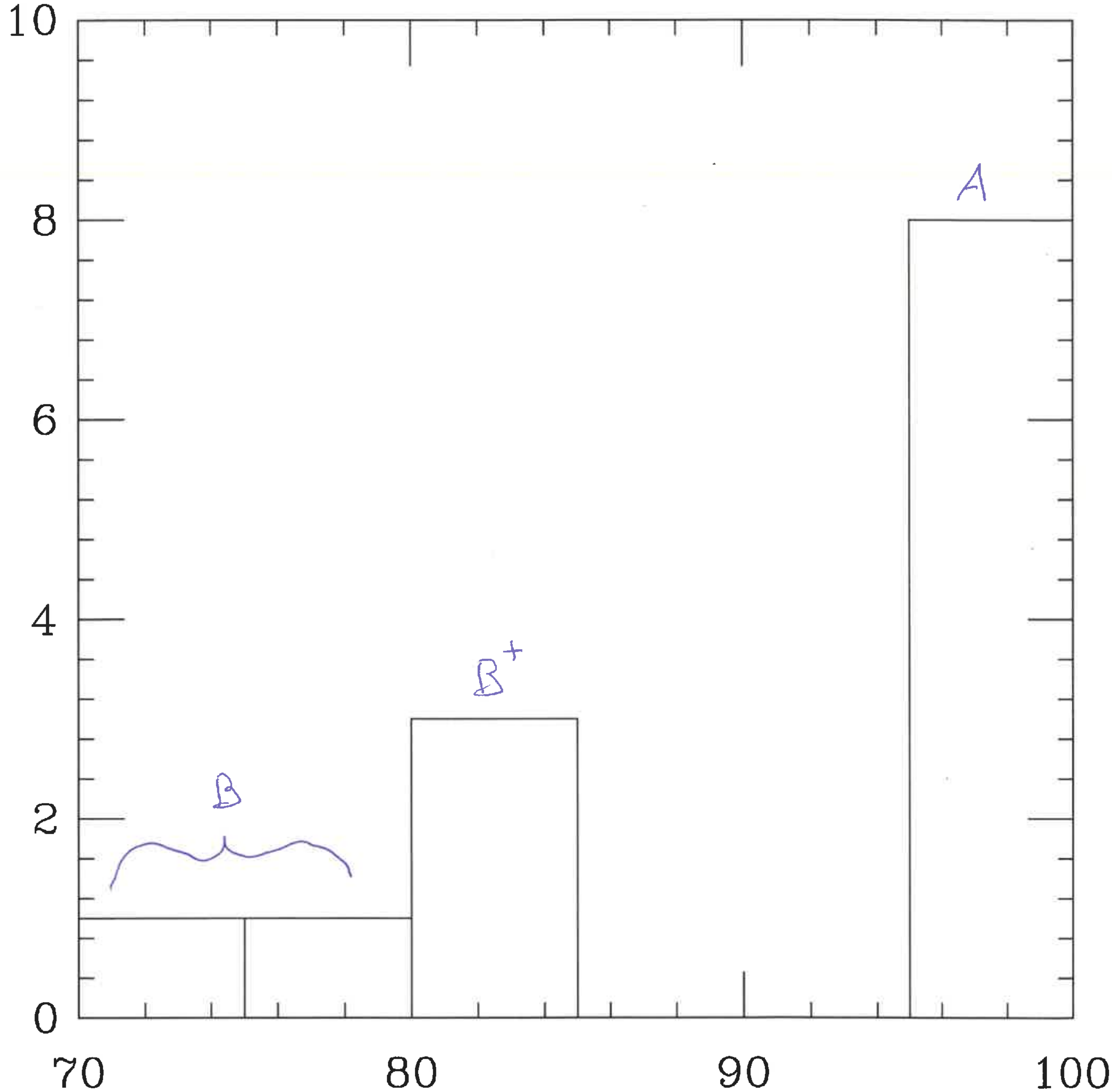
Fall 2025 7270

points: 80% homework
20% take home final



Set 9 – or Take Home Final – due Thursday December 11, 7 PM

This problem set is to be done by yourself, with no assistance from or discussion with others. You may ask or email me clarifying questions. Permitted materials are those from this class: your notes, my lecture notes, Srednicki's textbook, your homeworks and the solutions. You may use the Appendices in Peskin and Schroeder or Schwartz for Dirac or tensor identities (but please tell me when you do this). No other resources are permitted (no other books, notes from another class, Wikipedia, etc). A software package like Mathematica is permitted for simple mathematical operations like evaluating an integral. Please indicate when you have done so. No Feynman graph software packages are allowed.

