Solutions HW4 - Phys 7810-001

03/18/21 due 04/08/21

Problem 1 [80 pts]

Generalized BRST quantization of gauge theories. In class we obatained the path integral for a gauge theory based on the deWitt-Faddeev-Popov method. This showed us that the ratio $Z[\Phi]/Z[1]$ with Φ a product of gauge invariant operators, can be defined by fixing the gauge so that the volume of the gauge group cancels between the numerator and denominator leaving us with well-defined path integrals. The resulting expression is (after factoring out the gauge group volume and an irrelevant constant),

$$Z=\int[d\phi][dc][dc^+]\Omega[\phi]e^{ic^{+\alpha}.M_\alpha{}^\beta.c_\beta-\frac{i}{2\xi}f_\alpha.\kappa^{\alpha\beta}.f_\beta},$$

where I've used $\Omega[\phi]=e^{i\mathrm{S}[\phi]}\Phi[\phi]$, with ϕ being the full set of physical fields (those that define external states - may be gauge fields, matter fields both bosons and fermions but not ghosts). The only property of Ω that we need is that it's gauge invariant i.e, $\Omega[\phi^g]=\Omega[\phi]$ as is the case with the measure (defined as we did in class). f_α is the gauge fixing function and $M_\alpha{}^\beta(x,y)$ is the matrix we defined in class. We've also used a condensed notation so that repeated indices are summed over the discrete values of the gauge index as well as integrated over space-time. Thus for instance

$$c^{+\alpha}.M_\alpha{}^\beta.c_\beta\equiv\int_x\int_y c^{+\alpha}(x)M_\alpha{}^\beta(x,y)c_\beta(y),$$

etc. $\kappa_{\alpha\beta}$ is an invariant metric on the gauge group which for compact groups may be taken to be $\delta^{\alpha\beta}\delta^4(x-y)$ but again all we need here is that it is gauge invariant. Observe that up to an irrelevant (field independent) constant we can write

$$Z=\int[d\phi][dc][dc^+][dh]\Omega[\phi]e^{ic^{+\alpha}.M_\alpha{}^\beta.c_\beta-ih_\alpha.f^\alpha[\phi]+i\frac{\xi}{2}h_\alpha.\kappa^{\alpha\beta}.h_\beta}\tag{1}$$

This follows from doing the Gaussian integral over h_α.

Ghost number: Assign $+1$ to c_α, -1 to c^+_α and zero to all other fields. Clearly the gauge fixed action

$$\mathrm{S}[\phi]+c^{+\alpha}.M_\alpha{}^\beta.c_\beta-h_\alpha.f^\alpha[\phi]+\frac{\xi}{2}h_\alpha.\kappa^{\alpha\beta}.h_\beta\tag{2}$$

conserves ghost number

0.1 [30 pts] Killing vectors

a) [10 pts] Define the Killing vector field $\overrightarrow{K}_\alpha[\phi] = K^i_\alpha[\phi]\frac{\delta}{\delta\phi^i}$. Show that the commutator

$$[\overrightarrow{K}_\alpha[\phi], \overrightarrow{K}_\beta[\phi]] = (K^i_\alpha[\phi]K^j_{\beta,i}[\phi] - K^i_\beta[\phi]K^j_{\alpha,i}[\phi])\frac{\delta}{\delta\phi^j}. \tag{3}$$

Here $K^j_{\alpha,i} = \frac{\delta}{\delta\phi^i}K^j_\alpha$. Note that this diffential operator implements an infinitesimal transformation on the fields:

$$\delta_\theta\phi^i = \delta\theta^\alpha K^i_\alpha.$$

This is just an exercise in functional differentiation. Using the condensed notation defined above we have

$$[\overrightarrow{K}_\alpha[\phi], \overrightarrow{K}_\beta[\phi]] = [K^i_\alpha[\phi]\frac{\delta}{\delta\phi^i}, K^j_\beta[\phi]\frac{\delta}{\delta\phi^j}] = K^i_\alpha[\phi]\frac{\delta}{\delta\phi^i}K^j_\beta[\phi]\frac{\delta}{\delta\phi^j} - K^j_\beta[\phi]\frac{\delta}{\delta\phi^j}K^i_\alpha[\phi]\frac{\delta}{\delta\phi^i}$$

