Introduction to Lattice QCD

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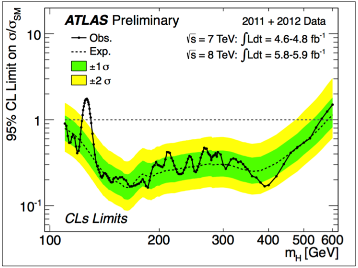
Why QCD?

Why QCD?

SU(3)xSU(2)xU(1) Standard Model describes physics (way too) well up to the TeV scale



The discovery of 125GeV nearly SM Higgs (?) & no SUSY give little constraint on BSM physics

Electroweak precision tests are more important than ever and depend on strong interactions

Why QCD?

QCD is a prototype model of gauge and fermion fields.
Adding scalars is trivial; SUSY not so much.

Models with different gauge groups, fermion representations and fermion numbers are candidates for beyond SM phenomenology - but all these candidates are strongly coupled and have to be studied non-perturbatively.

Why Lattice?

Strong interactions are

- Asymptotically free
- Confining
- Chirally broken

The latter two properties are non-perturbative.
Physical quantities are not analytic in the QCD coupling!
Lattice regularization is the only method that can describe non-perturbative QCD

Continuum QCD

Continuum Euclidean action:

$$
\begin{aligned}
S[A] &= \int d^d x \left(\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(x)\gamma_\mu D_\mu \psi(x) \right) \\
F_{\mu,\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \\
D_\mu &= \partial_\mu - i g A_\mu^a t^a
\end{aligned}
$$

Symmetries:

- SU(3) gauge symmetry
- Lorentz, C,P & T
- $\psi \to e^{i\alpha}\psi$
- In the $m = 0$ chiral limit $\psi \to e^{i\gamma_5 \alpha}\psi$

Chiral symmetry breaking

Flavor symmetry: in $m = 0$ chiral limit

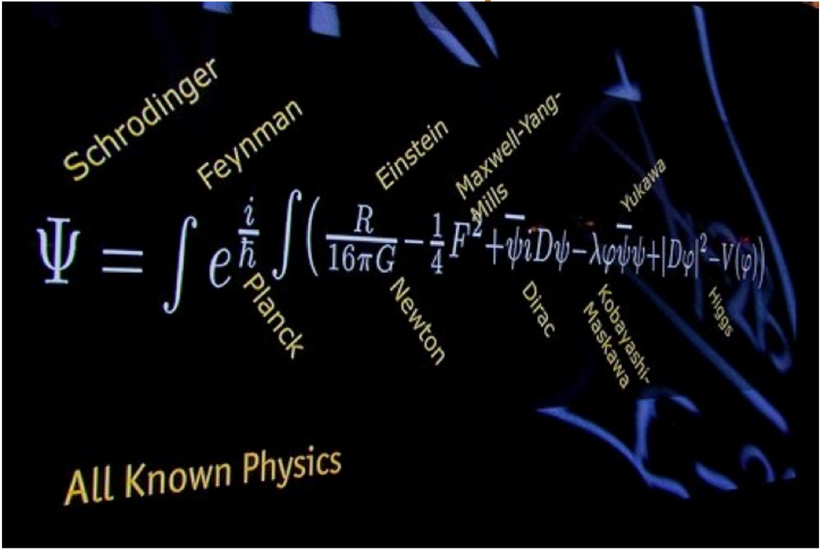
$$U(N_f)_V \times U(N_f)_A = U(1)_V \times SU(N_f)_V \times U(1)_A \times SU(N_f)_A$$

$U(1)_A$ gets broken by the anomaly
$SU(N_f)_A$ breaks spontaneously
$\rightarrow N_f^2 - 1$ massless Goldstone bosons (pions).
Fermion mass breaks the chiral symmetry explicitly