1) [10 points] Suppose we have a (massless) theory with a beta function with a linear zero,

$$s\frac{d\lambda}{ds} = a(\lambda^* - \lambda) \tag{1}$$

where $p \to sp$ so $s > 1$ carries us into the ultraviolet and $s < 1$ carries us into the infrared. In class I argued that for this beta function, $a > 0$ corresponds to an ultraviolet attractive fixed point and $a < 0$ corresponds to an infrared attractive fixed point. Integrate the β function, to find the running coupling $\lambda(s)$, and check that what I said is consistent in the two cases.

2) [10 points] If $\beta(\lambda) = -a\lambda^2$ (with $a > 0$), a system has an UV attractive fixed point. Suppose $\gamma_\phi = \lambda\gamma'_\phi$. Compute $\Gamma^{(n)}(sp, m = 0, \lambda, \mu)$ deep in the ultraviolet (that is, for very large s), show it scales like $s^{d_n}(\ln s)^Q$, and find Q.

Comments: Problem (1) is the situation for a simple second order phase transition in condensed matter physics. Problem (2) is the situation for asymptotic freedom, like QCD: note how, deep in the UV, $\Gamma(p)$ is given by naive free-field power counting, that is the s^{d_n}, which is only weakly altered by the logarithm. This is the origin of "scaling" in the strong interactions – at short distances, QCD behaves like a free theory of quarks and gluons, up to logarithmic corrections.

3. [20 points] In the last problem set you computed the decay width of the π^0 meson to a pair of photons. The Higgs is similar, but since it is a scalar, the effective Lagrangian is

$$\mathcal{L} = e^2 C H F_{\mu\nu}F^{\mu\nu} \tag{2}$$

where H is the Higgs field and C is a constant (with an interesting story, for later). Find the Higgs decay width to two photons, in terms of C.

4. [40 points] A simple model for dark matter, called the "Higgs portal," is that dark matter is a new real scalar field S which couples to the rest of the Standard Model through the Higgs boson H. One realization of this scenario uses the combined Higgs and S Lagrangian

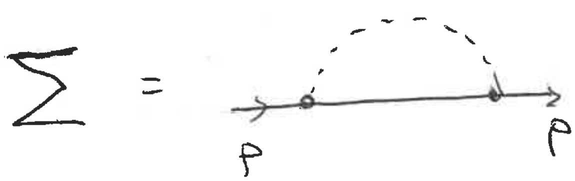
$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2}m_H H^2 + \frac{1}{2}(\partial_\mu S)(\partial^\mu S) - [\frac{1}{2}\mu_S^2 S^2 + \lambda_S S^4 + \frac{\lambda_3}{2}(H + v_H)^2 S^2] \tag{3}$$

(plus terms not relevant to this problem), where M_H is the Higgs mass, 125 GeV, λ_S and λ_3 are dimensionless couplings, and v_H is the vacuum expectation value of the Higgs, $v_H \sim 246$ GeV. (a) [3 points] What is the mass of the S particle, m_S? (b) [2 points] What are the interaction vertices between H and S? (c) [20 points] "Indirect detection"of dark matter involves seeing the annihilation of S in outer space by observing its decay products. Calculate the annihilation cross section for a pair of S's to convert to a fermion - antifermion pair, $SS \to \bar{f}f$. Work in the SS center of mass frame. Recall that the Higgs coupling to fermions is $(m_f/v_H)H\bar{f}f$ where m_f is the fermion mass. Nobody knows what the mass of a dark matter particle is, so make no approximations neglecting m_S, m_f or m_H. (d) [15 points] If the S is light enough, the decay $H \to SS$ could occur at the LHC. In the Standard Model, the ratio of Higgs total width to mass $\Gamma_H/M_H \sim 4 \times 10^{-5}$, and this is observed (indirectly) to be true to a nominal ten to twenty percent. Compute $\Gamma(H \to SS)$ and find a (rough) constraint on λ_3 such that the branching ratio for this decay is less than ten per cent.