$$= K^i_\alpha[\phi]K^j_\beta[\phi]_{,i}\frac{\delta}{\delta\phi^j} + K^i_\alpha[\phi]K^j_\beta[\phi]\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j}$$

$$- K^j_\beta[\phi]K^i_\alpha[\phi]_{,j}\frac{\delta}{\delta\phi^i} - K^j_\beta[\phi]K^i_\alpha[\phi]\frac{\delta}{\delta\phi^j}\frac{\delta}{\delta\phi^i}$$

$$=(K^i_\alpha[\phi]K^j_{\beta,i}[\phi] - K^i_\beta[\phi]K^j_{\alpha,i}[\phi])\frac{\delta}{\delta\phi^j}$$

In the last step we used the commutativity of functional partial derivatives. Note that the above is true even if some of the fields ϕ are fermionic though in that case one needs to keep track of the signs but you can check that it works though not required for this problem.

b)[10 pts] If this is an invariance of the action then we must have:

$$\overrightarrow{K}_\alpha \mathrm{S}[\phi] = 0. \tag{4}$$

Show that then

$$(K^i_\alpha[\phi]K^j_{\beta,i}[\phi] - K^i_\beta[\phi]K^j_{\alpha,i}[\phi])\frac{\delta}{\delta\phi^j}\mathrm{S}[\phi] = 0 \tag{5}$$

From (4) it follows that $\overrightarrow{K}_\beta\overrightarrow{K}_\alpha\mathrm{S}[\phi] = 0$ and $\overrightarrow{K}_\alpha\overrightarrow{K}_\beta\mathrm{S}[\phi] = 0$ and hence $[\overrightarrow{K}_\alpha, \overrightarrow{K}_\beta]\mathrm{S}[\phi] = 0$. Using (3) the result follows. Show that if this is true off-shell (i.e. is even when $\frac{\delta}{\delta\phi^j}\mathrm{S}[\phi] \neq 0$) and the set $\overrightarrow{K}_\alpha$ are a maximal set of symmetries which leave the action invariant (and are independent of the action) then the algebra is closed:

$$[\overrightarrow{K}_\alpha[\phi], \overrightarrow{K}_\beta[\phi] = c_{\alpha\beta}{}^\gamma[\phi]\overrightarrow{K}_\gamma[\phi]. \tag{6}$$

i.e. there is a non-linear constraint on the K's,

$$(K^i_\alpha[\phi]K^j_{\beta,i}[\phi] - K^i_\beta[\phi]K^j_{\alpha,i}[\phi]) = c_{\alpha\beta}{}^\gamma[\phi]K^j_\gamma[\phi]. \tag{7}$$

This follows since if $K'_\alpha s$ are a maximal set of symmetry generators then their commutator (which is also proportional to a $\delta/\delta\phi$) must also be some linear combination of the $K's$. In other words the operator acting on S in eqn. (5) must be a linear combination of the form $c_{\alpha\beta}{}^\gamma[\phi]\overrightarrow{K}_\gamma[\phi]$ (with $c_{\alpha\beta}{}^\gamma[\phi] = -c_{\beta\alpha}{}^\gamma[\phi]$) where in general the structure constants may be field dependent. Thus we get eqn. (7). The independence w.r.t. the action is required since otherwise one can write down a term which is dependent of the action (while being linear in the derivative operator) but still annihilates it.

c) [5+5 pts] Show that there is another constraint

$$c_{[\alpha\beta}{}^\delta c_{(\delta)\gamma]}{}^\sigma - \vec{K}_{[\alpha}c_{\beta\gamma]}{}^\sigma = 0, \tag{8}$$

where the instruction [...] on the indices means that the expression is summed over cyclic permutaions of the three indices.

This follows from the Jacobi identity $[\overrightarrow{K}_\alpha[\phi], [\overrightarrow{K}_\beta[\phi], \overrightarrow{K}_\gamma[\phi]]]$+cyclic permutations and the relation (7).