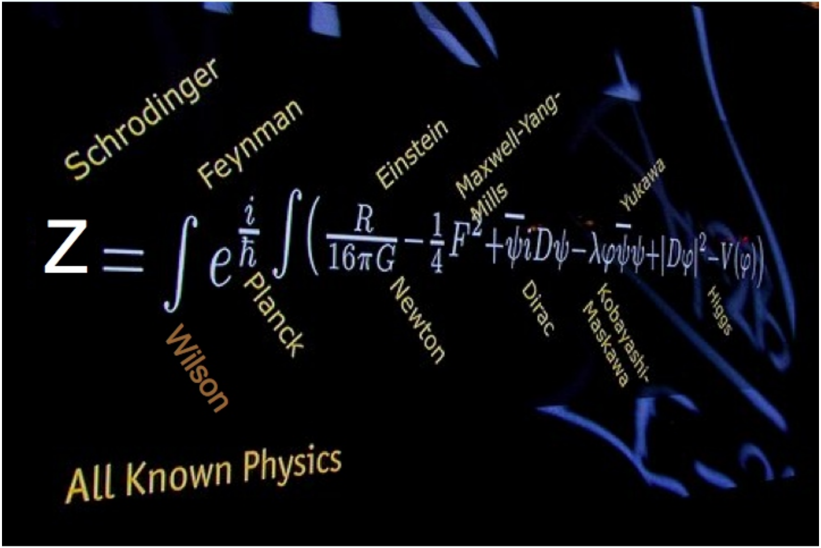
$$m_\pi^2 \sim m_q$$

$$m_p = m_{p0} + cm_q$$

All Known Physics



All Known Physics



Lattice action: Scalars

Continuum Euclidean system:

$$Z = \int \mathcal{D}\phi e^{-S[\phi]}$$

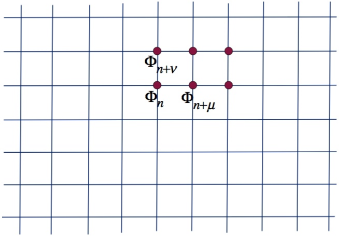
$$S[\phi] = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{1}{2} m^2 \phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right)$$

Need to

1. Interpret $\int \mathcal{D}\phi$
2. Regularize momentum integrals

Lattice discretization can do both.

Lattice action: Scalars



Discretize:

$$\phi(x) \longrightarrow \phi_n, \quad x = na$$

$$\int dx_i \longrightarrow a \sum_{n_i}$$

$$\int \mathcal{D}\phi \longrightarrow \prod_n d\phi_n$$

Discrete lattice derivative

$$\partial_\mu \phi(x) \longrightarrow \Delta_\mu \phi_n = \frac{1}{a}(\phi_{n+\hat{\mu}} - \phi_n)$$

$$\Delta^*_\mu \phi_n = \frac{1}{a}(\phi_n - \phi_{n-\hat{\mu}})$$

Lattice action: Scalars

The scalar lattice action:

$$
\begin{aligned}
S[\phi] &= a^4 \sum_n \left(\frac{1}{2} (\Delta_\mu \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 + \frac{\lambda}{4!} \phi_n^4 \right) \\
&= a^4 \sum_n \left(-\frac{1}{2} \phi_n (\Delta_\mu^* \Delta_\mu) \phi_n + \frac{1}{2} m^2 \phi_n^2 + \frac{\lambda}{4!} \phi_n^4 \right)
\end{aligned}
$$

Homework

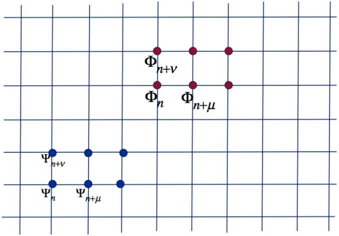
1) Show that in the $\lambda \to \infty$ the scalar lattice action reduces to the Ising model with $\phi_n = \pm 1$. (You will have to rescale the field to get ± 1)

2) Generalize the discussion of the scalar model to complex scalar and also to 2-component complex scalar. The latter one is the relevant model for the Standard Model Higgs.