5. [20 points] Consider the so-called "chiral rotation" of a free Dirac spinor ψ,

$$\psi \to \exp(i\epsilon\gamma_5)\psi \tag{4}$$

where γ_5 is the Dirac matrix which is the projector onto the chiral basis. (a) [2 points] What is the corresponding transformation of $\bar{\psi}$? (b) [5 points] Show explicitly that the massless Dirac Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$ is invariant under this rotation. (c) [5 points] What is the associated Noether current J_μ? (d) [3 points] If the fermion mass is nonzero, the Noether current is not conserved. Show that the size of its nonconservation is proportional to m. (e) [5 points] Now imagine a Yukawa theory, a free Dirac fermion ψ and a scalar ϕ, with an interaction term $\mathcal{L} = g\bar{\psi}\psi\phi$. The fermion mass is shifted by the self energy graph $\Sigma(p)$ shown in the figure, the mass shift is $m \to m - \Sigma(p = 0)$. Show that $\Sigma(p = 0)$ is proportional to m: chiral symmetry "protects" $m = 0$. You don't have to compute Σ itself.



1) $s\frac{d\lambda}{ds} = a(\lambda^* - \lambda)$. This is for $p \to sp$, so

$s > 1$ carries us into the UV, $s < 1$ into the IR

a) $a\int_{s=1}^{s}\frac{ds'}{s'} = \int_{\lambda(1)}^{\lambda(s)}\frac{d\lambda}{\lambda^*-\lambda}$ so $a\ln s = \ln\frac{\lambda^*-\lambda(1)}{\lambda^*-\lambda(s)}$.

This is $\lambda^* - \lambda(1) = [\lambda^* - \lambda(s)]\, s^a$

or $\lambda^* - \lambda(s) = s^{-a}[\lambda^* - \lambda(1)]$

or $\lambda(s) = \lambda^* - s^{-a}[\lambda^* - \lambda(1)]$

or $\lambda(s) - \lambda(1) = [\lambda^* - \lambda(1)][1 - s^{-a}]$

If $a > 0$ then as $s \to \infty$ the RHS becomes $\lambda^* - \lambda(1)$ and this says $\lambda(s)$ approaches λ^* - this is a UV attractive fixed point

If $a < 0$, it's easiest (for me) to see what happens by setting $a = -b$ and $s = 1/t$. $t \to \infty$ is flow to the IR. $1 - s^{-a} = 1 - \left(\frac{1}{t}\right)^b \to 1$ and

$\lambda(s = \frac{1}{t}) - \lambda(1) = \lambda^* - \lambda(1)$ or

$\lambda(s = \frac{1}{t}) = \lambda^*$. This is an IR attractive fixed point

2) $\beta(\lambda) = -a\lambda^2$, $\gamma_\phi = \lambda \gamma_\phi'$

$$\tilde{\Gamma}^{(n)}(sp, m, \lambda, \mu) = s^{d_n} \tilde{\Gamma}^{(n)}(p, \bar{m}(s), \bar{\lambda}(s), \mu) \cdot \exp(-n \cdot E)$$

