

Residue theorem and all that

$$z = x + iy \rightarrow f(z)$$

$f(z)$ is analytic at z if it has a derivative,

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

independent of how
limit is taken

non-analytic = singular: pole $\frac{c}{z-z_0}$, $\frac{c}{(z-z_0)^2}$ etc

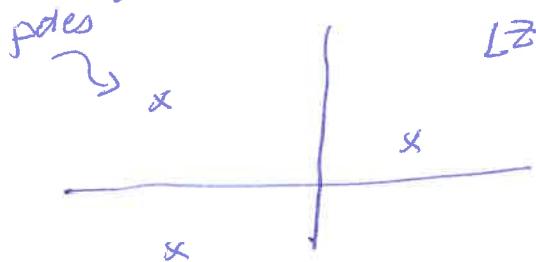
multiple valued: \sqrt{z} $z = e^{i\theta}$, $0 < \theta < 2\pi$

analytic $\sqrt{z} = \sqrt{r} e^{i\theta/2} = e^{\theta/2}, e^{i\pi/2}$

on real axis = double-valued
 $z^{1/3}$ etc, $\log z = (\log r + i\theta)$

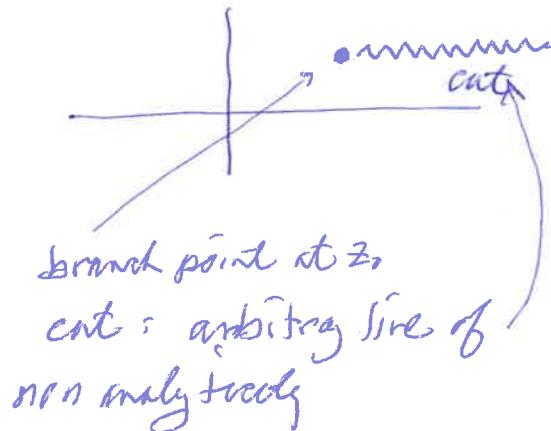
analytic, regular single valued \equiv regular

Map of complex plane



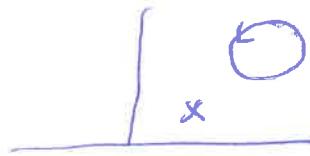
branch point, branch cut

$$f(z) = \sqrt{z-z_0} = \sqrt{r} e^{i\theta/2}$$



Cauchy's theorem : if $f(z)$ is regular in a region Ω , any integral around Ω closed path C is zero

$$\oint_C dz f(z) = 0$$



(proof uses Cauchy-Goursat eqn & Green's th)

Residue theorem : $\oint_C f(z) dz = 2\pi i \sum \text{residues}_{\text{inside } C}$

residue - if $f(z)$ has a simple pole at $z=z_0$,

$$\text{residue } R = \left[(z-z_0) f(z) \right]_{z=z_0}$$

$$\text{i.e. } f(z) = \frac{A}{z-z_0} \rightarrow R = A.$$

~~checks for $C = \text{circle}$~~

~~$f(z) = \text{constant}_0$~~

~~$I = \int f_0 e^{i\varphi} d\varphi$~~

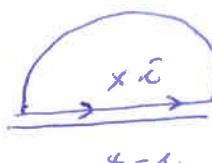
$$\begin{aligned} z &= z_0 + ce^{i\varphi} \\ dz &= ce^{i\varphi} d\varphi \end{aligned}$$

Check for a circle: $z-z_0 = ce^{i\varphi}$

integrate in φ : $dz = ie^{i\varphi} d\varphi$

$$\oint f(z) dz = \int_0^{2\pi} \frac{A}{ce^{i\varphi}} \cdot ie^{i\varphi} d\varphi = 2\pi i A$$

Use: $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} \rightarrow \oint \frac{dz}{z^2+1}$



← contour goes ~~not~~ $\text{around } \left(\frac{1}{2}, 0\right)$

$$= \oint dz \left(\frac{1}{z-i} - \frac{1}{z+i} \right) \frac{1}{2i} =$$

$$= \frac{2\pi i}{2i} = \pi$$

Physics issue pole lies on contour

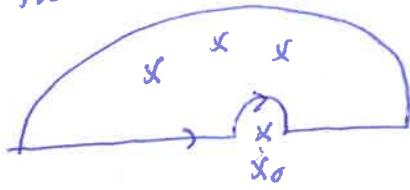
$$I = \int \frac{df}{p^2 - 2mE} e^{ipx}$$

usual resolution - think more about physics, shift path, use residues th.

Cauchy principal value

$$\text{PV} \int_a^b \frac{f(x)}{x-x_0} dx = \lim_{\delta \rightarrow 0} \int_a^{x_0-\delta} \frac{f(x)dx}{x-x_0} + \int_{x_0+\delta}^b \frac{f(x)dx}{x-x_0}$$

Turn into a contour \oint



$$\oint_C f(z) dz = \text{PV} \int f(z) dz + \text{small semicircle} + \text{big semicircle}$$

$$\text{semicircle} = -\pi i \times \text{residue} \left(\oint_C \frac{f(z)dz}{z-x_0} \right)$$

$$\text{PV} \int f(z) dz = 2\pi i \left[\frac{1}{2} \text{residue at } x_0 + \sum \text{res in } C \right]$$

$$\int \frac{f(x)}{x-x_0 \mp ie} dx = \text{PV} \int \frac{f(x)dx}{x-x_0} \pm i\pi f(x_0)$$

symbolic representation

$$\frac{1}{x-x_0 \mp ie} = \text{PV} \frac{1}{x-x_0} \pm i\pi \delta(x-x_0)$$