Some notes for quantum field theory

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Abstract

These are some fragmentary notes written in about 2017 for a graduate quantum field theory class. The first part is a discussion of conventions for fields and scattering amplitudes in quantum field theory. It introduces (and tries to motivate) the most used definitions in the literature. The second part describes Hamiltonian methods for scalar field theory: the Dyson series, the S-matrix, and then a set of calculations of scattering amplitudes. The goal is to introduce and motivate Feynman rules and Feynman diagrams. The third part describes the path integral for Abelian gauge theory (written so the non-Abelian generalization is obvious). I conclude with a reading list.

I. CONVENTIONS FOR FIELDS AND SCATTERING AMPLITUDES

A. Particles in a box

Begin with free particles in a box of volume $V = L^3$. A free particle amplitude is

$$\psi(x) = \langle x|k\rangle = \frac{1}{\sqrt{V}} e^{ikx} \tag{1}$$

The momentum in direction j is quantized as $k_j = 2\pi n_j/L$ where n_j is an integer for periodic boundary conditions. States are normalized so

$$\langle \vec{k}|\vec{k}'\rangle = \delta_{nn'} \tag{2}$$

and completeness is

$$1 = \sum_n |k\rangle\langle k| = \int \frac{V}{(2\pi)^3} d^3k |\vec{k}\rangle\langle k| \tag{3}$$

slightly abusing notation to treat k as continuous. (It is the usual $dn_x = L dk_x/(2\pi)$ story.) That is the phase space factor – the number of states between $\vec{k}$ and $\vec{k} + d\vec{k}$ is

$$dN = \frac{V}{(2\pi)^3} d^3k. \tag{4}$$

If $\psi(x)$ is a Schrödinger wave function, the probability density $\rho = \psi^*\psi = 1/V$. The current is

$$\vec{J} = \frac{1}{2im}\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi)^*\psi = \frac{k}{m}\frac{1}{V} = \frac{\vec{v}}{V} \tag{5}$$

where $\vec{v}$ is the particle's velocity.

The Golden Rule begins by defining the transition probability per unit time $d\Gamma$ in terms of the T-matrix $\langle f|\mathcal{T}|i\rangle$ from a state $|i\rangle$ at time $t = 0$ to a state $|f\rangle$ at time T,

$$d\Gamma = \frac{1}{T} \lim_{T\to\infty} |\langle f(T)|\mathcal{T}|i(0)\rangle|^2. \tag{6}$$

Consider a process where $|i\rangle$ is a two particle state, $i = 1$ to 2, and $|f\rangle$ is n_f outgoing particles, $j = 1$ to n_f. Let's assemble all the pieces for a formula for the differential cross section.

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First, in $d\Gamma$ there is a phase space factor for each outgoing particle,

$$\prod_{n=1}^{n_f} V \frac{d^3k_n}{(2\pi)^3}. \tag{7}$$

Next, in $\langle f(T)|\mathcal{T}|i(0)\rangle$ there will be a factor coming from the exponentials from the time dependence of the unperturbed states,

$$\int_0^T dt e^{i\Delta\omega t} = \frac{e^{i\Delta\omega T} - 1}{i\Delta\omega} \tag{8}$$

where $\Delta\omega = \sum_{i=1}^{2} E_i - \sum_{j=1}^{n_f} E_f$. This will square (the usual squaring of the delta function story) to give the Golden Rule expression

$$d\Gamma = 2\pi\delta(\Delta\omega)|\langle f|\mathcal{T}|i\rangle|^2 \prod_{n=1}^{n_f} \frac{V}{(2\pi)^3} d^3k_n. \tag{9}$$

The cross section is defined as the ratio

$$d\sigma = d\Gamma \times \frac{1}{flux} \tag{10}$$

where the flux factor is $flux = |J| = v_{rel}/V$ and $v_{rel} = v_1 - v_2$. Finally, it is often the case that momentum is conserved. Component by component the T-matrix will involve integrals of the form

$$\langle f|\mathcal{T}|i\rangle \propto \int_V d^3x e^{i\Delta kx} = V\delta_{\vec{n}_f,\vec{n}_i} = \prod_{i=x,y,z} \frac{e^{i\Delta k_i L} - 1}{i\Delta k_i}, \tag{11}$$

where the first identity involves the integer-counting for the wave numbers, quantized in the box, and the second equality comes from taking k as continuous and doing the integral over the box. We must square the T-matrix and take the volume to be large. The mathematics is identical to what is encountered in squaring the frequency delta function and going to long times, Eq. 8 from the time integral, to give Eq. 9. Putting all the pieces together,

$$\begin{aligned} d\sigma = & \frac{V}{v_{rel}} |\langle f|\hat{T}|i\rangle|^2 \times (2\pi)^4 \delta(\sum \omega_f - \sum \omega_i) \\ & \times V\delta^3(\sum \vec{k}_f - \sum \vec{k}_i) \\ & \times \prod_{n=1}^{n_f} V \frac{d^3k_n}{(2\pi)^3}. \end{aligned} \tag{12}$$

The leading V is from the flux and the V in the middle is from squaring the momentum conservation delta function. The reduced T-matrix $\langle f|\hat{T}|i\rangle$ is the full T-matrix $\langle f|\mathcal{T}|i\rangle$ with the overall momentum-conserving delta functions (pick the phrase you like best) snipped off, or divided out, or thrown away. Typically, it will involve an integral over all the relative coordinates, since overall space translation invariance is where over momentum conservation comes from. It is proportional to $(1/\sqrt{V})^{n_f+2}$ because there are $n_f + 2$ free particle wave functions in it, each carrying a factor of $1\sqrt{V}$ (recall Eq. 1). And it gets squared. Overall, all the volume factors cancel; the final result for the cross section will be independent of the volume (as it should be).

B. Delta function normalization

Alternatively, we can use delta-function normalization for plane wave states in an infinite box. The normalization convention is

$$\langle \vec{k}|\vec{k}'\rangle = \delta^3(k-k') \tag{13}$$

so that

$$\int d^3k \langle \vec{k}|\vec{k}'\rangle = 1 \tag{14}$$

and

$$1 = \int d^3k |k\rangle\langle k| \tag{15}$$

so

$$\langle x|k\rangle = \frac{1}{(2\pi)^{3/2}} e^{ikx} \tag{16}$$

Now phase space is just

$$dN = d^3k. \tag{17}$$

There is a translation dictionary between the "box" T-matrix element and the "$\delta-$function" one, just because the definitions of the states, Eqs.1 and 16, are different. Writing the dictionary in terms of volume-independent quantities, it is

$$[(2\pi)^3]^{(n_f+n_i)/2}\langle f|\mathcal{T}|i\rangle_{\delta-fn} = V^{(n_f+n_i)/2}\langle f|\mathcal{T}|i\rangle_{box}. \tag{18}$$

Either way this expression has no remaining 2π's in it. Generally, each side will be proportional to $\delta^3(\sum k_i - \sum k_f)$.

To pass to the cross section, we have to square the matrix element. Unfortunately, doing this while preserving the delta functions (note we are really squaring a delta function this time) requires using wave packets. That is a little too much for me, but it is worked out in Peskin and Schroeder, Sec. 4.6. The final answer involves

$$\langle f|\mathcal{T}|i\rangle = \langle\{k_f\}|\mathcal{T}|\{k_i\}\rangle = (2\pi)^4\delta^4(\sum k_f - \sum k_i)\frac{\tilde{T}}{(2\pi)^3} \tag{19}$$

and

$$d\sigma = \frac{1}{v_{rel}}|\tilde{T}|^2(2\pi)^4\delta^4(\sum k_f - \sum k_i)\prod_{j=1}^{n_f} d^3k_j. \tag{20}$$

Note $\tilde{T}$ has factors of 2π in it, from the normalization of the states. They can be pulled out into the phase space factor. This will give an expression identical to Eq. 11, but without the volume factors. (They would have cancelled anyway.)

C. The normalization everybody uses

Finally, suppose we choose to normalize our states so that

$$\langle \vec{k}|\vec{k}'\rangle = C\delta^3(k-k') \tag{21}$$

where C is some convenient real constant. Then

$$\int d^3k \langle \vec{k}|\vec{k}'\rangle = |C|^2 \tag{22}$$

and

$$\langle x|k\rangle = \sqrt{C}\frac{1}{(2\pi)^{3/2}}e^{ikx} \tag{23}$$

while the completeness relation is

$$1 = \int \frac{d^3k}{C}|k\rangle\langle k|. \tag{24}$$

The Schrödinger current (recall Eq. 5) becomes

$$\vec{J} = \frac{\vec{k}}{m}C. \tag{25}$$

This is a nonrelativistic formula, but in general the current is just C times a current defined with some ordinary normalization, as in the earlier sections. The phase space factor is

$$dN = \frac{d^3p}{C} \tag{26}$$

and the cross section is

$$d\sigma_C = d\sigma_{C=1} \times \frac{1}{C^{2+n_f}}. \tag{27}$$

Notice that the T-matrix element will rescale–the cross section can't change.

This convention seems stupid, but it is actually very clever. Why? It allows one to build in Lorentz invariance in an explicit way. The most commonly used choice in the literature is

$$C = 2E(k) \tag{28}$$

where $E(k) = \sqrt{k^2 + m^2}$ is the energy. That is, a momentum eigenstate is

$$|\vec{p}\rangle = \sqrt{2E(p)}a_p^\dagger|0\rangle. \tag{29}$$

(A quick check:

$$\langle\vec{q}|\vec{p}\rangle = \sqrt{2E(q)}\sqrt{2E(p)}\langle 0|a_q a_p^\dagger|0\rangle \tag{30}$$

and if we assume

$$[a_q, a_p^\dagger] = \delta^3(\vec{q} - \vec{p}), \tag{31}$$

the left hand side of Eq. 30 is $2E_p\delta^3(\vec{q} - \vec{p})$; compare Eq. 21.)

Why do this? It gives you several nice things:

First, the phase space factor is Lorentz invariant. It is

$$\frac{d^3p}{2E(p)(2\pi)^3} \tag{32}$$

per particle, evaluating the matrix element with pure exponentials and pulling out the 2π factors from the T-matrix into the phase space. Why is this Lorentz invariant? It is

$$\frac{d^3p}{2E(p)} = d^3pdE\delta(E^2 - p^2 - m^2)\theta(E). \tag{33}$$

The $\theta(E)$ picks the positive energy solution to the delta-function ($E = \pm\sqrt{p^2+m^2} = \pm E(p)$. The parts, $d^3pdE = d^4p_\mu$ and $\delta(p^2+m^2)$ are each separately Lorentz invariant, so the product is, also. Now integrate E over the delta function, you get Eq. 33.

Next, recall our definition of the field operator,

$$\phi(x)|\vec{k}\rangle = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[a_p e^{ipx} + \dots]. \tag{34}$$

Contract it on a state $|k\rangle$:

$$\begin{aligned}\phi(x)|\vec{k}\rangle &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[a_p e^{ipx} + \dots]\sqrt{2E(k)}a_k^\dagger|0\rangle \\ &= \frac{e^{ikx}}{(2\pi)^{3/2}}\frac{\sqrt{2\omega(k)}}{\sqrt{2\omega(k)}}|0\rangle.\end{aligned} \tag{35}$$

The algebra is simple, the $(2\pi)^{3/2}$ can be lumped in the phase space in Eq. 32. This will give nice answers when you contract field operators against states.

For the Klein-Gordon equation, the continuity equation is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{36}$$

where

$$\rho = \phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t} \tag{37}$$

and

$$\vec{J} = \phi^*\vec{\nabla}\phi - \phi(\vec{\nabla}\phi^*). \tag{38}$$

We expect ρ and $\vec{J}$ form a four vector: Eq. 36 is $\partial_\mu J^\mu = 0$.