(the Callen-Symanzik equation, see the notes around p. 178)

$$E = \int_1^s \frac{ds'}{s'} \gamma_\phi(\lambda(s'))$$

$$s\frac{d\lambda}{ds} = \beta(\lambda) \text{ so } \frac{ds'}{s'} = \frac{d\lambda'}{\beta(\lambda')} \text{ and here}$$

$$E = -\int_{\lambda(1)}^{\lambda(s)} \frac{d\lambda'}{a\lambda'^2} \cdot \gamma_\phi' \lambda = -\frac{\gamma_\phi'}{a}\int \frac{d\lambda'}{\lambda'}$$

$$= -\frac{\gamma_\phi'}{a} \ln\left[\frac{\lambda(s)}{\lambda(1)}\right] = \frac{\gamma_\phi'}{a} \ln\left[\frac{\lambda(1)}{\lambda(s)}\right].$$

$$\text{so } \tilde{\Gamma}^{(n)} \sim s^{d_n} \left[\frac{\lambda(1)}{\lambda(s)}\right]^{-n\gamma_\phi'/a}.$$

For the last term, integrate the β-function

$$s\frac{d\lambda}{ds} = -a\lambda^2 \text{ or } \int_{\lambda(1)}^{\lambda(s)} \frac{d\lambda}{\lambda^2} = -a\int_1^s \frac{ds'}{s'}$$

$$\text{or } \frac{1}{\lambda(s)} - \frac{1}{\lambda(1)} = a \ln s$$

$$\frac{1}{\lambda(s)} = \frac{1}{\lambda(1)} + a \ln s$$

$$\frac{\lambda(1)}{\lambda(s)} = 1 + a\lambda(1) \ln s \sim a\lambda(1) \ln s \text{ at big } s$$

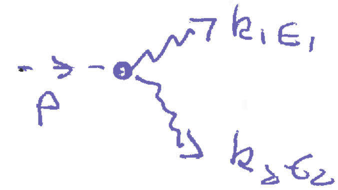
$$\Gamma^{(n)}(sp) \sim (sp)^{d_n} [\text{constant} \times \ln s]^{\varphi}$$

or $\Gamma^{(n)}(p) \sim p^n [\log p]^{d_n}$ – just a logarithmic correction to scaling via the engineering dimension

$\varphi = -n\gamma_\phi'/a$

3.1

3) $H = \gamma\gamma$, $\mathcal{L} = e^2 g\, H\, F_{\mu\nu}F^{\mu\nu}$



$$\Gamma = \frac{1}{2M_H} |\mathcal{M}|^2 \left[(2\pi)^4 \delta^4(p - k_1 - k_2) \frac{d^3k_1}{(2\pi)^3 2E_1} \frac{d^3k_2}{(2\pi)^3 2E_2} \right] \cdot \frac{1}{2}$$

The $\frac{1}{2}$ is for Bose statistics on the two identical γ's.
$|\mathcal{M}|^2$ is isotropic. This is a problem involving 2 body phase space. We have another such problem, so let's pause to work out the phase space factor for 2 massive particles (here the photons are massless, of course)

$$PS = \frac{(2\pi)^4}{(2\pi)^6} \int \frac{d^3k_1}{2E_1}\, d^4k_2\, \delta(k_2^2 - m^2)\, \delta^4(p - k_1 - k_2)$$

$$= \frac{1}{4\pi^2} \int \frac{d^3k_1}{2E_1}\, \delta((p - k_1)^2 - m^2)$$

(I forgot to say - the two particles have identical mass)

$E_1^2 = k_1^2 + m^2$ so $k_1^2\, dk_1 = k_1 k_1\, dk_1 = k_1 E_1\, dE_1$.

In the CM, $p = (M_H, \vec{0})$,
$k_1 = (E_1, \vec{k}_1)$

$$(p - k_1)^2 - m^2 = M_H^2 - 2M_H E_1 + m^2 - m^2,$$

$$PS = \frac{1}{8\pi^2} \int d\Omega\, \frac{k_1}{E_1} E_1\, dE_1\, \delta(M_H^2 - 2M_H E_1)$$

$$= \frac{1}{8\pi^2} \frac{k_1}{2M_H} \int d\Omega = \frac{4\pi}{16\pi^2} \frac{k_1}{M_H}$$

(if $|\mathcal{M}|^2$ is isotropic) - And $E_1 = \frac{M_H}{2}$.

