^4 \delta^4(P-k_1-k_2) \times \delta(k_1^2)\,\delta(k_2^2)$$

where $M = m_{\pi^0}$. This is

$$d\Gamma = \frac{|T|^2}{2M} \frac{d^3k_1}{2k_1^0} \frac{\delta\left((P-k_1)^2\right)}{(2\pi)^2}$$

We ~~construct~~ evaluate the kinematics in the rest ~~~~ frame of ~~the~~ the π^0

$$P = (M, \vec{0})$$

$$k_1 = \omega(1, \hat{n}) \;:\; \hat{n} = \text{direction of photon}.$$

We already know $k_1^2 = 0 \ldots$

$$\frac{d^3k_1}{2k_1^0} = \frac{\omega\, d\omega}{2} d\Omega$$

$$d\Gamma = \frac{|T|^2}{16\pi^2 M} \omega\, d\omega\, d\Omega\, \delta(M^2 - 2M\omega)$$

The δ-function tells us $\omega = \frac{M}{2}$, of course - ~~~~

$$\frac{d\Gamma}{d\Omega} = \frac{|T|^2}{16\pi^2 M} \cdot \frac{M}{2} \cdot \frac{1}{2M} = \frac{|T|^2}{64\pi^2 M}$$

There is another $\frac{1}{2}$ for 2 identical bosons in the final state.

And if $|T|^2$ is angle-independent, (it will be, after we sum final state polarizations), $\int d\Omega = 4\pi$ and

$$\Gamma = \frac{|T|^2}{64\pi^2 M} \cdot \frac{1}{2} \cdot 4\pi = \frac{|T|^2}{32\pi M}.$$

Back to $|T|^2$ - $e^2 = 4\pi\alpha$

$$\sum_{\lambda_1,\lambda_2} |T|^2 = \sum_{\lambda_1,\lambda_2} \frac{(4\pi\alpha)^2}{16\pi^4 f_\pi^2} \left(\epsilon^{\mu\nu\rho\sigma} k_{1\mu} \epsilon_{1\nu} k_{2\rho} \epsilon_{2\sigma}\right)$$

$$\times \left(\epsilon^{\alpha\beta\gamma\delta} k_{1\alpha} \epsilon^*_{1\beta} k_{2\gamma} \epsilon^*_{2\delta}\right)$$

The prefactor is $\frac{\alpha^2}{\pi^2 f_\pi^2}$. Call the rest R.

We sum polarizations with the Feynman gauge trick

$$\sum_\lambda \epsilon_\mu(\lambda) \epsilon_\nu^*(\lambda) = -g_{\mu\nu}$$

$$R = \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} k_{1\mu} k_{2\rho} k_{1\alpha} k_{2\beta}\, g_{\nu\beta}\, g_{\sigma\delta}$$

$$= \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\nu\gamma\sigma} k_{1\mu} k_{2\rho} k_1^\alpha k_2^\gamma .$$

I have raised + lowered indices to prepare to use the last magic identity. I'll permute indices to line everything up...

$$R = \epsilon^{\nu\sigma\rho\mu}\epsilon_{\nu\sigma\gamma\alpha} k_{1\mu} k_{2\rho} k_1^{\alpha} k_2^{\gamma}$$

$$= -2\left[\delta^{\rho}{}_{\gamma}\delta^{\mu}{}_{\alpha} - \delta^{\rho}{}_{\alpha}\delta^{\mu}{}_{\gamma}\right] k_{1\mu} k_{2\rho} k_1^{\alpha} k_2^{\gamma}$$

$$= -2\left[k_2^2 k_1^2 - (k_1 \cdot k_2)^2\right]$$

$$= 2(k_1 \cdot k_2)^2 \quad \text{since } k_1^2 = k_2^2 = 0$$

And with $(k_1 + k_2)^2 = M^2 = 2 k_1 \cdot k_2$

$$\Gamma = \frac{\alpha^2}{\pi^2 f_\pi^2} \frac{M^4}{2} \cdot \frac{1}{32\pi M}$$

$$= \frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2} \quad (!)$$

and the units check ...