$$S = -K \sum_{n,\mu} \phi_n \phi_{n+\mu}$$

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Lattice action: naive fermions



Discretize:

$$\psi(x) \longrightarrow \psi_n, \qquad x = na$$

$$\int dx_i \longrightarrow a \sum_{n_i}$$

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \longrightarrow \prod_n d\psi_n d\bar{\psi}_n$$

The naive fermion lattice action:

$$S[\psi] = a^4 \sum_n \left(\frac{1}{2} \bar{\psi}_n \gamma_\mu (\Delta^*_\mu + \Delta_\mu) \psi_n + m_q \bar{\psi}_n \psi_n \right)$$

$$\bar{\psi}_n \gamma_\mu \psi_{n+\mu} - \bar{\psi}_n \gamma_\mu \psi_{n-\mu}$$

Lattice action: gauge fields

The most important feature is local gauge symmetry.
Gauge transformation:

$$\psi_n \to V_n \psi_n, \quad \bar{\psi}_n \to \bar{\psi}_n V_n^\dagger \quad V_n \in SU(3)$$

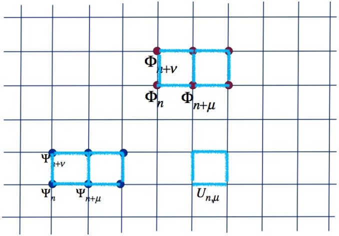
and the role of the gauge field is to make derivatives gauge invariant

$$\bar{\psi}_n \psi_{n+\hat{\mu}} \longrightarrow \bar{\psi}_n U_{n,\mu} \psi_{n+\hat{\mu}}$$

with $U_{n,\mu} \in SU(3)$ transforming as

$$U_{n,\mu} \longrightarrow V_n U_{n,\mu} V_{n+\hat{\mu}}^\dagger$$

Lattice action: gauge fields



Relation to continuum gauge field

$$A_\mu(x) \longrightarrow U_{n,\mu} = e^{-iagA^b_{n,\mu} t^b}$$

Gauge invariant quantities:

$$\bar\psi_n \gamma_\mu U_{n,\mu} \psi_{n+\mu}, \quad \phi_n U_{n,\mu} \phi_{n+\mu},$$

$$\prod_{\mathcal{C}} U_{n,\mu} \quad \text{for any closed loop}$$

The gauged lattice action could be

$$S[\psi] \sim \sum_n \left(\mathrm{Tr}(\prod_{\mathcal{C}} \mathrm{U_{n,\mu}} + \mathrm{hc}) + \mathrm{a}^3 (\bar\psi_\mathrm{n} \gamma_\mu \mathrm{U_{n,\mu}} \psi_{\mathrm{n}+\mu} + \mathrm{h.c.} + \mathrm{am} \bar\psi_\mathrm{n} \psi_\mathrm{n}) \right)$$

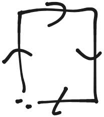
Is it that simple?

Lattice action

The simplest gauge action is the plaquette (Wilson gauge)

$$S[\psi] = \sum_n \quad \frac{\beta}{6}\sum_p \mathrm{Tr}(\mathrm{U}_\square + \mathrm{U}_\square^\dagger)$$
$$+a^3(\bar{\psi}_n\gamma_\mu U_{n,\mu}\psi_{n+\mu} + \bar{\psi}_n\gamma_\mu U^\dagger_{n-\mu,\mu}\psi_{n-\mu} + am\bar{\psi}_n\psi_n)$$

but we could take any other closed loop (or combination of loops).