It's convenient to introduce the velocity, $\frac{|\vec{k}_1|}{E} = v_f$

or $|k_1| = \frac{M_H v_f}{2}$, $PS = \frac{1}{4\pi} \frac{1}{M_H} \frac{M_H}{2} v_f = \frac{1}{8\pi} v_f$.

$$\Gamma = \frac{1}{2M_H}\cdot\frac{1}{2}\times\sum_{pol's}|\mathcal{M}|^2\cdot\frac{v_F}{8\pi} \quad ; v_F = 1 \text{ here!}$$

Now for $\mathcal{M}$.

$$F_{\mu\nu} = \sum_\lambda \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega_q}}\left[i(q_\mu\epsilon_\nu^\lambda - q_\nu\epsilon_\mu^\lambda)\, a_{q\epsilon}\, e^{iq\cdot x} + h.c.\right],$$

$$\langle 0| a_{k_1\epsilon_1} a_{k_2\epsilon_2} \left(a^\dagger_{q_a\epsilon_a} a^\dagger_{q_b\epsilon_b}\right)|0\rangle$$

gives a "2" from two contractions,

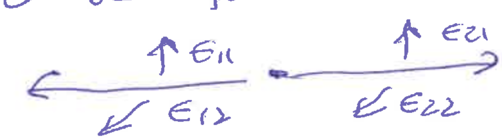
$$\mathcal{M} = -ie^2 d\cdot 2\cdot (k_{1\mu}\epsilon_{1\nu} - k_{1\nu}\epsilon_{1\mu})(k_2^\mu\epsilon_2^\nu - k_2^\nu\epsilon_2^\mu)$$

$$= -ied\cdot 2\cdot 2\cdot\left[(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_2) - (k_1\cdot\epsilon_2)(k_2\cdot\epsilon_1)\right]$$

$$\sum_{\epsilon_1\epsilon_2}|\mathcal{M}|^2 = 16\cdot(4\pi\alpha)^2|d|^2\sum_{\epsilon_1\epsilon_2}\left[(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_2) - (k_1\cdot\epsilon_2)(k_2\cdot\epsilon_1)\right]^2$$

There are 2 ways to do the polarization sums, either directly, or with the Feynman gauge trick,

$$\sum_\lambda \epsilon_\mu(\lambda)\epsilon_\nu(\lambda) = -g_{\mu\nu}.$$

Direct: $\vec{k}_i\cdot\vec{\epsilon}_i = 0$ or $k_\mu\epsilon^\mu = 0$. Kinematics are back to back: k_1  k_2 so

$k_i\cdot\epsilon_j = 0$ for all i, j, $\epsilon_{1i}\cdot\epsilon_{2j} = \delta_{ij}$,

$$\sum_{\epsilon_1\epsilon_2}|\mathcal{M}|^2 = 16(4\pi\alpha)^2|d|^2\cdot 2\cdot(k_1\cdot k_2)^2$$

$$(k_1+k_2)^2 = M_H^2 = 2k_1\cdot k_2$$

$$\sum_{\epsilon_1\epsilon_2}|\mathcal{M}|^2 = 16\cdot(4\pi\alpha)^2|d|^2\cdot 2\cdot\left(\frac{M_H^2}{2}\right)^2.$$

Alternatively, $\sum_{\epsilon_1\epsilon_2} |\mathcal{M}|^2 = 16(4\pi\alpha)^2 |\zeta|^2 \sum_{\epsilon_1\epsilon_2} (T_1+T_2+T_3)$

$$\sum_{\epsilon_1\epsilon_2} T_1 = \sum_{\epsilon_1\epsilon_2} (k_1\cdot k_2)^2 (\epsilon_1\cdot\epsilon_2)^2 = (k_1\cdot k_2)^2 \underbrace{\epsilon_{1\mu}\overbrace{\epsilon_2^{\mu}\,\epsilon_{1\nu}}^{-g^{\mu\nu}}\epsilon^{2\nu}}_{-g_{\mu\nu}}$$

