Set 9 – or Take Home Final – due Thursday December 11, 7 PM

This problem set is to be done by yourself, with no assistance from or discussion with others. You may ask or email me clarifying questions. Permitted materials are those from this class: your notes, my lecture notes, Srednicki's textbook, your homeworks and the solutions. You may use the Appendices in Peskin and Schroeder or Schwartz for Dirac or tensor identities (but please tell me when you do this). No other resources are permitted (no other books, notes from another class, Wikipedia, etc). A software package like Mathematica is permitted for simple mathematical operations like evaluating an integral. Please indicate when you have done so. No Feynman graph software packages are allowed.

1) [10 points] Suppose we have a (massless) theory with a beta function with a linear zero,

$$s\frac{d\lambda}{ds} = a(\lambda^* - \lambda) \tag{1}$$

where $p \to sp$ so $s > 1$ carries us into the ultraviolet and $s < 1$ carries us into the infrared. In class I argued that for this beta function, $a > 0$ corresponds to an ultraviolet attractive fixed point and $a < 0$ corresponds to an infrared attractive fixed point. Integrate the β function, to find the running coupling $\lambda(s)$, and check that what I said is consistent in the two cases.

2) [10 points] If $\beta(\lambda) = -a\lambda^2$ (with $a > 0$), a system has an UV attractive fixed point. Suppose $\gamma_\phi = \lambda\gamma'_\phi$. Compute $\Gamma^{(n)}(sp, m = 0, \lambda, \mu)$ deep in the ultraviolet (that is, for very large s), show it scales like $s^{d_n}(\ln s)^Q$, and find Q.

Comments: Problem (1) is the situation for a simple second order phase transition in condensed matter physics.. Problem (2) is the situation for asymptotic freedom, like QCD: note how, deep in the UV, $\Gamma(p)$ is given by naive free-field power counting, that is the s^{d_n}, which is only weakly altered by the logarithm. This is the origin of "scaling" in the strong interactions – at short distances, QCD behaves like a free theory of quarks and gluons, up to logarithmic corrections.

3. [20 points] In the last problem set you computed the decay width of the π^0 meson to a pair of photons. The Higgs is similar, but since it is a scalar, the effective Lagrangian is

$$\mathcal{L} = e^2 CHF_{\mu\nu}F^{\mu\nu} \tag{2}$$

where H is the Higgs field and C is a constant (with an interesting story, for later). Find the Higgs decay width to two photons, in terms of C.

4. [40 points] A simple model for dark matter, called the "Higgs portal," is that dark matter is a new real scalar field S which couples to the rest of the Standard Model through the Higgs boson H. One realization of this scenario uses the combined Higgs and S Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{1}{2}m_H H^2 + \frac{1}{2}(\partial_\mu S)(\partial^\mu S) - [\frac{1}{2}\mu_S^2 S^2 + \lambda_s S^4 + \frac{\lambda_3}{2}(H + v_H)^2 S^2] \tag{3}$$

(plus terms not relevant to this problem), where M_H is the Higgs mass, 125 GeV, λ_S and λ_3 are dimensionless couplings, and v_H is the vacuum expectation value of the Higgs, $v_H \sim 246$ GeV. (a) [3 points] What is the mass of the S particle, m_S? (b) [2 points] What are the interaction vertices between H and S? (c) [20 points] "Indirect detection"of dark matter involves seeing the annihilation of S in outer space by observing its decay products. Calculate the annihilation cross section for a pair of S's to convert to a fermion - antifermion pair, $SS \to \bar{f}f$. Work in the SS center of mass frame. Recall that the Higgs coupling to fermions is $(m_f/v_H)H\bar{f}f$ where m_f is the fermion mass. Nobody knows what the mass of a dark matter particle is, so make no approximations neglecting m_S, m_f or m_H. (d) [15 points] If the S is light enough, the decay $H \to SS$ could occur at the LHC. In the Standard Model, the ratio of Higgs total width to mass $\Gamma_H/M_H \sim 4 \times 10^{-5}$, and this is observed (indirectly) to be true to a nominal ten to twenty percent. Compute $\Gamma(H \to SS)$ and find a (rough) constraint on λ_3 such that the branching ratio for this decay is less than ten per cent.

5. [20 points] Consider the so-called "chiral rotation" of a free Dirac spinor ψ,

$$\psi \to \exp(i\epsilon\gamma_5)\psi \tag{4}$$

where γ_5 is the Dirac matrix which is the projector onto the chiral basis. (a) [2 points] What is the corresponding transformation of $\bar{\psi}$? (b) [5 points] Show explicitly that the massless Dirac Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$ is invariant under this rotation. (c) [5 points] What is the associated Noether current J_μ? (d) [3 points] If the fermion mass is nonzero, the Noether current is not conserved. Show that the size of its nonconservation is proportional to m. (e) [5 points] Now imagine a Yukawa theory, a free Dirac fermion ψ and a scalar ϕ, with an interaction term $\mathcal{L} = g\bar{\psi}\psi\phi$. The fermion mass is shifted by the self energy graph $\Sigma(p)$ shown in the figure on the next page; the mass shift is $m \to m - \Sigma(p = 0)$. Show that $\Sigma(p = 0)$ is proportional to m: chiral symmetry "protects" $m = 0$. You don't have to compute Σ itself.

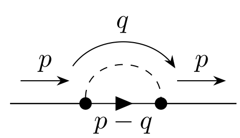

Figure 1: Fermion self energy.