The state

$$\begin{aligned}\phi(x)\,|0\rangle &= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[a_p e^{ipx} + \dots]\,|0\rangle \\ &= \int \frac{d^3p}{2\omega(p)}\,|p\rangle\end{aligned} \tag{39}$$

(use Eqs. 31 and 29) is a linear superposition of single particle states . It's almost the same as the nonrelativistic convention, just with an extra $1/\omega(p) \sim 1/m$ in the nonrelativistic limit. We can define a plane wave solution

$$\begin{aligned}\Phi_k(x) &= \langle 0|\phi(x)|k\rangle \\ &= \exp(ikx).\end{aligned} \tag{40}$$

Obviously, this is "what we would expect" for the position space wave function for the single particle state $|p\rangle$.

Compute ρ and $\vec{J}$ using $\Phi_k(x)$. We find $\rho \propto E(p)$, $\vec{J} \propto \vec{p}$, that is, the probablilty current $(\rho, \vec{J}) \propto p^\mu$ obviously forms a four vector, because the components are of J^μ are proportional to p^μ.

Next, look ahead to consider Lorentz transformations. $p = (E, \vec{p})$ in one frame becomes $\Lambda p = (E', \vec{p}')$ in another frame. States must transform unitarily,

$$|\Lambda p\rangle = U(\Lambda)|p\rangle \tag{41}$$

and operators must transform as

$$O \to U(\Lambda) O U^{-1}(\Lambda). \tag{42}$$

$U(\Lambda)$ is some unitary transformation.

What is awkward is that $\delta^3(\vec{p}-\vec{q})$ is not Lorentz invariant. However, $E\delta^3(\vec{p}-\vec{q})$ is Lorentz invariant:

$$E\delta^3(\vec{p}-\vec{q}) = E'\delta^3(p'-q'). \tag{43}$$

This will mean that $\langle q|p\rangle = 2E(p)\delta^3(\vec{p}-\vec{q})$, the inner product of two states, will be Lorentz invariant. To check the Lorentz invariance, perform a Lorentz transformation along direction p_3. Change variables in the delta function, and use the chain rule,

$$\delta^3(p-q) = \delta^3(p'-q')\frac{dp_3'}{dp_3} \tag{44}$$

where $p_3' = \gamma(p_3 + \beta E)$ and $E' = \gamma(E + \beta p_3)$. Then

$$\delta^3(p-q) = \delta^3(p'-q')\gamma(1 + \beta\frac{\partial E}{\partial p_3}). \tag{45}$$

Since $E^2 = p_3^2 + \ldots$, this is

$$\delta^3(p-q) = \delta^3(p'-q')\frac{\gamma}{E}(E + \beta\frac{p_3}{E}E) = \delta^3(p'-q')\frac{E'}{E} \tag{46}$$

as required.

Creation operators are operators, and it happens that

$$U(\Lambda)a_p^\dagger U^{-1}(\Lambda) = \sqrt{\frac{E(\Lambda p)}{E(p)}}a_{\Lambda p}^\dagger. \tag{47}$$

This follows from

$$U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle \tag{48}$$

or (insert $1 = U^{-1}(\Lambda)U(\Lambda)$)

$$U(\Lambda)a_p^\dagger\sqrt{E(p)}U^{-1}(\Lambda)U(\Lambda)|0\rangle = \sqrt{E(\Lambda p)}a_{\Lambda p}^\dagger|0\rangle. \tag{49}$$

This uses $U|0\rangle = |0\rangle$ – we are assuming that the vacuum is simple, the same in all frames Anyway, now look at a field operator,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[a_p e^{ipx} + \ldots] \tag{50}$$

Transform it: $x' = \Lambda x$, $p' = \Lambda p$, substitute in $a_{\Lambda p}$,

$$\phi(\Lambda x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[\sqrt{\frac{\omega(p')}{\omega(p)}}a_{\Lambda p}e^{ik(\Lambda x)} + \dots] \tag{51}$$

Change variables from p to p', use $d^3p/(2\omega(p)) = d^3p'/(2\omega(p'))$, and you discover that

$$\phi(\Lambda x) = \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega(p')}}[a_{p'}e^{ip'x')} + \dots]. \tag{52}$$

Look at this closely: it says that $\phi(x)$ really is a scalar operator. Under a Lorentz transformation, it remains unchanged. (Only its argument, x, changes.)

Box normalization is much easier to work with, but then $\phi(x)$ is not a true scalar field, in the sense of Lorentz transformations. Think about it, what is the box doing? Of course, if the original Lagrange density is Lorentz invariant, like

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi), \tag{53}$$

all predictions of the theory will be Lorentz invariant. However, with a choice of normalization for states which does not know about Lorentz symmetry, the intermediate steps of a calculation will not look Lorentz invariant. The Lorentz invariance of the final result will seem to be a miracle, if you used noncovariant definitions for fields or states. It is better to have a formalism where everything you work with has simple, known transformation properties so that Lorentz invariance can be observed every step of the way. Miracles are not always good things!

Finally, in this convention, the differential cross section is

$$d\sigma = \frac{1}{4E_1E_2v_{rel}}|\tilde{T}|^2(2\pi)^4\delta^4(\sum k_f - \sum k_i)\prod_{j=1}^{n_f}\frac{d^3k_j}{(2\pi)^3 2E(k_j)} \tag{54}$$

and $\tilde{T}$ is the T-matrix with all extraneous factors of 2π, or $\sqrt{2E}$, and the overall momentum conserving delta function stripped off. This looks very robotic (and it is). But now you can use it, and you don't need to think about where it came from. $\tilde{T}$ is often called the "invariant ampltude" M. It is a scalar function of the momenta and other four vectors (like polarizations) characterizing the scattering amplitude.

II. TIME DEPENDENT PERTURBATION THEORY, THE $S-$MATRIX, LORENTZ INVARIANCE, ANTIPARTICLES

A. Time dependent perturbation theory

We need approximate solutions for the time dependent Schrödinger equation. We assume that the Hamiltonian can be written as $H = H_0 + H_1$, where H_0 is "large" in some sense and can be diagonalized. We further assume that H_1 may be regarded as a perturbation. The time evolution will be developed in terms of the eigenfunctions of H_0.

It is very useful to replace the time dependent wave function by a time evolution operator, and to find matrix elements of that operator. It is defined through

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle \tag{55}$$

and it obeys an equation of motion

$$i\hbar\frac{\partial U(t,t_0)}{\partial t} = HU(t,t_0). \tag{56}$$

Matrix elements of U connect initial states at some early time with final states at some later time. What they encode is called the "transition amplitude" from the initial state to the final state. It is defined as

$$U_{\beta\alpha}(t,t_0) = \langle\beta(t)|U(t,t_0)|\alpha(t_0)\rangle \tag{57}$$

To find a practical expression for the evolution operator, let us define the "interaction representation" perturbation

$$\hat{H}_1(t) = e^{iH_0t/\hbar}H_1(t)e^{-iH_0t/\hbar} \tag{58}$$

and the "interaction representation" evolution operator

$$\hat{U}(t_f,t_i) = e^{iH_0t_f/\hbar}U(t_f,t_i)e^{-iH_0t_i/\hbar}. \tag{59}$$

We want to solve

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = (H_0+H_1)\,U(t,t_0). \tag{60}$$

In interaction representation, Eq. 60 becomes

$$i\hbar\frac{\partial}{\partial t}\left[e^{-iH_0t/\hbar}\hat{U}e^{iH_0t/\hbar}\right] = (H_0+H_1)\,e^{-iH_0t/\hbar}\hat{U}e^{iH_0t_0/\hbar}. \tag{61}$$

The left hand side is

$$i\hbar\frac{\partial}{\partial t}\left[e^{-iH_0t/\hbar}\hat{U}e^{iH_0t_0/\hbar}\right] = \left[H_0e^{-iH_0t/\hbar}\hat{U} + e^{-iH_0t/\hbar}i\hbar\frac{\partial\hat{U}}{\partial t}\right]e^{iH_0t_0/\hbar}, \tag{62}$$

so

$$\left[H_0e^{-iH_0t/\hbar}\hat{U} + e^{-iH_0t/\hbar}i\hbar\frac{\partial\hat{U}}{\partial t}\right]e^{iH_0t_0/\hbar} = (H_0+H_1)e^{-iH_0t/\hbar}\hat{U}e^{iH_0t_0/\hbar} \tag{63}$$

or

$$i\hbar\frac{\partial\hat{U}}{\partial t} = \left(e^{iH_0t/\hbar}H_1^{-iH_0t/\hbar}\right)\hat{U} = \hat{H}_1\hat{U}. \tag{64}$$

This is just a first order differential equation. Integrating it, we find

$$i\hbar\left[\hat{U}(t,t_0) - \hat{U}(t_0,t_0)\right] = \int_{t_0}^{t}dt'\hat{H}_1(t')\hat{U}(t',t_0). \tag{65}$$

Because $\hat{U}(t_0,t_0) = 1$, the evolution operator satisfies the integral equation

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t}dt'\hat{H}_1(t')\hat{U}(t',t_0). \tag{66}$$

Like all Volterra integral equations, it can be solved by iteration:

$$\begin{aligned}\hat{U}(t_f,t_i) =& 1-\frac{i}{\hbar}\int_{t_i}^{t_f} dt_1 \hat{H}_1(t) + \left(-\frac{i}{\hbar}\right)^2 \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_2} dt_1 \hat{H}_1(t_2)\hat{H}_1(t_1) \\ &+ \left(-\frac{i}{\hbar}\right)^3 \int_{t_i}^{t_f} dt_3 \int_{t_i}^{t_3} dt_2 \int_{t_i}^{t_2} dt_1 \hat{H}_1(t_3)\hat{H}_1(t_2)\hat{H}_1(t_1) + \cdots \end{aligned} \tag{67}$$

The operators $H_1(t)$ may not commute at different times, and so it is important to preserve their time ordering.

This is called the "Dyson series" for the evolution operator. Note the nested time integrals, preserving the ordering in the multiple time integrals of the temporal points where the potential acts.

We can write this even more compactly by introducing the "time ordering operator" $T(A(t_1)B(t_2)\dots)$. The time ordering operation takes the operators and evaluates them in ascending order from the right; with the operator at earliest time farthest right, then the next earliest one, and so on until the operator evaluated at the latest time sits at the far left. Then we can think of the times t_i in Eq. 67 as a set of dummy labels and write the evolution operator as

$$\hat{U}(t_f,t_i) = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2 \dots \int_{t_n}^{t_f} dt_n T(\hat{H}_1(t_1)\hat{H}_1(t_2)\dots\hat{H}_1(t_n)) \tag{68}$$

For useful insight, look at the second order term in the sum, and return to Schrödinger representation. Consider a transition from one eigenstate of H_0, $|\alpha\rangle$, to another one, $|\beta\rangle$. Inserting complete sets of states the transition amplitude becomes

$$\begin{aligned} U_{\beta\alpha} = & \left(-\frac{i}{\hbar}\right)^2 \sum_{\gamma} \int_{t_i}^{t_f} dt_2 \int_{t_i}^{t_2} dt_1 e^{-iE_\beta(t_f-t_i)/\hbar} \\ & \times \langle\beta|H_1(t_2)|\gamma\rangle\, e^{-iE_\gamma(t_2-t_1)/\hbar}\, \langle\gamma|H_1(t_1)|\alpha\rangle\, e^{-iE_\alpha(t_1-t_i)/\hbar}. \end{aligned} \tag{69}$$

Notice the structure (moving in from the right): the system begins in state $|\alpha\rangle$ and the time evolution is the trivial exponential of an energy eigenstate. Then the perturbation acts at one time t_1, driving the system into a superposition of eigenstates of H_0, the $|\gamma\rangle$ states. Lastly, the perturbation acts again at t_2 to carry the system into $|\beta\rangle$.

B. The S-matrix

The S-matrix is defined as the limit

$$S = \lim_{t\to\infty} \lim_{t_0\to-\infty} U(t,t_0). \tag{70}$$

We only have eigenstates of H_0 so we always compute matrix elements of S between them, $\langle\beta|S|\alpha\rangle$. We make a set of demands on S, and these demands constrain possible Hamiltonians. We set $\hbar = 1$ from here on.