$$= 4(k_1\cdot k_2)^2$$

$$\sum_{\epsilon_1\epsilon_2} T_2 = -2(k_1\cdot k_2)\,\underbrace{\epsilon_{1\mu}\,\overbrace{\epsilon^{2\mu}\,\epsilon_{1\nu}}_{-g_{\mu\nu}}\,k^{2\nu}\,\epsilon_2^{\lambda}}^{-g^{\mu\lambda}}\,k_{1\lambda}$$

$$= -2(k_1\cdot k_2)\, g_{\mu\nu}\, g^{\mu\lambda}\, k^{2\nu} k_{1\lambda} = -2(k_1\cdot k_2)^2$$

$$\sum_{\epsilon_1\epsilon_2} T_3 = (k_{1\mu} k_{1\nu} \underbrace{\epsilon_2^{\mu}\epsilon_2^{\nu}}_{-g^{\mu\nu}})(k_{2\rho} k_{2\sigma} \underbrace{\epsilon_1^{\rho}\epsilon_1^{\sigma}}_{g^{\rho\sigma}})$$

$= k_1^2 \times k_2^2 = 0$ - photons are massless!

$$\sum_{pol} |\mathcal{M}|^2 = 16(4\pi\alpha)^2 |\zeta|^2 (k_1\cdot k_2)^2 \cdot (4-2)$$

$$= 16(4\pi\alpha)^2 |\zeta|^2 \left(\frac{M_H^2}{2}\right)^2 \cdot 2$$

$$= 8(4\pi\alpha)^2 |\zeta|^2 M_H^4$$

$$\Gamma = \frac{1}{2M_H}\cdot\frac{1}{2}\cdot\underbrace{\frac{1}{8\pi}}_{\text{phase space, } v_F=1}\cdot 8(4\pi\alpha)^2|\zeta|^2 M_H^4$$

$$= 4\pi\alpha^2 |\zeta|^2 M_H^3$$

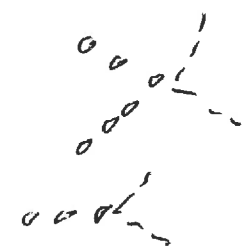
Note the units in $\mathcal{L}$: $F_{\mu\nu}F^{\mu\nu}$ is ~~as~~ an energy density
H has units of energy, $[\zeta]$ must be $[\text{energy}]^{-2}$.

4) Higgs portal:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)\partial^\mu H - \frac{1}{2} m_H^2 H^2$$
$$+ \frac{1}{2}(\partial_\mu S)(\partial^\mu S)$$
$$- \left[\frac{1}{2}\mu_S^2 S^2 + \lambda_S S^4 + \frac{\lambda_3}{2}(H+v_H)^2 S^2 \right]$$
$$+ \cdots$$
$$+ \frac{m_f}{v} H \bar{f} f .$$

a) $m_S^2 = \mu_S^2 + \lambda_3 v^2$

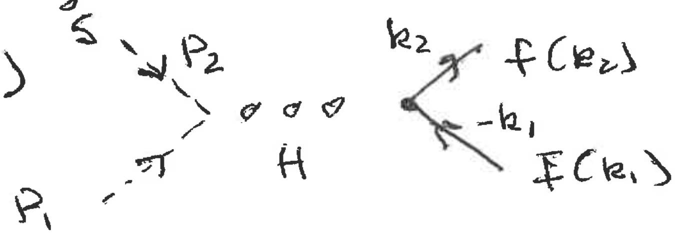
b) H-S interactions are $\frac{\lambda_3}{2} H^2 S^2$



(note: there will be symmetry factors when the vertices go into amplitudes) and $\lambda_3 v_H H S^2$

(~~something~~ ~~...~~ ~~...~~ where to ~~put them~~)

c)



