Set 9 – due 7 April

This homework is due the week after spring break. I'll be out of town next week at a workshop and we will have a guest lecturer, John Bohn, giving three lectures on molecules. He's an expert so this should be interesting. I'm putting the problem set out (very) early because I don't want to burden him with handing it out etc.

1) [40 points] Nuclei sometimes decay from their excited states to their ground state by a process called "internal conversion," a process in which one of the $1s$ electrons is emitted instead of a photon. Suppose the initial and final nuclear wave functions be $\phi_I(\vec{r}_1, \vec{r}_2 \ldots, \vec{r}_A)$ and $\phi_F(\vec{r}_1, \vec{r}_2 \ldots, \vec{r}_A)$ where $\vec{r}_i$ for $i = 1$ to Z label the protons and $i = Z + 1$ to $i = A$ label the neutrons. The perturbation giving rise to the transition is just the nucleus - electron Coulomb interaction

$$V = -\sum_{i=1}^{Z} \frac{e^2}{|\vec{r} - \vec{r}_i|} \tag{1}$$

where $\vec{r}$ is the electron coordinate. Thus the matrix element is (here V in $1/\sqrt{V}$ is the volume, of course)

$$\langle f|T|i\rangle = -\int d^3r \prod_{j=1}^{A} d^3r_j \phi_F^* \frac{1}{\sqrt{V}} e^{-i\vec{p}\cdot\vec{r}/\hbar} \left(\sum_{i=1}^{Z} \frac{e^2}{|\vec{r} - \vec{r}_i|} \right) \phi_I \psi_{100}(r). \tag{2}$$

Compute the rate for the process for a dipole transition in terms of

$$\vec{d} = \int \prod_{j=1}^{A} d^3r_j \phi_F^* (\sum_i^Z \vec{r}_i) \phi_I \tag{3}$$

by making use of the expansion

$$\frac{1}{|\vec{r} - \vec{r}_i|} \simeq \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}_i}{r^3}. \tag{4}$$

Assume the outgoing electron is nonrelativistic. You can leave the final radial integral undone if you pull off all the dimensionful factors (leaving a dimensionless integral), and then if you show that the dimensions for Γ the inverse lifetime work out.

Deal with the outgoing state using the useful identity

$$\exp(i\vec{p}\cdot\vec{x}/\hbar) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 4\pi (i)^l j_l(pr/\hbar) Y_l^m(\theta, \phi)^* Y_l^m(\theta', \phi') \tag{5}$$

where (θ, ϕ) are the angles for $\vec{r}$, (θ', ϕ') are the angles for $\vec{p}$, and $j_l(x)$ is a spherical Bessel function.

If you are curious, you can keep going: The radial integral left undone is

$$\hat{I} = \int_0^\infty dr \exp(-Zr/a_0) j_1(pr/\hbar) \tag{6}$$

which we can write as either

$$\hat{I} = \frac{a_0}{Z} \int_0^\infty dy e^{-y} j_1(\frac{pa_0}{\hbar Z} y) = \frac{a_0}{Z} I(\frac{pa_0}{\hbar Z}) \tag{7}$$

or

$$\hat{I} = \frac{\hbar c}{pc} \int_0^\infty dw j_1(w) \exp(-\frac{Z\hbar}{a_0 p} w) = \frac{\hbar c}{pc} J(\frac{Z\hbar}{a_0 p}). \tag{8}$$

And there is some physics I didn't tell you: nuclear energy differences are much greater than electronic ones, so $pa_0/(\hbar Z)$ is huge. You can use this fact to simplify the integral and get a cute answer.

2) [10 points] Consider a sample of diatomic molecules which are polarized so that the atoms lie along the $\hat{y}$ axis. Neutrons are scattered off the nuclei. (The beam comes along the $\hat{z}$ direction.) The potential is very short range so we can regard it as a delta-function

$$V(r) = V_0 \delta(y - b)\delta(x)\delta(z) + V_0 \delta(y + b)\delta(x)\delta(z).$$

Compute the differential cross section in Born approximation. Where does it have nodes? What is going on?