1. Unitarity

We want $\langle\beta|S^\dagger S|\alpha\rangle = \delta_{\alpha\beta}$. This is the same thing as asking whether the evolution operator is unitary. To satisfy this requirement, H_I must be Hermitian. The proof is simple. $U(t_0,t_0) = \hat{U}(t_0,t_0) = 1$. Then

$$i\frac{d}{dt}[\hat{U}(t,t_0)^\dagger \hat{U}(t,t_0) = [-\hat{U}^\dagger \hat{H}_I^\dagger]\hat{U} + \hat{U}^\dagger[\hat{H}_I\hat{U}] = \hat{U}^\dagger[-\hat{H}_I^\dagger + \hat{H}_I]\hat{U} \tag{71}$$

where we have used

$$\frac{d\hat{U}}{dt} = -i\hat{H}_I\hat{U} \tag{72}$$

and its Hermitian conjugate. If $\hat{H}_I$ is Hermitian, $U^\dagger U$ is a constant and the constant is unity from $U(t_0,t_0) = \hat{U}(t_0,t_0) = 1$.

2. Energy conservation

Is $S_{\beta\alpha} = 0$ if $E_\alpha \neq E_\beta$? This is tricky! Recall $H_0\,|\alpha\rangle = E_\alpha\,|\alpha\rangle$, so we might consider

$$\langle\beta|[H_0,S]|\alpha\rangle = (E_\beta - E_\alpha)S_{\beta\alpha}. \tag{73}$$

Is this zero? If so, we have energy conservation. But there is a problem: S depends on H_I and generally $[H_0,H_I] \neq 0$.

The escape is dirty. We have to (formally) adiabatically switch on and off H_I at early and late times, so that we can actually construct our "in" and "out" states. Then (writing formally)

$$U(t,t_0) = T\exp(i(H_o + H_I^{-\lambda|t|})t) \tag{74}$$

so that

$$\lim_{t\to\pm\infty}[H_0,U] \to 0. \tag{75}$$

This says that we can only have energy conservation between the beginning and the end of the reaction – the intermediate states can be anything. So much for being formal. The algebra will take care of energy conservation for us, modulo technical problems associated with squaring delta functions. For example

$$S^{(1)}_{\alpha\beta} = -i\int_{-\infty}^{\infty} dt e^{iE_\alpha t}\,\langle\alpha|H_I|\beta\rangle\,e^{-iE_\beta t} \propto \delta(E_\alpha - E_\beta). \tag{76}$$

In fact, it will turn out, depending on how time ordered products are handled, that either

1. Energy isn't conserved throughout the reaction, but all particles are on shell, $p^2 = m^2$

2. OR E is conserved, but intermediate state particles do not have $p^2 = m^2$.

3. Momentum conservation

We want $S_{\beta\alpha} = 0$ if $\sum_{i\in\alpha} p_i \neq \sum_{j\in\beta} p_j$ or more formally $[P, S] = 0$ where P is the momentum operator. We will get this if $[P, H_I] = 0$. The momentum P is not only an operator, it is the generator of translations $x \to x + a$. Then S will be invariant under space translations if H is, and momentum will be conserved. As you know, scattering in an external field does not conserve momentum, but in that case there is no translation invariance. We have to be careful about treating the momentum as a four vector and asking for time translation invariance, because in Hamiltonian quantum mechanics, time is special: it is "what we evolve in."

4. Lorentz invariance

Consider a Lorentz transformation (some combination of boosts and rotations) parametrized by a matrix Λ (with six perameters, three for rotations, three for boosts). We have observers in two frames:

- O sees state $|\alpha\rangle$
- O' sees state $|\Lambda\alpha\rangle$

In analogy with rotation matrices, states are transformed by a unitary operator $U(\Lambda)$,

$$|\Lambda\alpha\rangle = U(\Lambda)|\alpha\rangle \tag{77}$$

and operators must transform as

$$O \to U(\Lambda)OU^{-1}(\Lambda). \tag{78}$$

That the transformation is unitary means that $\langle\alpha|\beta\rangle = \langle\Lambda\alpha|\Lambda\beta\rangle = \delta_{\alpha\beta}$. Different observers must see S-matrices which do the same thing, so $S_{\alpha,\beta} = S_{\Lambda\alpha,\Lambda\beta}$, or

$$\begin{aligned}\langle\beta|S|\alpha\rangle &= \langle U(\Lambda)\beta|S|U(\Lambda)\alpha\rangle \\ &= \langle\beta|U(\Lambda)^{-1}SU(\Lambda)|\alpha\rangle\,.\end{aligned} \tag{79}$$

So, we want $S = U^{-1}SU$ or $[S, U(\Lambda)] = 0$.

This just looks like formalism, until we realize that the Hamiltonian perturbative S-matrix treats time in a special way – there is the time ordered product. (Go back and look at Eq. 68.) How can we solve this constraint? There is one possibility everyone knows (more arcane versions, for particles with higher spin, or derivative interactions, are described in Weinberg – keep it simple, now): First, write the perturbing Hamiltonian as an integral over a Hamiltonian density

$$H_I(t) \to \int d^3x\mathcal{H}(x,t) \tag{80}$$

where H is a Lorentz scalar. Being a scalar means that the transformation rule for $\mathcal{H}$ is

$$U(\Lambda)^{-1}\mathcal{H}(x)U(\Lambda) = \mathcal{H}(\Lambda^{-1}x). \tag{81}$$

(the argument changes but $\mathcal{H}$ does not change; Λ^{-1} because the order of the U's in Eq. 81 is reversed compared to Eq. 77.) Then the S-matrix is

$$S = \sum_n \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n T(\mathcal{H}(x_1) \dots \mathcal{H}(x_n)). \tag{82}$$

This looks promising: all the integrations, and $\mathcal{H}$, are invariants. Unfortunately, we are not out of the woods yet. The time ordering operator might not be Lorentz invariant. Do a Lorentz transformation:

$$U^{-1}SU = \sum_n \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n T_x(\mathcal{H}(\Lambda^{-1}x_1) \dots \mathcal{H}(\Lambda^{-1}x_n)). \tag{83}$$

I wrote T_x to remind that the time ordering is in the frame of x, NOT in $\Lambda^{-1}x$. To get this, I have inserted unity as $U(\Lambda)^{-1}U(\Lambda) = 1$ between the $\mathcal{H}$'s in the time ordered product. Now change variables, writing $x_i = \Lambda y_i$. The integration measure is invariant, $d^4x = d^4y$ so

$$U^{-1}SU = \sum_n \frac{(-i)^n}{n!} \int d^4y_1 \dots d^4y_n T_x(\mathcal{H}(y_1) \dots \mathcal{H}(y_n)). \tag{84}$$

T_x contains terms like

$$\theta((\Lambda^{-1}y_1)^0 - (\Lambda^{-1}y_2)^0)\mathcal{H}(y_1)\mathcal{H}(y_2) + \theta((\Lambda^{-1}y_2)^0 - (\Lambda^{-1}y_1)^0)\mathcal{H}(y_2)\mathcal{H}(y_1). \tag{85}$$

Now there are two possibilities. If Λ does not change the ordering of times, this expression will be the same in all frames. If the two points x_1 and x_2 are separated by a time-like interval, the time ordering will not change, and $S = U^{-1}SU$. We have Lorentz invariance.

However, if the two points are space-like separated, one can move from a frame where $t_1 > t_2$ to one where $t_1 < t_2$. To maintain Lorentz invariance, we must be able to exchange the order of the two $\mathcal{H}$'s in the time ordering. To do this, we need $\mathcal{H}(y_1)\mathcal{H}(y_2) = \mathcal{H}(y_2)\mathcal{H}(y_1)$, or

$$[\mathcal{H}(y_1), \mathcal{H}(y_2)] = 0 \; for \; (y_1 - y_2)^2 < 0. \tag{86}$$

We say "our Hamiltonian commutes outside the light cone." Eqs. 80 and 86 are the requirements Lorentz invariance imposes on any quantum theory.

C. Relativistic spin-zero quantum fields

We want to consider scattering processes in perturbation theory from a state $|\Phi_{k_1,k_2\dots}\rangle \propto a^\dagger(k_1)a^\dagger(k_2)\dots|0\rangle$ to some final state $|\Phi_{k_1',k_2'\dots}\rangle$. We will build interactions Hamiltonians out of fields and their derivatives, since that is all we have in a field theory. Our states are created and destroyed by field operators. In position space, a potential scalar field operator could be

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}}[a(p)e^{ipx}], \tag{87}$$

where we temporarily assume that we only have a field annihilation operator in ϕ. The factors of π and $E(p)$ are a convenient choice (see the notes on conventions), and it lets us quantize with $[a(p), a^\dagger(p')] = \delta^3(p - p')$.

Now, we know we need two things:

- $\mathcal{H}$ must be Hermitian, so $\mathcal{H} = \mathcal{H}(\phi, \phi^\dagger)$
- $[\mathcal{H}(y_1), \mathcal{H}(y_2)] = 0$ for space-like separation, $(y_1 - y_2)^2 < 0$.

Is Eq. 87 good enough to satisfy these requirements? If $[\phi(x), \phi(y)^\dagger] = 0$ for $(x-y)$ space-like, we're good. Let's see:

$$[\phi(x), \phi(y)^\dagger] = \frac{1}{(2\pi)^3} \int \frac{d^3k d^3k'}{\sqrt{4E(k)E(k')}} e^{-i(kx-k'y)} [a(k), a^\dagger(k')] \tag{88}$$

(four dimensional dot products in the exponential, $kx = E(k)x^0 - \vec{k} \cdot \vec{x}$), or, from the commutator $[a(k), a^\dagger(k')] = \delta^3(k - k')$,

$$[\phi(x), \phi(y)^\dagger] \equiv \Delta_+(x - y) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2E(k)} e^{-ik(x-y)}. \tag{89}$$

This is NOT an equal time commutator ($x_0 \neq y^0$). It is a scalar function of $(x - y)$. Let's evaluate it.

(a) $x - y$ time-like: Pick a frame where $\vec{x} - \vec{y} = 0$, $x^0 - y^0 \equiv x = \pm\sqrt{(x-y)^2}$:

$$\begin{aligned} \Delta_+(x) &= \frac{4\pi}{2(2\pi)^3} \int \frac{k^2 dk}{\sqrt{k^2 + m^2}} e^{\mp ix\sqrt{k^2+m^2}} \\ &= \frac{m}{8\pi x} (N_1(mx) \pm J_1(mx)) \end{aligned} \tag{90}$$

using tables,and these are Bessel functions.

(b) $x - y$ space-like. Work in a frame where $t = 0$, $|x| = \sqrt{-(x-y)^2}$

$$\begin{aligned} \Delta_+(x) &= \frac{1}{(2\pi)^3} \int \frac{k^2 dk d\Omega}{\sqrt{k^2 + m^2}} e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{1}{(2\pi)^2} \int \frac{k^2 dk}{\sqrt{k^2 + m^2}} \frac{\sin kx}{kx} \\ &= \frac{m}{4\pi^2 x} K_1(mx) \end{aligned} \tag{91}$$

Tables again, and this is a Bessel function of imaginary argument, $K_1(mx) \sim \sqrt{\pi/(2mx)} \exp(-mx)$.

Although we evaluates these expressions at particular points, we can slide the results up and down the light cone by doing boosts. Eq. 90 is benign; the time ordering of time-like separated events is preserved by Lorentz transformations. But Eq. 91 is bad news: it says that $[\phi(x), \phi(y)^\dagger] \neq 0$ outside the light cone. Our guess for $\phi(x)$, Eq. 87, is inconsistent with special relativity.