$s = (P_1+P_2)^2$

$$\mathcal{M} = \bar{u}(k_2)\left[\frac{-i m_f}{v_H}\right] v(k_1)$$
$$\times \frac{-i}{(k_1+k_2)^2 - m_H^2} \cdot (-2i\lambda_3 v)$$

the "2" from 2 contractions of S

$$|\mathcal{M}|^2 = \frac{4\lambda_3^2 m_F^2}{(s-m_H)^2} \sum_{\text{spins}} \bar{u}(k_2)v(k_1)\bar{v}(k_2)u(k_2)$$

$$= \frac{4\lambda_3^2 m_F^2}{(s-m_H)^2} \operatorname{Tr}(\not{k}_2 + m_F)(\not{k}_1 - m_F)$$

$$= \frac{4\lambda_3^2 m_F^2}{(s-m_H^2)^2} \cdot 4\left[k_1 \cdot k_2 - m_F^2\right].$$

$$s = (p_1+p_2)^2 = (k_1+k_2^2) = 2m_F^2 + 2k_1 \cdot k_2$$

$$k_1 \cdot k_2 - m_F^2 = \frac{s}{2} - 2m_F^2 = \frac{s}{2}\left(1 - \frac{4m_F^2}{s}\right)$$

$$= \frac{s}{2} v_F^2$$

$$d\sigma = \frac{1}{2E_1E_2 v_{rel}} |\mathcal{M}|^2 \times \left[\frac{d^3k_1}{2E_1(2\pi)^3}\frac{d^3k_2}{2E_2(2\pi)^3} \times (2\pi)^4 \delta^4(p_1+p_2-k_1-k_2)\right]$$

the object in brackets is the expression for 2-body phase space. $|\mathcal{M}|^2$ is isotropic, we have computed it elsewhere, it is equal to

$$\frac{1}{8\pi} v_F, \quad v_F^2 = 1 - \frac{4m_F^2}{s}$$

v_{rel} is $2v_S$, with $p_1 = (E, p)$, $p_2 = (E, -p)$ ~~$s = (p_1+p_2)^2$~~ (in the CM, $E_1 = E_2 = E$).

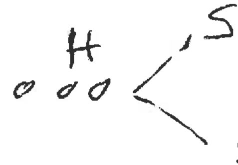
$$s = 4E^2, \quad v_S^2 = 1 - \frac{4m_S^2}{s}$$

$$\sigma = \frac{1}{4E_1E_2 v_S} |\mathcal{M}|^2 \cdot \frac{v_F}{8\pi}$$

$$= \frac{1}{s\, v_S} \frac{8\lambda_3^2 m_F^2}{(s-m_H^2)^2} \, 8s\, v_F^2 \cdot \frac{v_F}{8\pi}$$

$$\sigma = \frac{\lambda_3^2 m_F^2}{\pi (s - m_H^2)^2} \frac{v_F^3}{v_S}$$ $SS \to F\bar{F}$ in the SS CM.

d) $H \to SS$



$\mathcal{M} = -i \lambda_3 v \times 2$ for 2 contractions

$\Gamma(H \to SS) = \frac{1}{2 m_H} |\mathcal{M}|^2 \times$ (2 body phase space) $\times \frac{1}{2}$

($\frac{1}{2}$ for 2 idential final state particles)

phase space again $\frac{v}{8\pi}$, $v^2 = 1 - \frac{4 m_S^2}{m_H}$

$$\Gamma(H \to SS) = \frac{4 \lambda_3^2 v^2}{4 M_H} \frac{v}{8\pi}$$

$$= \frac{\lambda_3^2 v^2}{8 \pi m_H} \sqrt{1 - \frac{4 m_S^2}{m_H^2}}$$

For $\frac{\Gamma(H \to SS)}{m_H} < 0.1 \times$ Standard Model $\frac{\Gamma(H)}{m_H}$

$< \cancel{4 \times 10^{-5}}\ 0.1 \times 4 \times 10^{-5}$

$$\frac{\lambda_3^2}{8\pi} \left(\frac{v = 246 \text{ GeV}}{m_H = 125 \text{ GeV}} \right)^2 \sqrt{1 - \frac{4 m_S^2}{m_H^2}} < 4 \times 10^{-6}$$

$\Longrightarrow \lambda_3 < 10^{-3}$ or so - depends on $\frac{m_S}{m_H}$, of course!