Lattice action

Does it reproduce at least the naive $a \to 0$ continuum limit?
Expand:

$$U_{n,\mu} = e^{-aA_\mu(n)} = 1 - aA_\mu(n) + \frac{a^2}{2}A_\mu(n)^2 + \ldots$$

leads to

$$U_\Box = e^{-a^2 G_{\mu\nu}}, \quad G_{\mu\nu} = F_{\mu\nu} + \mathcal{O}(a)$$

so

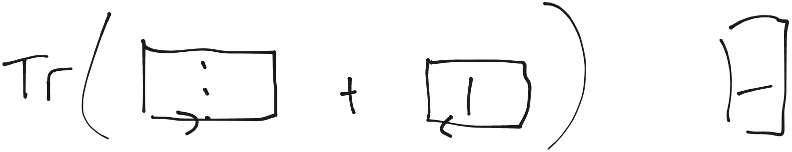
$$\mathrm{Tr}(U_\Box + U_\Box^\dagger) = 2\mathrm{Tr}\mathbf{1} + a^4\mathrm{Tr}(F_{\mu\nu})^2 + \mathcal{O}(a^6)$$

i.e. correct continuum form if $\beta = 2N/g^2$.

Homework

3) Prove the relations on the previous slide. Be careful: how many terms are there in the $\sum_{n_i} \mathrm{Tr}(\mathrm{U}_{\square} + \mathrm{U}^{\dagger}_{\square})$? What is the trace of the SU(3) generators?

4) Can you derive similar expressions for the 1x2 and other small loops? These terms show up in improved actions, like the Symanzik gauge action.



Observables

The expectation value of any operator is given

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathcal{D} U_{n,\mu} \mathcal{O} e^{-S_g[U]},$$

$$Z = \int \prod_{n,\mu} \mathcal{D} U_{n,\mu} e^{-S_g[U]}$$

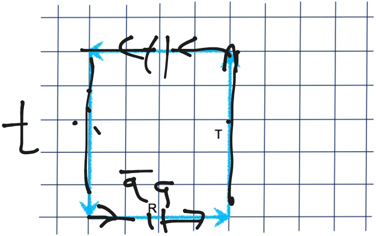
Operators of physical quantities are called observables.
Expectation value of a non-gauge invariant operator vanishes.
Numerical simulations: create configurations with probability

$$e^{-S_g[U]},$$

and calculate expectation values.

Wilson loops & the static potential

The Wilson loop: closed gauge loop



An RxT Wilson loop describes a quark-antiquark pair propagating at distance R for time T

$$W(R,T) \sim e^{-V(R)T}$$

where $V(R)$ is the static potential

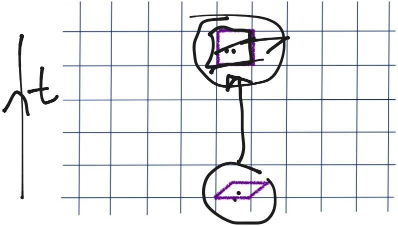
In a confining theory

$$V(R) = c + \frac{e}{R} + \sigma R$$

Coulomb + linear terms (σ is the string tension)

Glueballs

Glueballs are pure gauge bound states



Plaquettes can create glueball states (different combinations are taken to describe different quantum numbers). If they are separated at distance T

$$\langle \Box(0) \Box'(T) \rangle \sim e^{-m_G T}$$

where m_G is the glueball mass.

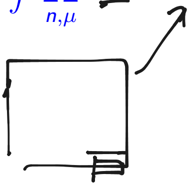
Glueballs are notoriously difficult to calculate. (Tricks, tricks and more tricks are needed.)

Strong coupling expansions

Before powerful computers there was strong coupling expansion . . .
At small β

$$e^{-S_g[U]} = e^{-\beta/6 \sum_p \mathrm{Tr}(U_\square + U_\square^\dagger)} = \prod_p \left(1 - \frac{\beta}{6}\mathrm{Tr}(U_\square + U_\square^\dagger) + \ldots\right)$$

$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathcal{D}U_{n,\mu} \mathcal{O}(U) \prod_p \left(1 - \frac{\beta}{6}\mathrm{Tr}(U_\square + U_\square^\dagger) + \ldots\right)$$