What to do? We need something additional to cancel Δ_+ outside the light cone. Guess number two:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega(p)}} [a(p)e^{ipx} + \eta a_c^\dagger(k) e^{-ipx}], \tag{92}$$

Now, if we assume that a and a_c commute, and $[a_c(p), a_c(p')^\dagger] = \delta^3(p - p')$ the field commutator is

$$[\phi(x), \phi(y)^\dagger] \sim [a + a_c^\dagger, a^\dagger + a_c] = \Delta_+(x - y) - |\eta|^2 \Delta_+(y - x). \tag{93}$$

Since Δ_+ is an even function outside the light cone, we can achieve cancellation by setting $\eta = 1$ (and you can absorb phases into a_c if you want).

So (punch line) to satisfy Lorentz invariance, for every particle annihilated by ϕ there must be another particle created by ϕ, and it has to have the same mass, so that the two Δ_+ functions cancel. Hmm... what about its quantum numbers?

If some quantity is conserved, its charge operator commutes with the Hamiltonian,

$$[Q, \mathcal{H}] = 0 \tag{94}$$

or, if $\mathcal{H}$ is a product of fields,

$$[Q, \prod_{i \in \mathcal{H}} \phi_i(x)] = 0. \tag{95}$$

Now,

$$\begin{aligned} [Q, a_i(k)] \left|\psi\right\rangle &= Q[a_i(k) \left|\psi\right\rangle] - a_i(k) Q \left|\psi\right\rangle \\ &= (q_\psi - q_i) a_i \left|\psi\right\rangle - q_\psi a_i \left|\psi\right\rangle \\ &= -q_i a_i \left|\psi\right\rangle \end{aligned} \tag{96}$$

or $[Q, a_i] = -q_i a_i$. Similarly, $[Q, a_{ci}^\dagger] = +q_{ic} a_{ic}^\dagger$. The physical interpretation of Eq. 95 is that we want the total charge annihilated at any interaction to be zero. This means that the total charge never changes; it is conserved. To achieve this, it must be that

$$q_i = -q_{ic} \tag{97}$$

for all i. The particle annihilated by a_{ic} has to have the opposite charge as the particle annihilated by a, in addition to having the same mass. This is the antiparticle of i. The field operator creates antiparticles and annihilates particles with exactly the same strength.

Antiparticles do not exist because the Dirac equation has negative energy solutions. Antiparticles exist to make the S-matrix Lorentz invariant.

Some comments:

- a_c is called the "charge conjugate field." (we will describe charge conjugation later on).
- $\eta = 1$ for scalars, for convenience. We will discuss spinors ($j = 1/2$) later.

D. Example 1: ϕ^4 field theory

For our first example let us set $a_c = a$ so ϕ is Hermitian. (This is called a "real scalar field" in the literature.) Pick an interaction Hamiltonian density

$$\mathcal{H}(x) = \frac{g}{4!} \phi(x)^4. \tag{98}$$

Examples of such systems are the Higgs boson, or in statistical mechanics, the field theoretic analog of the Ising model. Now calculate a scattering process $k_1 + k_2 \to k_3 + k_4$. We can do this in first order perturbation theory,

$$S(k_1 + k_2 \to k_3 + k_4) = 1 - i \int d^4x \, \langle \Phi_{k_3,k_4} | \mathcal{H}(x) | \Phi_{k_1,k_2} \rangle \tag{99}$$

where

$$|\Phi_{k_1,k_2)} = \sqrt{2E_i 2E_j} a(k_i\rangle^\dagger \, a(k_k)^\dagger \, |0\rangle \, . \tag{100}$$

The "1" term does not contribute to scattering so we discard it.

Let us schematically evaluate the matrix element. Write the field operator four times using Eq. 95, insert it in Eq. 99, and inside the thicket of math symbols you will find the expression

$$\frac{1}{4!} \langle 0|a(k_3)a(k_4)[a_a + a_a^\dagger][a_b + a_b^\dagger][a_c + a_c^\dagger][a_d + a_d^\dagger]a(k_1)^\dagger a(k_2)^\dagger|0\rangle \tag{101}$$

Assume all the $k_a \ldots k_d$ are unequal (the equal ones form a set of measure zero). The only non-vanishing terms are those for which two of the a's, and two of the $a^\dagger$'s, in the brackets, contract against the in and out states. Make a table: there are 4 ways to contract a_d or $a_d^\dagger$ against an $a(k)$ or an $a(k)^\dagger$. This leaves 3 ways to contract the a_c or $a_c^\dagger$, two ways for the a_b (etc) and one way is left. This gives an overall 4! combinitorial factor. Each contraction has the same weight. So we are left with

$$S(k_1 + k_2 \to k_3 + k_4) = -ig \int \frac{d^4x}{(2\pi)^6} e^{i(k_1+k_2-k_3-k_4)x} \frac{\sqrt{2E_1 2E_2 2E_3 2E_4}}{\sqrt{2E_1 2E_2 2E_3 2E_4}} \tag{102}$$

or

$$S(k_1 + k_2 \to k_3 + k_4) = -ig \frac{(2\pi)^4}{(2\pi)^6} \delta^4(k_1 + k_2 - k_3 - k_4) \tag{103}$$

If you have read the notes on "Conventions for fields and scattering amplitudes," you know what comes next. With $S = 1 + \mathcal{T}$, we can strip off the delta-function and the factors of 2π, leaving the "effective T-matrix T" or "invariant amplitude M" to plug into the formula for the cross section. I'll recopy it, to save you time:

$$d\sigma = \frac{1}{4E_1 E_2 v_{rel}} |M|^2 (2\pi)^4 \delta^4(\sum k_f - \sum k_i) \prod_{j=1}^{n_f} \frac{d^3k_j}{(2\pi)^3 2E(k_j)}. \tag{104}$$

In this case, the invariant amplitude is very simple,

$$M = -ig. \tag{105}$$

What is the cross section? We have to integrate the momenta over the delta function. We can get this with (yet) another trick: writing $E = p^0$ we convert the three dimensional integral to a four dimensional one

$$\frac{d^3p}{2E} = d^4p \delta^+(p^2 - m^2) \tag{106}$$

The δ^+ means to take only the positive square root. Then the phase space integral is

$$d^4k_3 d^4k_4 \delta^4(k_3 + k_4 - k_1 - k_2) \delta^+(k_3^2 - m^2) \delta^+(k_4^2 - m^2) = \frac{d^3k_3}{2E_3} \delta^+((k_1 + k_2 - k_3)^2 - m^2) \tag{107}$$

Let's evaluate this in the center of mass frame:

$$\begin{aligned} k_1 &= (E, \vec{k}) \\ k_2 &= (E, -\vec{k}) \\ k_3 &= (E_3, \vec{k}) \end{aligned} \tag{108}$$

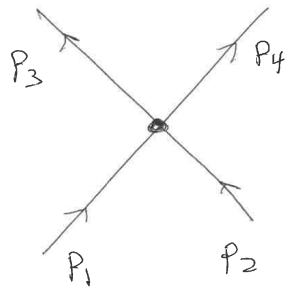

FIG. 1. Lowest order scattering Feynman diagram for $\mathcal{H} = g\phi^4/4!$.

where $k_3^2 + m^2 = E_3^2$. The delta function is $\delta((k_1 + k_2)^2 - 2(k_1 + k_2) \cdot k_3) = \delta(s - 4EE_3)$ where s is called a "Mandelstam invariant': it is a generalization of the squared center of mass energy, $s = (k_1 + k_2)^2 = 4E_{CM}^2$. Incidentally, the relative velocity is $v_{rel} = 2k/E$. Introducing the angular dependence through $d^3k = k^2 dk d\Omega$, a few steps of algebra give

$$\frac{d\sigma}{d\Omega} = \frac{g^2}{8\pi^2 s} \tag{109}$$

(isotropic scattering) and the total cross section is

$$\sigma = \frac{g^2}{2\pi s}. \tag{110}$$

Many ingredients in this calculation will be repeated over and over. To keep track of these ingredients, it is convenient to introduce a diagrammatic language – Feynman diagrams and Feynman rules. Our little exercise gives us some proto-rules:

1) For this theory, each order in perturbation theory contributes a weight $-ig/4!$. The $-i$ is from the Dyson series, the 4! from the $g/4!$ in $\mathcal{H}$. We display this weight as a vertex with four lines meeting. Note that momentum is conserved at the vertex.

2) Each external line (incoming or outgoing particle) contributes a weight 1 to M or $1/((2\pi)^3\sqrt{2E(k)}$ to the T-matrix.

3) There is an overall combinitorial factor of 4! contracting field in $\mathcal{H}$ against those in the in - and out - states

The associated diagram is shown in Fig. 1. Feynman diagrams are not cartoons! Each line or vertex refers to some specific mathematical function.

E. Example 2: Discovering the propagator

For our next example, imagine three species of scalar fields A, B, and C, none of which is its own antiparticle. Let

$$\mathcal{H}(x) = g[A(x)B(x)C(x) + h.c.] \tag{111}$$

Let's calculate $BC \to BC$ scattering. We need second order perturbation theory.

$$S(k_b + k_c \to k_b' + k_c') = \frac{(-i)^2}{2!}\int d^4x d^4y \, \langle \Phi_{k_b',k_c'}|T(\mathcal{H}(x)\mathcal{H}(y))|\Phi_{k_b,k_c}\rangle \tag{112}$$

The two vertices are at space-time points x and y. At one of these points, b' and c' are created, and at the other point b and c are annihilated. Each choice gives the same weight to S, so pick one of them and cancel the 2!. As before, we have to evaluate

$$\langle 0|b(k_b')c(k_c')\{T((a_1^\dagger + a_{c1})(b_2^\dagger + b_{c2})(c_3^\dagger + c_{c3})(a_4 + a_{c4}^\dagger)(b_5 + b_{c5}^\dagger)(c_6 + c_{c6}^\dagger))\}b(k_b)^\dagger c(k_c)^\dagger|0\rangle \, . \tag{113}$$

b_2 annihilates $b(k_b)^\dagger$, c_3 annihilates $c(k_c)^\dagger$, and only

$$\langle 0|T((a_1^\dagger + a_{c1})(a_4 + a_{c4}^\dagger)|0\rangle \tag{114}$$

is left. In full gory detail we have

$$S = -g^2\int d^4x d^4y \frac{e^{-ik_b y}}{(2\pi)^{3/2}}\frac{e^{-ik_c y}}{(2\pi)^{3/2}}\frac{e^{ik_b' x}}{(2\pi)^{3/2}}\frac{e^{ik_c' x}}{(2\pi)^{3/2}}\langle 0|T(A(x)^\dagger A(y))|0\rangle \tag{115}$$

The quantity $\langle 0|T(A(x)^\dagger A(y))|0\rangle$ is called the Propagator. Anticipating the calculation, it is a function of $(x-y)$. We can perform the x and y integrals by shifting variables to x and $x-y$. The x integral gives a delta function leaving

$$S = -g^2\frac{(2\pi)^4\delta^4(k_{b'} + k_{c'} - k_b - k_c)}{(2\pi)^6}[-i\Delta_F(k_b + k_c)] \tag{116}$$

where

$$-i\Delta_F(k) = \int d^4z e^{ikz}\langle 0|T(A(z)^\dagger, A(0))|0\rangle \, . \tag{117}$$

Of course,

$$M = (-ig)^2[-i\Delta_F(k_b + k_c)]. \tag{118}$$

Again we can draw a picture. See Fig. 2. And also note that x^0 is not necessarily later (or earlier) than y^0.

