**Set 1– due 27 January**

Two “check perturbation theory formulas” problems, and then a real physical one.

1) [15 points] Consider the two state system

$$H = \begin{pmatrix} \epsilon_1 & \Delta \\ \Delta & \epsilon_2 \end{pmatrix} \tag{1}$$

with all entries real, $\epsilon_1 > \epsilon_2$, and $\epsilon_1 \neq \epsilon_2$. (a) [4 points] Diagonalize it and find the exact energies and eigenstates. (b) [7 points] Using perturbation theory, find the energies through order $\Delta^2$ and eigenfunctions through order $\Delta$. (c) [4 points] Expand your solutions from (a) in a power series in $\Delta$ and show that they agree with your answers in (b).

2) [25 points] Consider the Hamiltonian for two spin-1/2 particles

$$H = \lambda \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \epsilon(\sigma_{1z} - \sigma_{2z}) \tag{2}$$

(a) [2 points] show $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_{1z}\sigma_{2z} + 2[\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}]$; $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$. (b) [3 points] What are the $\epsilon = 0$ eigenstates and eigenvalues? The $\lambda = 0$ eigenstates and eigenvalues? (c) Compute the eigenvalues of $H$ perturbatively to lowest nontrivial order for (i) [7 points] $\epsilon/\lambda$ small; (ii) [7 points] $\epsilon/\lambda$ large. (d) [6 points] Diagonalize $H$ exactly and show that your answers agree with those of (c) in the appropriate limits.

3) [25 points] (a) 20 points] Suppose (and this is true) that the proton is not a point particle; instead, that it is a sphere of radius $R$ with uniform charge density (an approximation, of course). Compute the shift in the ground state energy of Hydrogen. Hint: write $H_1 = V_{true} + e^2/r$. Find a numerical value if $R/a_0 = 10^{-5}$ or $R \simeq 10^{-13}$ cm.

Comment 1: Green’s functions are quite useful things!

Comment 2: to do the integral exactly is a horrible mess, and a numerical evaluation of the horrible mess turns out to be very unstable, since there are delicate cancellations. $R/a_0$ is tiny. Think like a physicist and use this fact to your advantage!

(b) [5 points] The numerical value in part (a) shows that this is basically invisible for hydrogen. However, where this expression is used is for muonic atoms: the electron is replaced by a muon (mass 100 MeV, roughly), and the nucleus is heavy (lead, with charge $Z = 82$ and atomic number $A = 206$). In that case, what is the analog Bohr radius for the muon, what are "typical" Rydberg-like energies, and what is the energy shift from part (a)? (A nuclear radius is roughly $A^{1/3}$ times 1 fm where one fm is $10^{-13}$ cm.) To be honest, the approximation you made in part (a) to do the integral really doesn't work any more, but it will give you all the correct orders of magnitude.

The problem is of some current interest: see H. Gao and M. Vanderhaeghen, "The proton charge radius," [arXiv:2105.00571 [hep-ph]].