

45. Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...

$$1/2 \times 1/2$$

1	0
+1/2	1/2
-1/2	1/2

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

m_1	m_2	Coefficients
m_1	m_2	
\vdots	\vdots	
\vdots	\vdots	

$$1 \times 1/2$$

3/2	1/2
+3/2	1/2
1	1/2

$$2 \times 1$$

3	2
+3	2
1	2

$$1 \times 1$$

2	1
+2	1
1	1

$$3/2 \times 1$$

5/2	3/2
+5/2	3/2
1	3/2

$$3/2 \times 1/2$$

2	1
+2	1
1	1

$$d_{\ell m,0}^{\ell} = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell}^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle$$

$$d_{m',m}^j = (-1)^{m-m'} d_{-m,-m'}^j = d_{-m,-m'}^j$$

$$3/2 \times 3/2$$

3	2
+3	2
1	2

$$d_{0,0}^1 = \cos \theta$$

$$d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$$

$$d_{1,1}^1 = \frac{1 + \cos \theta}{2}$$

$$2 \times 3/2$$

7/2	5/2
+7/2	5/2
1	5/2

$$3/2 \times 1/2$$

1/2	1/2
+1/2	1/2
1	1/2

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$$

$$d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$$

$$2 \times 2$$

4	3
+4	3
1	3

$$2 \times 1$$

3	2
+3	2
1	2

$$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left(\frac{1 + \cos \theta}{2} \right)^2$$

$$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$$

$$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$$

$$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$$

$$d_{2,-2}^2 = \left(\frac{1 - \cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$$

$$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$$

$$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$$

$$d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

Figure 45.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).