We have to calculate the propagator. We begin with the coordinate space version

$$-i\Delta_F(x-y) = \theta(x^0-y^0)\,\langle 0|A(x)^\dagger A(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|A(y)A(x)^\dagger|0\rangle \tag{119}$$

The first term is

$$\frac{\theta(x^0-y^0)}{(2\pi)^3}\int \frac{d^3kd^3k'}{\sqrt{4E(k)E(k')}}\,\langle 0|[a(k)^\dagger e^{ikx} + a_c(k)e^{-ikx}][a(k')e^{-ik'y} + a_c(k')^\dagger e^{ik'y}]|0\rangle\,. \tag{120}$$

The $a_c a_c^\dagger$ term is the only non-vanishing one, and $k = k'$ is needed, so the first term collapses to

$$\frac{\theta(x^0-y^0)}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3 2E(k)}e^{-ik(x-y)}. \tag{121}$$

The second term is similar. The coordinate space propagator is

$$-i\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(k)}[\theta(x^0-y^0)e^{-ik(x-y)} + \theta(y^0-x^0)e^{ik(x-y)}] \tag{122}$$

Let's write this as a four dimensional Fourier transform. We use the integral expression with a convergence factor

$$\int_{-\infty}^{\infty} dt e^{i\omega t}\theta(t) = \int_0^\infty dt e^{i\omega t}e^{-\epsilon t} = \frac{i}{\omega + i\epsilon} \tag{123}$$

and so

$$\theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\epsilon} \tag{124}$$

Convert this to a contour integral and close the contour in the upper half plane if $t < 0$, in the lower half plane if $t > 0$, pick up the residue, and it checks.

Then (here note the $(t, \vec{x})$ and define $E(k) = \sqrt{k^2 + m^2}$)

$$-i\Delta_F(x) = \frac{i}{(2\pi)^4}\int \frac{d^3kd\omega}{2E(k)(\omega + i\epsilon)}[e^{-iE(k)t + i\vec{k}\cdot\vec{x} - i\omega t} + e^{iE(k)t - i\vec{k}\cdot\vec{x} + i\omega t}] \tag{125}$$

The Fourier transform is

$$\begin{aligned} -i\Delta_F(q) &= \int d^4x(-i\Delta_F(x))e^{iqx} \\ &= i\int \frac{d^4x}{(2\pi)^4}\frac{d^3kd\omega}{2E(k)(\omega + i\epsilon)}[e^{-iE(k)t + i\vec{k}\cdot\vec{x} - i\omega t} + e^{iE(k)t - i\vec{k}\cdot\vec{x} + i\omega t}]e^{iqx} \end{aligned} \tag{126}$$

and so $\vec{q} = \pm\vec{k}$ from the $\vec{x}$ integral (the sign doesn't matter). The $\int dx^0$ gives a $2\pi\delta(q^0 - \omega - E(k))$ for the first term and a $2\pi\delta(q^0 + \omega + E(k))$ for the second one. The expression collapses to

$$-i\Delta_F(q) = i\int \frac{d\omega}{2E(q)(\omega + i\epsilon)}[\delta(\omega + E(q) - q^0) + \delta(\omega + E(q) + q^0)] \tag{127}$$

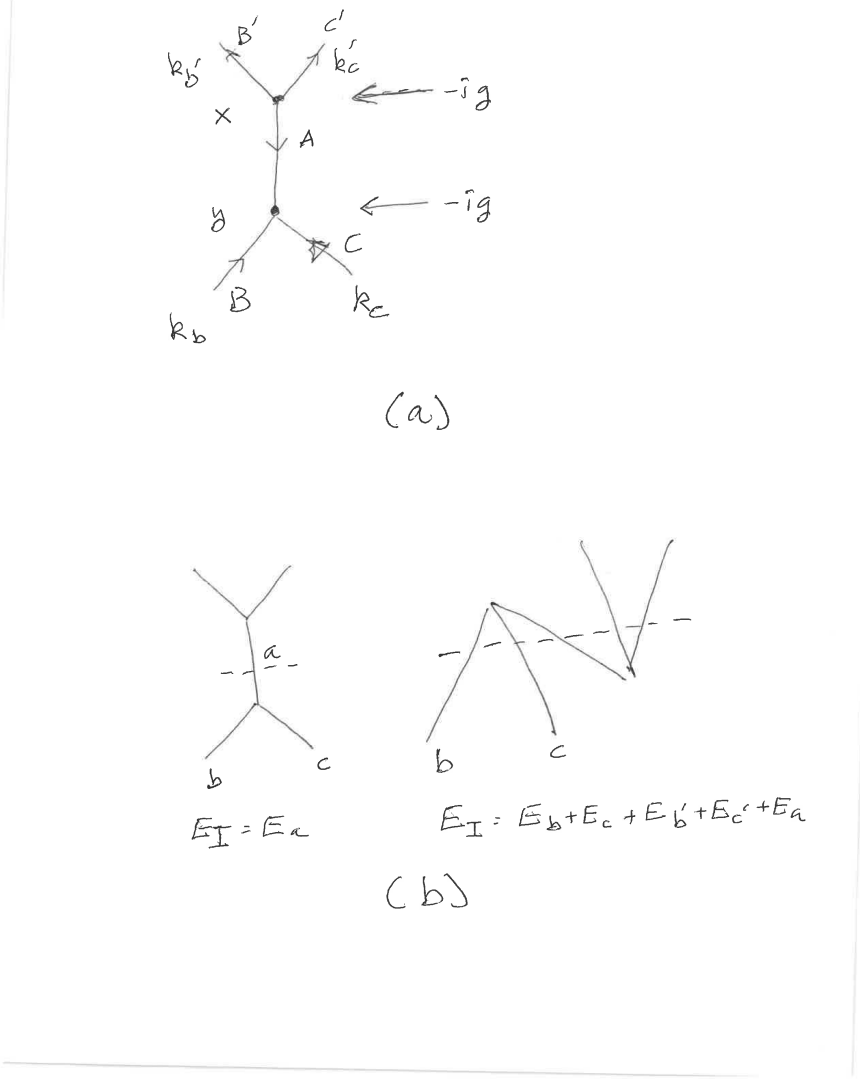

FIG. 2. Lowest order scattering Feynman diagram for $BC \to BC$ with $\mathcal{H} = ABC + h.c.$

and so we can write the result in two equivalent ways: First, separate the terms:

$$-i\Delta_F(q) = \frac{i}{2E(q)}[\frac{1}{q^0 - E(q) + i\epsilon} + \frac{1}{-q^0 - E(q) + i\epsilon}] \tag{128}$$

and second, combine them,

$$-i\Delta_F(q) = \frac{i}{q^2 - m^2 + i\epsilon}. \tag{129}$$

This is a very important object, the propagator for a scalar field.

There is an associate Feynman diagram, Fig. 2.

We are not going to use Eq. 128 any more, but let's look at it before moving on. The first term is the antiparticle piece, where the time ordering is that first the antiparticle is produced, and later it annihilates to produce the outgoing BC pair. The second term is

the opposite time order, emission preceded absorption. The energy denominators are the ones in the "usual" quantum mechanical second-order perturbation theory formula for the T-matrix

$$T = \sum_I \frac{\langle f|V|I\rangle \langle I|V|i\rangle}{E_i - E_I}. \tag{130}$$

This you can see if you write $\vec{q} = \vec{k}_b + \vec{k}_c = -p_a$, $q^0 = E_b + E_c$, and change notation so $E_a = E(q)$. Then

$$M \propto 0g^2[\frac{i}{2E_a}\{\frac{1}{E_b + E_c - E_a + i\epsilon} + \frac{1}{-E_a - E_b - E_c + i\epsilon}\}] \tag{131}$$

The second term is trickier until you realize that the initial energy is $E_b + E_c$ and the intermediate energy $E_I = E_b + E_c + E_{b'} + E_{c'} + E_a$ so the denominator is also the correct energy difference.

In nonrelativistic situations, the intermediate states from the different time orderings are just different, and you can't use all our tricks to get a beautiful answer, like Eq. 129. This is all people had, before Feynman (1948-ish). How would you like to do quantum field theory this way? It was horrible... Imagine higher order calculations! Lorentz invariance is hidden, too. There is an old book, Heitler, "The quantum theory of radiation," take a look at it, on a rainy day.

Now go back to the S-matrix

$$\begin{aligned} S(k_b + k_c \to k'_b + k'_c) &= \frac{(-i)^2}{2!}\int d^4x d^4y \frac{e^{-i(k_b+k_c)y} e^{-i(k_{b'}+k_{c'})x}}{(2\pi)^6}[-i\Delta_F(x-y)] \\ &= -g^2\frac{(2\pi)^4}{(2\pi)^6}\delta^4(k_{b'} + k_{c'} - k_b - k_c)\frac{i}{(k_{b'} + k_{c'})^2 - m_a^2 + i\epsilon}] \end{aligned} \tag{132}$$

Here, $k_a = k_b + k_c = k_{b'} + k_{c'}$ so four-momentum is conserved at each vertex, but $k_a^2 = (k_{b'} + k_{c'})^2 \neq m_a^2$. We say "the intermediate particle is off mass shell."

Finally, what's the cross section? In a CM frame, $(k_b + k_c)^2 = E_{CM}^2$ and

$$\frac{d\sigma}{d\Omega} \sim |\frac{1}{E_{CM}^2 - m_a^2}|^2 \tag{133}$$

This diverges at $E_{CM} = m_a$. This is the mark of a resonance, when the CM energy is exactly right to produce the intermediate particle on shell, the cross section becomes very large. The divergence is not physical, it is an artifact of working to lowest order the perturbative expansion, where the resonance has zero width. The resonance is the physics!

F. Example 3: Another amplitude

For our next example, keep $\mathcal{H} = g(ABC + A^\dagger B^\dagger C^\dagger)$, but compute the reaction $B\bar{C} \to B\bar{C}$.

$$S = -g^2 \int\int d^4x d^4y e^{-(k_b - k_{c'})x} e^{-(k_{b'} - k_c)y} \langle 0|T(A(x)^\dagger A(y))|0\rangle \tag{134}$$

This gives

$$M = -g^2 \frac{i}{(k_b - k_{c'})^2 - m_a^2 + i\epsilon} \tag{135}$$

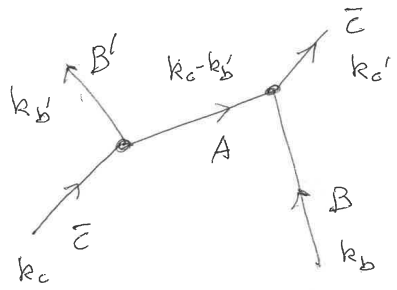

FIG. 3. Lowest order scattering Feynman diagram for $B\bar{C} \to B\bar{C}$ with $\mathcal{H} = ABC + h.c.$

Let's write the S-matrix a little more suggestively,

$$\begin{aligned} S &= -g^2 \int\int d^4x d^4y e^{-(k_b - k_{c'})ix} e^{-i(k_{b'} - k_c)y} \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_a^2 + i\epsilon} e^{iq(x-y)} \\ &= -g^2 \frac{1}{(2\pi)^6} \int \frac{d^4q}{(2\pi)^4} [(2\pi)^4 \delta^4(k_b - k_{c'} + q)][(2\pi^4)\delta^4(k_{b'} - k_c + q)] \frac{i}{(k_b - k_{c'})^2 - m_a^2 + i\epsilon} \end{aligned} \tag{136}$$

Writing this expression tells us that we have conservation of four-momentum at each vertex. We also integrate over all internal momenta (in this case, just q). One of the delta-functions will give overall four momentum conservation when all the integrals are done.

The associated Feynman diagram is shown in Fig. 3.