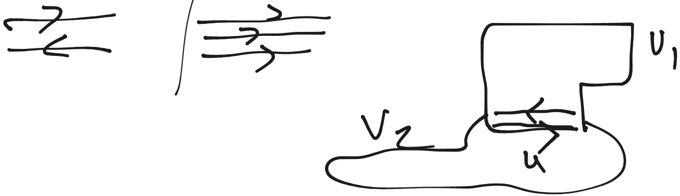
$$\int dU_{n\mu} \, U_{n\mu} = 0$$

Strong coupling expansions

Which terms survive $\int \mathcal{D}U$? Where U and $U^\dagger$ or three U's meet

$$\int dU \mathrm{Tr}(\mathrm{U}\mathrm{V}_1)\mathrm{Tr}(\mathrm{U}^\dagger \mathrm{V}_2) = \mathrm{Tr}(\mathrm{V}_1\mathrm{V}_2)$$

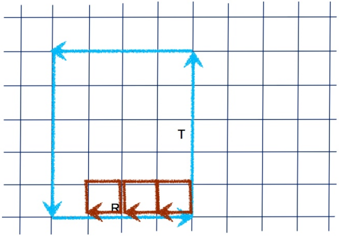
(OK, this should really be done with group characters.)



Wilson loop at strong coupling

$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \prod_{n,\mu} \mathcal{D}U_{n,\mu} \mathcal{O}(U) \prod_p \left(1 - \frac{\beta}{6} \mathrm{Tr}(U_\Box + U_\Box^\dagger) + \dots \right)$$

Cover every link with an opposite directional one:



Need RxT plaquettes to cover it all

$$W(R,T) \sim \beta^{RT} \sim e^{T(R\log(\beta))}$$

The potential

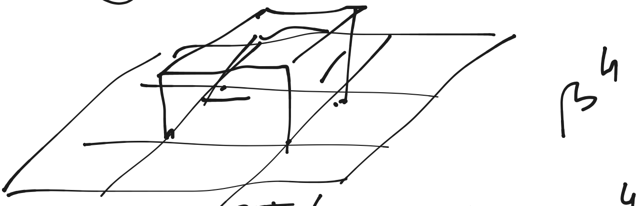
$$V(R) = -R\log(\beta)$$

linear in R.

The strong coupling gauge model is confining!

Wilson loop at strong coupling

Next order : plaquette sticking out:
β^4, multiplicity 4*RxT, etc



$$W = \beta^{RT}\left(1 + 4RT\beta^4 + \dots\right)$$

The string tension:($u = \beta/6$)

$$-\sigma = \ln(\mathrm{u}) + 4\mathrm{u}^4 + 12\mathrm{u}^5 - 10\mathrm{u}^6 \dots$$

Glueballs in strong coupling

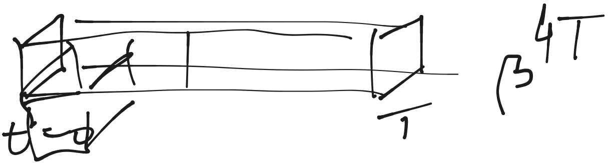
Lowest order : connect the two plaquettes with a tube:

$$\langle \mathcal{O} \rangle \sim (\frac{\beta}{6})^{4T} \sim e^{-T \times 4 \ln(\beta/6)}$$

giving

$$m_G = -4 \ln u \pm 3u \ldots$$

The rest depends on the glue ball quantum numbers.



Message from strong coupling expansion

The strong coupling pure gauge system is

- Confining
- Has massive glueballs
- Meson spectrum shows chiral symmetry breaking (but that's not exact)

Is there any use for the strong coupling expansion in the are of supercomputers?

Homework

3) Calculate the next order term to the strong coupling expansion of the potential. (It is known to order 14)
4) Calculate the glueball mass in next order strong coupling expansion. Take two plaquettes, parallel to each other for simplicity.
5) Feeling ambitious? Calculate the potential between two Polyakov lines at finite temperature in the strong coupling.
References: Montvay&Munster has extensive discussion about the strong coupling expansion.