Chiral symmetry : $\psi' = e^{i\epsilon\gamma_5}\psi$ or $\delta\psi = i\epsilon\gamma_5\psi$

a) What is the transformation for $\bar{\psi}$?

$\psi^{+\prime} = \psi^+ e^{-i\epsilon\gamma_5}$ to start, note γ_5 is Hermitian, in Weyl basis it is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Also $\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 0$.

$\bar{\psi}' = \psi^{+\prime}\gamma^0 = \psi^+ e^{-i\epsilon\gamma_5}\gamma_0 = \psi^+\gamma_0 e^{i\epsilon\gamma_5}$ from the anticommutation relation, so

$$\bar{\psi}' = \bar{\psi} e^{i\epsilon\gamma_5}$$

b) $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$ so $\mathcal{L}' = i\bar{\psi}'\gamma^\mu\partial_\mu\psi'$ –

$\mathcal{L}' = i\bar{\psi} e^{i\epsilon\gamma_5}\gamma^\mu e^{i\epsilon\gamma_5}\partial_\mu\psi$

$= i\bar{\psi} e^{i\epsilon\gamma_5} e^{-i\epsilon\gamma_5}\gamma^\mu\partial_\mu\psi$ (anticommutation again!)

$= i\bar{\psi}\gamma^\mu\partial_\mu\psi = \mathcal{L}.$

c) Conserved current?

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta(\partial_\mu\bar{\psi}) + \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi).$$

$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] = \frac{\partial\mathcal{L}}{\partial\psi}$ and the same is so for $\bar{\psi}$,

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right] = 0 \quad \text{(from (b))}$$

This is $\partial_\mu J^\mu = 0$, $J^\mu = 0 + i\bar{\psi}\gamma^\mu(i\epsilon\gamma_5\psi)$

$J^\mu \propto \bar{\psi}\gamma^\mu\gamma_5\psi.$

d) With a mass term the Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

Take Hermitian conjugate, use $\gamma^0 \gamma^{\mu +} \gamma^0 = \gamma^\mu$

$$-i(\partial_\mu \psi^+)\gamma^{\mu +} - m\psi^+ = 0$$

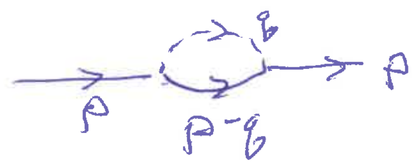
$$[-i(\partial_\mu \psi^+)\gamma^0\gamma^\mu\gamma^0 - m\psi^+]\gamma^0 = 0$$

$$-i\partial_\mu \bar\psi \gamma^\mu - m\bar\psi = 0.$$

$$\partial_\mu J^\mu = \partial_\mu [\bar\psi \gamma^\mu \gamma_5 \psi]$$

$$= (\partial_\mu \bar\psi)\gamma^\mu \gamma_5 \psi - \bar\psi \gamma_5 \gamma^\mu \partial_\mu \psi$$

$$= -\frac{m}{i}\bar\psi\gamma_5\psi - \bar\psi\gamma_5 \frac{m}{i}\psi = 2im\bar\psi\gamma_5\psi.$$

e) $\mathcal{L}_I = g\phi\bar\psi\psi$. $\Sigma =$ 

$$\Sigma(p) = -ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \frac{i(\not p - \not q + m_F)}{(p-q)^2 - m_F^2 + i\epsilon}$$

$$\Sigma(p=0) = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{i(\not q - m_F)}{(q^2 - m^2 + i\epsilon)(q^2 - m_F^2 + i\epsilon)}$$

All the terms inside the integral are even in q (they depend on q^2) - except for the $\not q = \gamma^\mu q_\mu$ term. It integrates to zero, so $\Sigma(0)$ is proportional to m_F!

Many people made the argument that the complete symmetry of the massless Yukawa model is $\psi \to e^{i\theta\gamma_5}\psi$, $\bar\psi \to \bar\psi e^{-i\theta\gamma_5}$, $\phi \to -\phi$, quantum corrections respect symmetries $\Rightarrow \Sigma = 0$. I'm an engineer!