G. Wick's theorem

Wick's theorem is used to reduce the nasty time ordered products of field operators into simple expressions, by recasting the time ordered product of many fields into a product of time ordered products of two fields (which are just propagators). Let me describe it for bosons; fermions have some permutation factors (no surprise). Write each field in its creation and annihilation parts, $\phi = \phi^+ + \phi^-$ where $\langle 0| \phi^+ = \phi^- |0\rangle = 0$. Recall the normal ordering operator,

$$: \phi_i \phi_j := \phi_i^+ \phi_j^+ + \phi_i^+ \phi_j^- + \phi_j^+ \phi_i^- + \phi_i^- \phi_j^- . \tag{137}$$

We can write the time ordered product of two fields as the normal ordered product plus a constant C, since the rearrangements needed to go from one to the other just involve commutators of fields.

$$T(\phi_1 \phi_2) =: \phi_1 \phi_2 : + C. \tag{138}$$

Now take the vacuum expectation value of this expression. The vacuum expectation value of the normal ordered product vanishes, so

$$\langle 0|T(\phi_1 \phi_2)|0\rangle = 0 + C. \tag{139}$$

and so we have

$$T(\phi_1 \phi_2) =: \phi_1 \phi_2 : + \langle 0|T(\phi_1 \phi_2)|0\rangle . \tag{140}$$

This is for two fields. Wick's theorem generalizes this to n fields. For even n, it says

$$\begin{aligned} T(\phi_1 \phi_2 \dots \phi_n) = &: \phi_1 \phi_2 \dots \phi_n : \\ &+ \langle 0|T(\phi_1 \phi_2)|0\rangle : \phi_3 \dots \phi_n : + \text{order } n-2 \text{ permutations} \\ &+ \sum \langle 0|T(\phi_i \phi_j)|0\rangle \langle 0|T(\phi_k \phi_l)|0\rangle : \phi \dots \phi : \\ &+ \dots \\ &+ \sum \langle 0|T(\phi_i \phi_j)|0\rangle \langle 0|T(\phi_k \phi_l)|0\rangle \dots \langle 0|T(\phi_m \phi_n)|0\rangle \end{aligned} \tag{141}$$

The proof is not so interesting. It is done by induction, and it is in Bjorken and Drell, "Relativistic quantum fields" (and probably other books). But the formula is quite useful. Return to $\mathcal{H} = ABC + h.c.$ where we need

$$\begin{aligned} \langle B'\bar{C}'|T(\mathcal{H}(x)\mathcal{H}(y)|B\bar{C}\rangle = \langle B'\bar{C}| \{ &: A_i^\dagger B_1^\dagger C_1^\dagger A_2 B_2 C_2 : \\ &+ \langle 0|T(A_1^\dagger A_2|0\rangle : B_1^\dagger C_1^\dagger B_2 C_2 : \\ &+ \langle 0|T(B_1^\dagger B_2|0\rangle : A_1^\dagger C_1^\dagger A_2 C_2 : \\ &+ \dots \\ &\} |B\bar{C}\rangle \end{aligned} \tag{142}$$

Only the $\langle B'\bar{C}| : B_1^\dagger C_1^\dagger B_2 C_2 : |B\bar{C}\rangle$ term has the right combination of operators to annihilate $B\bar{C}$ and create $B'\bar{C}'$. Everything else gives zero.

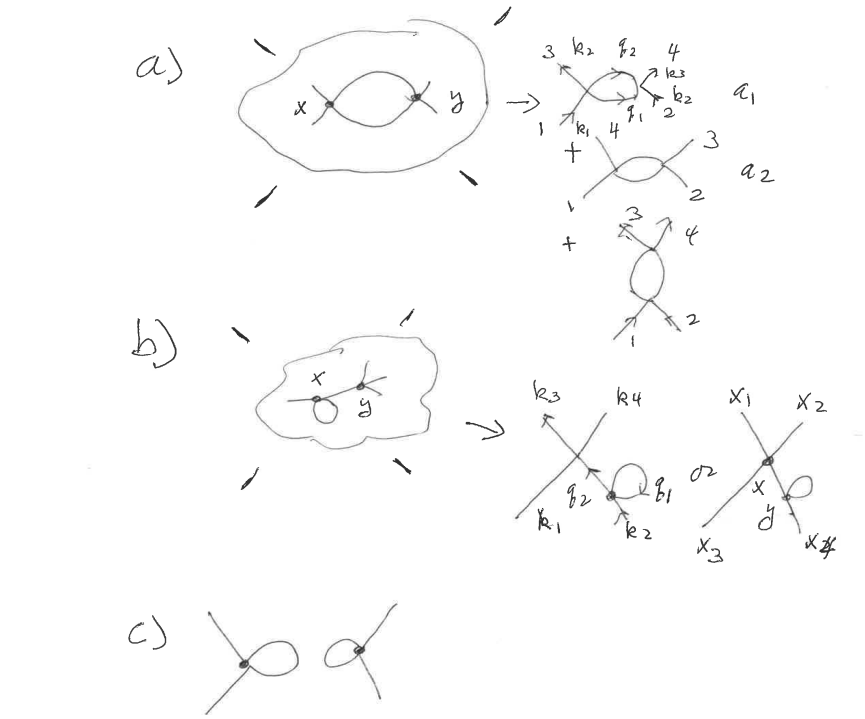

FIG. 4. Higher order scattering processes with $\mathcal{H} = g\phi^4/4!$.

H. A glance at higher orders

We return to

$$\mathcal{H} = \frac{g}{4!}\phi^4(x) \tag{143}$$

where ϕ is real, so $a = a_c$. Let us look at the order g^2 contributions to scatttering $\phi_1 + \phi_2 \to \phi_3 + \phi_4$.

$$S^{(2)} = \frac{1}{2!}\left(\frac{-ig}{4!}\right)^2 \int d^4x d^4y \, \langle \Phi_{34}|T(\phi(x)^4\phi(x)^4)|\Phi_{12}\rangle \tag{144}$$

When we use Wick's theorem to clean this up, we want to pull off terms with normal products of four fields.

$$T(\phi(x)^4\phi(x)^4) = \sum_{ijkl} \langle 0|T(\phi_j\phi_k)||0\rangle \, \langle 0|T(\phi_k\phi_l)|0\rangle : \phi_m\phi_n\phi_o\phi_p : \tag{145}$$

Four of $(ijklmnop)$ are at x, the other four are at y.

We have three generic kinds of terms (see Fig. 4):

- a) $i = k = x$, $j = l = y$ and permutations
- b) $i = x$, $j = y$, $k = l = x$ OR y
- c) In addition, we may have $i = j = x$, $k = l = y$. These are called "disconnected diagrams." They don't contribute to scattering (they won't be proportional to $\delta^4(k_1 + k_2 - k_3 - k_4)$. We will neglect them.

There are three "type (a)" graphs. They all have a common combinitorial factor C, which we will ignore for now. For eaxmple,

$$\begin{aligned} S_{a_1}a = C(-ig)^2 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-k_1-k_3) \\ \times(2\pi)^4\delta^4(q_1+q_2-k_2-k_4) \times \frac{-i}{q_1^2-m^2+i\epsilon} \times \frac{-i}{q_2^2-m^2+i\epsilon} \end{aligned} \tag{146}$$

so that the invariant amplitude is

$$M_{a1} = -Cg^2 \int \frac{d^4q_1}{(2\pi)^4}[\frac{-i}{q_1^2-m^2+i\epsilon}][\frac{-i}{(k_1-k_3-q_1)^2-m^2+i\epsilon}]. \tag{147}$$

This is called a graph with a "loop integral." The momentum q_1 is unconstrained and must be integrated over.

There are four type (b) graphs.

$$\begin{aligned} (2\pi)^4\delta^4(k_1+k_2-k_3-k_4)M_{b_1} \propto (-ig)^2 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}(2\pi)^4\delta^4(q_2-k_2)\delta^4(k_3+k_4-q_2-k_1) \\ \times\frac{-i}{q_1^2-m^2+i\epsilon} \times \frac{-i}{q_2^2-m^2+i\epsilon} \end{aligned} \tag{148}$$

This says that

$$M_{b_1} \propto (-ig)^2 \frac{-i}{k_2^2-m^2+i\epsilon}\int \frac{d^4q_1}{(2\pi)^4}\frac{-i}{q_1^2-m^2+i\epsilon}. \tag{149}$$

This is getting mechanical, but the result is worrisome: the integral is divergent!

$$\int_{|q|<\Lambda} \frac{d^4q_1}{(2\pi)^4}\frac{-i}{q_1^2-m^2+i\epsilon} \propto \int^{\Lambda} \frac{d^3q}{q^2} \propto \Lambda^2 \tag{150}$$

where I have put in a cutoff to make sense of the expression.

This is our first encounter with the "divergences" of quantum field theory. They arise from the short distance sector of the theory. This is most easily seen by working in coordinate space and keeping all the propagators,

$$M(x_1,x_2,x_3,x_4) \propto (-ig)^2 \int d^4x d^4y [-i\Delta_F(x_1-x)][-i\Delta_F(x_2-x)][-i\Delta_F(x_3-x)] \\ [-i\Delta_F(x-y)][-i\Delta_F(y-x_4)][-i\Delta_F(y-y)] \tag{151}$$

The divergence comes from the last term: emission and absorption at the same point,

$$-i\Delta_F(x=0) = \int \frac{d^4q_1}{(2\pi)^4} \frac{-i}{q_1^2 - m^2 + i\epsilon}. \tag{152}$$

This contribution is present even though we might only be interested in scattering (or other processes) taking place at very low energies, or at long distances. To proceed further at this point, we have to do several things:

1. Enumerate all possible divergent contributions: are there a finite number, or an infinite number? It turns out that in ϕ^4 theory, only two point functions and four point functions contain divergences.

2. Regulate the divergences: do something to make them well defined. In the example, I cut off the momentum integral, restricting $|q| < \Lambda$.

3. Do calculations in the regularized theory and make sense of them.

This is a project for later in the semester.

I. Feynman rules

You may have noticed a certain robotic similarity in all our examples. All our manipulations can be absorbed into a set of rules for constructing invariant amplitudes, in terms of Feynman diagrams.

Here are the Feynman rules for scalar field theory:

Given an interaction Hamiltonian density which is a product of fields (for example $\mathcal{H} = g\phi^4/4!$)

1) Draw all connected diagrams allowed by $\mathcal{H}$. For each vertex there is a factor $-i$, a factor of the coupling constant, and there is an overall $1/n!$ from the Dyson series. Each diagram will typically have some associated combinitorial factor.

2) Each internal boson carrying a momentum q contributes a $-i\Delta_F(x-y)$ in coordinate space, or

$$\int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \tag{153}$$

in momentum space.

3) Each vertex counts either (orient all the momenta in or out for this)

$$(2\pi)^4 \delta^4(\sum q) \tag{154}$$

or

$$\int d^4x e^{i\sum qx} \tag{155}$$

As an alternative to rules 2 and 3,
2'-3') Conserve momentum at every vertex, integrate

$$\int \frac{d^4q}{(2\pi)^4} \tag{156}$$

over all unconstrained momenta

And compute the invariant amplitude from the resulting expression by snipping off delta functions and 2π's associated with the incoming and outgoing particles

$$M\frac{(2\pi)^4}{(2\pi)^{3(n_i+n_f)/2}}\delta^4(\sum p_{out} - \sum p_{in}) \to M \tag{157}$$

The differential cross section is

$$d\sigma = \frac{1}{4E_1E_2v_{rel}}|M|^2(2\pi)^4\delta^4(\sum k_f - \sum k_i)\prod_{j=1}^{n_f}\frac{d^3k_j}{(2\pi)^3 2E(k_j)} \times \frac{1}{n_f!} \tag{158}$$

The new $1/n_f!$ is a counting factor for identical particles, basically Bose statistics. It is discussed in Bjorken and Drell, "Relativistic Quantum Mechanics," p. 136, or in Zee, a hint on p. 54.

J. Conclusion

This chapter has been a terse but fairly complete introduction to quantum field theory based on Hamiltonians. We only looked at scalar particles. The interesting (and more physical, at least before 2012) quantum field theories contain fermions and gauge fields. Perturbation theory for such systems is very similar to what we have done: there will be vertices and propagators, which only differ in detail (important details, true) from what we have found.

However, before we push on to these systems, I want to go back to the drawing board. You recall, that there are two ways of introducing quantum mechanics, either with Hamiltonians, or with the path integral. Path integrals are not so useful for practical problems in single particle quantum mechanics, but they really come into their own when they are applied to quantum field theories. We have to become comfortable with them. That is the subject of the next chapter.

III. CONCLUSION FOR ALL THE NOTES TO THIS POINT

References: Late in the class I discovered David Tong's lectures [1], which are close to what I taught.

Otherwise I had a hard time finding a book that does things exactly like I have presented them here. I think there are two reasons for this:

First, quantum field theory can be developed right from the start using path integrals. That is the approach of Srednicki and of Zee (mostly). We are going to do path integrals next. I started with Hamiltonians because I thought it was closer to what you already knew – nonrelativistic quantum mechanics.

Second, to actually do a computation there are a lot of things you have to know all at once. Picking what to teach first is a judgement call.

Having said that, there are bits of the following books which come close to these lectures: Schwartz 7.2-7.4, Srednicki Ch. 3, Peskin and Schroeder Sec. 2.4. Weinberg Volume 1, chapter 6 parallels a lot of what I did in these notes. I was surprised to see that Sidney Coleman's lecture notes [2] basically parallel what I am doing. They are different because his audience was Harvard theoretical physics graduate students, and because he was much more creative than I am.

IV. GAUSSIAN INTEGRALS WITH ZERO EIGENVALUES

Gauge redundancy appears in the path integral formulation of a gauge theory through "flat directions" in the functional integral.

Consider the naive formula for the partition function for a gauge theory

$$Z = \int DA_\mu \exp(iS) \tag{159}$$

where dA is a shorthand for integration over the vector potential,

$$\int DA = \int \prod_{x,\mu} dA(x)_\mu \tag{160}$$

and the action is

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{161}$$

Go to p-space, (incorrect 2π's)

$$A_\mu(x) = \int d^4p e^{ipx} A_\mu(p) \tag{162}$$

and the action becomes

$$\begin{aligned} S &= \frac{1}{4}\int d^4p(p_\mu A_\nu(-p) - p_\nu A_\mu(-p))(p_\mu A_\nu(p) - p_\nu A_\mu(p)) \\ &= \frac{1}{2}\int d^4p A_\mu(-p)A_\nu(p)[g^{\mu\nu}p^2 - p^\mu p^\nu]. \end{aligned} \tag{163}$$

The propagator is the inverse of the object in brackets, a 4×4 matrix. Unfortunately, it has zero eigenvalues for all values of p and so it is not invertable. (As an example, write the matrix out for $p^\mu = (p,0,0,0)$.) The naive formula has issues.

Let's consider, more generally, a Gaussian integral

$$G(A) = \int_{-\infty}^{\infty} dx_1 \dots dx_N \exp(-x^T A x) = \frac{1}{\sqrt{\det A}} \tag{164}$$

(again neglecting factors of π). This integral is well defined as long as the determinant does not vanish. If the determinant does vanish, we have a problem – the integral blows up.

We can see the origin of this blowup if we change variables to a basis where A is diagonal. If A is a real, symmetric $N \times N$ matrix, then we can rotate x with an orthogonal matrix R,

$$y = Rx \tag{165}$$

and

$$\prod_{i=1}^{N} dx_i = \prod_{i=1}^{N} dy_i \tag{166}$$

so

$$\begin{aligned} G(A) &= \int \prod dy \exp(-x^T R^T A R x) \\ &= \int \prod dy_i \exp(-y_i^2 d_i) \\ &= \sqrt{\frac{\pi}{\prod_i d_i}} \end{aligned} \tag{167}$$

We have problems (we get an infinity) of one of the d_i's is zero. In that case, it comes from

$$\int_{-\infty}^{\infty} dy = \infty. \tag{168}$$

Can we get a sensible answer for the integral even when the determinant vanishes? If we can divide out the culprit infinite integral, the answer is Yes. How can we do this? Suppose we have n zero eigenvalues. Let's define the restricted Gaussian integral

$$G_R(A) = \int dy_1 \dots dy_{N-n} \exp(-x^T A x). \tag{169}$$

This is awkward, since we have to know the y's before hand. Instead, define new variables y_{N-n+1} to y_N and write

$$G_R(A) = \int dy_1 \dots dy_{N-n} dy_{N-n+1} \dots dy_N \delta(y_{N-n+1}) \dots \delta(y_N) \exp(-x^T A x). \tag{170}$$

We can now change variables back from y to x,

$$dy_1 \dots dy_N = dx_1 \dots dx_N \det|\frac{\partial y}{\partial x}| \tag{171}$$

so that

$$G_R(A) = \int \prod_{i=1}^{N} dx_i \{\det|\frac{\partial y}{\partial x}| \prod_{j=N-n+1}^{N} \delta(y_j)\} \exp(-x^T A x). \tag{172}$$

The extra factors, the terms inside the curly brackets, restrict the integration from its original $N-$dimensional space to an $N-n$ dimensional one. Of course, one must cleverly choose the y's to restrict the integration measure, or else the Jacobian $\det|\partial y/\partial x|$ will be singular.

A. More about gauge theories

I've just described the mechanical construction we will follow, to construct the path integral for the photon. But we can do better. Let's try to derive some more general formulas for gauge theories, and then really look closely at gauge invariance. I will restrict all my discussions to the case of ordinary electrodynamics, but the language will be general enough to consider more complicated situations (like non-Abelian gauge symmetry).

Let's begin by looking at constant functions in a gauge theory. They actually don't exist! the closest one can come to a constant function ϕ is to define a function whose covariant derivative is zero,

$$D_\mu \phi = (\partial_\mu + iA_\mu)\phi = 0 \tag{173}$$

(Note I have set $q = 1$ or redefined $A_\mu = qA_\mu$.) Now

$$\phi(x + dx) = \phi(x) + dx^\mu \partial_\mu \phi \tag{174}$$

which is

$$\phi(x + dx) = \phi(x) - idx^\mu A_\mu \phi(x) \tag{175}$$

if $D_\mu \phi = 0$, and this is

$$\phi(x + dx) = \exp(-idx^\mu A_\mu(x))\phi(x) + O(dx^2) \tag{176}$$

Under a local gauge transformation $\phi(x) \to \exp(-i\Lambda(x))\phi(x) \equiv V(x)\phi(x)$. The exponential in Eq. 176 becomes

$$\begin{aligned} \exp(idx^\mu A_\mu(x)) &\to \exp(idx^\mu(A_\mu(x) + \partial_\mu \Lambda(x))) \\ &= \exp(-i(\Lambda(x + dx) + A_\mu(x)dx^\mu - \Lambda(x))) \\ &= V(x + dx)\exp(-idx^\mu A_\mu(x))V(x)^\dagger \end{aligned} \tag{177}$$

with a slight overkill of notation, $V^\dagger = V^*$ here – it's just a phase rotation. (The more complicated notation allows an immediate generalization to non-Abelian gauge theories like QCD or the Standard Model.)

Note that $\exp(-idx^\mu A_\mu(x))$ transforms by rotations V and $V^\dagger$ "at its ends."

Now iterate Eq. 176 along any curve P from initial point x to final point y:

$$\phi(y) = \exp(-i \int_{P,x}^{y} dx \cdot A\phi(x)). \tag{178}$$

As $\phi(x)$ is carried ("parallel transported," is the jargon borrowed from General Relativity) from point x to point y along a path P, it picks up a phase, the line integral of A along the path. Calling $\phi(y) = U(y, x)\phi(x)$, we notice that under a gauge transformation $\phi(y)' = V(y)\phi(y)$ or

$$\phi(y)' = V(y)U(y, x)V^\dagger(x)V(x)\phi(x) = U(y, x)'\phi(x)'. \tag{179}$$

$U(x, y)$ is often called a "Wilson line," named after Ken Wilson. Under a gauge transformation, it rotates with a phase factor at each end. Notice that this expression implies that $U(x, x)$, a "Wilson loop," is gauge invariant. (For a non-Abelian gauge theory, $\phi(x)$ would be a column vector and U would become a matrix; the gauge invariant object is Tr $U(x, x)$.)

Consider next an infinitesimal path of length a in the $\hat{\mu}$ direction, connecting the point $x - a\hat{\mu}$ to x:

$$\exp(-iaA_\mu(x)') = V(x)\exp(-iaA_\mu(x))V^\dagger(x - a\hat{\mu}) \tag{180}$$

or, expanding,

$$\begin{aligned} 1 - iaA_\mu(x)' &= V(x)[1 - iaA_\mu(x)][V(x)^\dagger - a\partial_\mu V^\dagger] \\ &= 1 - iaV\partial_\mu V^\dagger - aVAV^\dagger + \ldots \end{aligned} \tag{181}$$

or

$$A(x)'_\mu = V(x)A_\mu(x)V(x)^\dagger - iV(x)\partial_\mu V(x)^\dagger. \tag{182}$$

This slightly complicated language comes into its own when considering non-Abelian gauge theories (where $\phi(x)$ becomes a column vector and U becomes a matrix).

Now let's focus on the gauge transformation itself

$$V(x) = \exp i\omega(x). \tag{183}$$

The V's are the elements of an Abelian group G parametrized by the ω's. There is an identity element, $\omega = 0$ or $V = 1$, an inverse, $VV^\dagger = 1$, and a closure relation

$$V(x)'' = V(x)'V(x) \in G. \tag{184}$$

Mathematicians call this G the group $U(1)$ – the group of unitary transformations in one dimension. The group is "compact:" the size of all the elements $||V|| = 1$. Notice that the $U(y, x)$'s are elements of G, too:

$$U(x, x') = U(x, x'')U(x'', x) \in G. \tag{185}$$

Now, remember all our issues with $\int_{-\infty}^{\infty} dA$? If we had compact variables, we would not encounter such integrals.

All this discussion suggests that we should define gauge theories *not* directly in terms of the A_μ's, but in terms of the U's. We do this in the following (roundabout) way:

First, replace the four dimensional space time continuum by a four dimensional lattice of points, with a lattice spacing a. (This probably only makes sense for an Euclidean path integral.) Matter fields (fermions and bosons) are replaced by fields defined on the sites of the lattice, $\psi(x) \to \psi(x_i)$. Gauge transformations are also defined only on the sites (as $V(x_i)$).

Gauge information is transmitted by the $U(x, y)$'s. The minimal size U will span one lattice site

$$U(x + a\hat{\mu}, x) \equiv U_\mu(x) \tag{186}$$

and it transforms as

$$U(x + a\hat{\mu}, x)' = V(x + a\hat{\mu})U(x + a\hat{\mu}, x)V(x)^\dagger \tag{187}$$

Notice $U(x, x + a\hat{\mu}) \equiv U_\mu(x)^\dagger$. Thus, the U_μ's live on the links of the lattice.

Third, we define the integration measure for the path integral as an integration over all the allowed values of the U's. This is called the "Haar measure" (or "invariant integration over the group elements") and is denoted as dU. It is invariant in the sense that if $U' = VUV^\dagger$, then $dU' = dU$. If G is compact, the integration is over a finite range.

Two examples: in our case, where $G = U(1)$, we can parametrize $U = \exp(i\theta)$ and

$$\int dU = \int_{-\pi}^{\pi} d\theta, \tag{188}$$

or geometrically, Haar measure is a line integral around a unit radius circle. Another simple group is $SU(2)$. Any element of $SU(2)$ can be parametrized in terms of the Pauli matrices and four real numbers,

$$U = U_0 1 + i\vec{U} \cdot \vec{\sigma} \tag{189}$$

in terms of which

$$\int dU = \int \prod_{i=0}^{3} dU \delta(1 - \sum_i U_i^2). \tag{190}$$

Then the gauge part of the (Minkowski space) partition function is defined as

$$Z = \int \prod_{i,\mu} dU_\mu(i) \exp iS(U) \tag{191}$$

The Euclidean path integral would have no i in the exponent.

Now, what could $S(U)$ be? It must be gauge invariant, and so the most general S will be a sum of closed paths of U's, multiplied by a set of arbitrary coefficients,

$$S(U) = \sum_x \sum_{path\ j} U_{P_j}(x,x) c_{P_j} \tag{192}$$

where $U_{P_j} = U(x,x_1)U(x_1,x_2)\ldots U(x_n,x)$.

The theory ought to have an interesting continuum limit, that is, if we take the lattice spacing a to zero, the action ought to reduce to something proportional to

$$S = \int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{193}$$

Actually, that is easy to achieve. Any closed path will reduce to this form. Consider the smallest one, a path around a 1×1 plaquette,

$$U = \exp(-a[A_\mu(x) + A_\nu(x + a\hat{\mu}) - A_\mu(x + a\hat{\nu}) - A_\nu(x)]) \tag{194}$$

Write this as $\exp(i\Delta)$ and Taylor expand Δ for small a:

$$\begin{aligned} \Delta &= A_\mu(x) + (A_\nu(x) + a\partial_\mu A_\nu(x)) - (A_\mu(x) + a\partial_\nu A_\mu(x)) - A_\nu(x) \\ &= a(\partial_\mu A_\nu - \partial_\nu A_\mu). \end{aligned} \tag{195}$$

Then

$$U = i + ia^2 F_{\mu\nu} - \frac{a^4}{2} F_{\mu\nu}^2 + \ldots. \tag{196}$$

Clearly

$$\sum_j (U(j) + U^\dagger(j) - 2) = -(a^4 \sum_j) F_{\mu\nu}^2 = -\int d^4x F_{\mu\nu}^2. \tag{197}$$

We could just take the action to be the left hand side of this expression, a sum over all the "plaquettes" of U's. This looks like stupendous overkill, but notice that the compact measure precludes infinities a la

$$\int_{-\infty}^{\infty} dx \exp(-0 \times x^2). \tag{198}$$

We could use this formalism to study the nonperturbative behavior of gauge theories. In fact, that is what people like me do for a living. Instead, let's use it to study gauge theories in perturbation theory. That's easy here. The Haar measure for a $U(1)$ gauge theory is, if $U = \exp(i\theta)$,

$$\int dU = \int_{-\pi}^{\pi} d\theta \tag{199}$$

so if we take $U = \exp(iaA)$,

$$dU = \int_{-\pi}^{\pi} d(aA) \to \lim_{a\to 0} \int_{-\pi/a}^{\pi/a} dA_\mu \to \int_{\infty}^{\infty} dA_\mu \tag{200}$$

and we are back to integration over all the values of a classical field (just like we had for scalar fields).

Unfortunately, this is the integral which had the flat directions.

B. A well behaved path integral

The resolution to our difficulties is to eliminate the flat directions from the functional integral. This is done precisely as was discussed in Sec. IV, specifically in Eq. 172. Gauge fixing involves a constraint $g(A) = 0$ for some function g. Recall the gauge transformation formula Eq. 182,

$$A_\mu^V = VA_\mu(x)V^\dagger - iV\partial_\mu V^\dagger \tag{201}$$

and consider the quantity

$$\Delta_g^{-1}(A_\mu) = \int DV\delta[g(A_\mu^V)] \tag{202}$$

that is, integrate the gauge fixing $\delta-$function over all the gauge transformations V at every site. Then involve the identity

$$1 = \Delta_g(A_\mu) \int DV\delta[g(A_\mu^V)]. \tag{203}$$

Our partition function is

$$Z(J) = \int DA_\mu \{\Delta_g(A_\mu) \int DV\delta[g(A_\mu^V)]\} \exp(-iS(J)). \tag{204}$$

The terms in the curly brackets are our "1". Of course $S(J) = S + \int A_\mu J^\mu$ for use in generating Green's functions.

Now notice that $\Delta_g(A_\mu)$ is, in fact, gauge invariant:

$$\Delta_g^{-1}(A_\mu^{V'}) = \int DV\delta[g(A_\mu^{V'V})], \tag{205}$$

but $DV = D(V'V)$ (Haar measure at work) so

$$\Delta_g^{-1}(A_\mu^{V'}) = \int DV''\delta[g(A_\mu^{V''})] = \Delta_g^{-1}(A_\mu). \tag{206}$$

This means that we can perform a gauge transformation in $Z(J)$

$$Z(J) = \int dV \int DA_\mu \Delta_g(A_\mu)\delta[g(A)]\exp(-iS(J)). \tag{207}$$

The $\int dV$ just factors out of the expression. We can drop it. What does the rest of the expression mean? Integrate over all the A_μ's, subject to the gauge fixing constraint $\delta(g)$, and with the Jacobian $\Delta_g(A_\mu)$.

All we have to do is groom this expression to make it useful. We start (improbably) by changing variables from V to g itself,

$$DV = Dg\det(\frac{\delta V}{\delta g}) \tag{208}$$

so that

$$\Delta_g^{-1}(A) = \int Dg\det(\frac{\delta V}{\delta g})\delta(g) = \det(\frac{\delta V}{\delta g})_{g=0} \tag{209}$$

or

$$\Delta_g(A) = \det(\frac{\delta g}{\delta V})_{g=0}. \tag{210}$$

If we parametrize $V = \exp(i\omega)$, this is

$$\Delta_g(A) = \det(\frac{\delta g}{\delta \omega})_{g=0}. \tag{211}$$

(Be patient, an example is coming.)

We are nearly done. It would be very useful to get everything into the exponential. Let's pick a gauge choice, with a free parameter α, which will do that for us:

$$\begin{aligned}\tilde{g}(A) &= \int Dc(x)\delta(g-c)\exp(-\frac{i}{2\alpha}\int d^4x(C(x)^2) \\ &= \exp(-\frac{i}{2\alpha}\int d^4x(g(A)^2).\end{aligned} \tag{212}$$

For example, $g(A) = \partial^\mu A_\mu$ corresponds to

$$\tilde{g}(A) = \exp(-\frac{i}{2\alpha}\int d^4x(\partial_\mu A^\mu)(\partial_\nu A^\nu). \tag{213}$$

Finally, *if* (this is a very big IF) $\det(\frac{\delta g}{\delta \omega})$ does not depend on A, it also factors out of the functional integral and we can drop it. Again, an example: $g(A) = \partial^\mu A_\mu$. Its gauge transformation is

$$g(A^V(x)) = \partial_\mu(A^\mu + \partial^\mu\omega) \tag{214}$$

so

$$\frac{\delta g}{\delta \omega} = \delta^4(x-y)\Box \tag{215}$$

What is this? Pick some boundary conditions, solve $\Box\phi = \lambda\phi$ ($\Box$ is the d'Alembertian, $\partial^\mu\partial_\mu$) and then

$$\det\frac{\delta g}{\delta \omega} = \prod_j \lambda_j. \tag{216}$$

Even more explicitly,

$$\det\frac{\delta g}{\delta\omega} = \det\sum_{n,m}\int d^4x d^4y\ \phi_n^\dagger(x)\delta^4(x-y)\Box\phi(y) \tag{217}$$

which is $\det\delta_{nm}\lambda_n$. End of the example. The point is, $\det\frac{\delta g}{\delta\omega}$ has no A dependence, so we can factor it out and forget about it.

Coulomb gauge is similar. The gauge fixing term is $\delta(\vec{\nabla}\cdot\vec{A})\det\nabla^2$.

This was a mess. I dragged you through it, because there are cases where the factorization does not occur, and the determinant remains as part of the functional integral. Gauge choices for non-Abelian gauge theories pick up extra terms. Even for QED, it is possible to invent nonlinear gauge choices. In the literature, they are called "ghosts." We will not discuss them any more.

We are nearly done. Let's focus on the action including the gauge fixing term:

$$\begin{aligned}
S &= \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A_\mu)(\partial^\nu A^\nu)] \\
&= -\int d^4x[\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\nu) + \frac{1}{2\alpha}(\partial_\mu A_\mu)(\partial^\nu A^\nu)] \\
&= -\int d^4x[\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu + \frac{1}{2\alpha}(\partial_\mu A_\mu)(\partial^\nu A^\nu)] \\
&= -\frac{1}{2}\int d^4x A_\rho[-\partial^\mu\partial_\mu g^{\rho\nu} + (1-\frac{1}{\alpha})\partial^\rho\partial^\nu]A_\nu
\end{aligned} \tag{218}$$

integrating the derivative terms by parts. Now we can go into momentum space,

$$S = \frac{1}{2}\int d^4p A_\mu(-p)A_\nu(p)[g^{\mu\nu}p^2 - (1-\frac{1}{\alpha}p^\mu p^\nu] \tag{219}$$

and we see we just have a quadratic form. The propagator will be

$$-iD_{\mu\nu}(p) = i[g_{\mu\nu}X(p) + p_\mu p_\nu Y(p)], \tag{220}$$

where X and Y are scalar functions of p. Going through the index-pushing exercise (a homework problem) of inverting the differential operator,

$$g_{\mu\nu} = [p^2 g_{\mu\rho} - (1-\frac{1}{\alpha})p_\mu p_\rho][g_{\rho\nu}X(p) + p_\rho p_\nu Y(p)], \tag{221}$$

gives the photon propagator

$$-iD_{\mu\nu}(p) = \frac{i}{p^2}[g_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}]. \tag{222}$$

The parameter α is a gauge choice. Invariant amplitudes correspond to physical processes and they will be gauge invariant. Thus the parameter α will not be present in the final answer. Sometimes, one carries α through the calculation, as a check against mistakes: if it survives till the end, you have a mistake. More often, one picks a "convenient" choice of gauge; "convenience" meaning that with that choice the calculation simplifies. Two such choices are

- Feynman gauge: $\alpha = 1$

$$-iD_{\mu\nu}(p) = \frac{ig_{\mu\nu}}{p^2}. \tag{223}$$

- Landau gauge: $\alpha = 0$

$$-iD_{\mu\nu}(p) = \frac{i}{p^2}[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}]. \tag{224}$$

Both are obviously covariant.

V. POSSIBLE READING LIST

Here are a collection of books and papers I have found useful for describing various aspects of quantum field theory. The preprints with "arXiv" labels come from the data base `http://arXiv.org/` (which is the standard data base for most of theoretical physics).

The two books I debated between as a text (not that I will follow anything very closely) were Zee [3] and Srednicki [4]. They are quite complementary (with identical advice for the reader). I used Zee the last time I taught this course and the students didn't care for it so I will try Srednicki this time. Two very particle-physics oriented books are Schwartz [5] and Peskin and Schroeder [6]. Ramond [7] has things other books don't, specifically the use of generating functionals for perturbation theory and a nice calculation of beta functions. Ryder [8] is concise but complete; I often put it on reserve when I teach the formal parts of electrodynamics. Weinberg's books [9, 10] are very complete, but maybe they aren't really textbooks. (My first QFT course was from Weinberg.) Finally we are far in the past with Bjorken and Drell [11, 12]. In their day they were quite influential and for a few special subjects (solving the Dirac equation, doing tree-level QED) they are still worth consulting. I've put Refs. [3–8] on reserve in the Engineering Library, which is in the basement of the math building. Try to violate social norms and go in the stacks there to browse the QFT book section. You will find a lot of books, including (hopefully) the one for you.

I added Martin's supersymmetry primer [13] for its Sec. 2, which has a terse description of Dirac, Weyl and Majorana fermions.

I included some statistical mechanics oriented texts, Refs. [14–19] on my list. I have a problem with condensed matter field theory books in that I find their treatment of path integrals and perturbation theory somewhat unsystematic, but then I am a particle physicist. Maybe I am missing something! The classic review article on the renormalization group by Wilson and Kogut [20] is not to be missed. Polyakov [21] had interesting things to say. Finally I include my own book [22] (not to be selfish, but bits of this course washed up there) and for BEC aficionados, Pethick and Smith[23].

Various "inspirational" articles [24–28] and some introductions to effective field theory [29–33] round out the list.

Remember, you can't learn quantum field theory from any one book or any one course. Bohr said that "The opposite of truth is falsehood but the opposite of a great truth is another great truth." Quantum field theory contains many great truths.

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