If the symmetry is linearly realized (i.e. $\delta_\theta\phi^i = i\delta\theta^\alpha t^i_{\alpha j}\phi^j$ with $[\mathbf{t}_\alpha, \mathbf{t}_\beta] = if_{\alpha\beta}{}^\gamma\mathbf{t}_\gamma$, find the K's and show that $c_{\alpha\beta}{}^\gamma[\phi] = if_{\alpha\beta}{}^\gamma$. i.e. the c's are field independent. This remains true for Yang-Mills gauge transformations on gauge fields (which have an inhomogeneous piece) as well as for gravity (diffeomorphisms), but not for example in supergravity.

If the action of the symmetry is of the form $\delta_\theta\phi^i = i\delta\theta^\alpha t^i_{\alpha j}\phi^j$ then $K^i_\alpha = it^i{}_{\alpha l}\phi^l$, $K^i_{\alpha,j} = it^i{}_{\alpha j}$ and so the LHS of (7) becomes

$$i^2\left(t^i_{\alpha l}\phi^l t^j_{\beta i} - t^i_{\beta l}\phi^l t^j_{\alpha i}\right) = \phi^l\left(t^j_{\alpha i}t^i_{\beta l} - t^j_{\beta i}t^i_{\alpha l}\right) = \phi^l[\mathbf{t}_\alpha, \mathbf{t}_\beta]^j{}_l = \phi^l if{}_{\alpha\beta}{}^\gamma it^j_{\gamma l} = f{}_{\alpha\beta}{}^\gamma K^j{}_\gamma.$$

Comparing with the RHS of (7) we have $c_{\alpha\beta}{}^\gamma = if{}_{\alpha\beta}{}^\gamma$.

0.2 [50 pts]BRST charge.

a)[10 pts] Define the Slavnov operator,

$$\mathcal{S} = c^\alpha K^i_\alpha\frac{\delta}{\delta\phi^i} - \frac{1}{2}c^\beta c^\gamma c^\alpha_{\beta\gamma}[\phi]\frac{\delta}{\delta c^\alpha} - h^\alpha\frac{\delta}{\delta c^{+\alpha}}. \tag{9}$$

Note that this operator has ghost number +1. A BRST transformation is defined to be the action of $\theta\mathcal{S}$ where θ is a Grassman number (so $\theta^2 = 0$) on the set of fields $\{\phi^i, c_\alpha, c^{+\alpha}, h^\alpha\}$. Show that the gauge fixed action can be written as

$$\mathrm{S}_{\mathrm{gf}} = \mathrm{S}[\phi] + \mathcal{S}\Psi[\phi, c, c^+, h], \tag{10}$$

where the so-called gauge fixing fermion (with ghost number -1) is given by

$$\Psi = c^{+\alpha} f_\alpha[\phi] - \frac{1}{2}\xi c^+_\alpha \kappa^{\alpha\beta} h_\alpha. \tag{11}$$

Let us work out $\mathcal{S}\Psi$. First note the following results.

$$\mathcal{S}\phi^j = c^\alpha K^j_\alpha,\ \mathcal{S}c^\alpha = -\frac{1}{2}c^\beta c^\gamma c^\alpha_{\beta\gamma}[\phi],\ \mathcal{S}c^{+\alpha} = -h^\alpha,\ \mathcal{S}h^\alpha = 0.$$

We also have $\overrightarrow{K}_\alpha f^\beta(\phi) = K^\gamma_\alpha f^\beta_{,\gamma} \equiv M^\beta_\alpha$. Thus we have $S + \mathcal{S}\Psi[\phi, c, c^+, h] = S + c^{+\alpha}.M_\alpha^{\ \beta}.c_\beta - h_\alpha.f^\alpha[\phi] + \frac{\xi}{2}h_\alpha.\kappa^{\alpha\beta}.h_\beta$ which is the gauge fixed action of eqn. (2).

b) [15 pts] It turns out that any gauge theory once it is gauge fixed can be written in the form of eqn. (10) with Ψ an arbitrary (i.e. not necessarily of the form (11)) fermionic functional of ghost no. -1. You don't need to prove this here (for a proof see for example Weinberg Quantum Theory of Fields vol 2 page 40). Using the relations for the Killing vector derived in subsection 0.1 show that the Slavnov operator is nilpotent. i.e.

$$\mathcal{S}^2 = 0$$

This is a straightforward exercise using the fermionic nature of c, c^+ and the Jacobi identity relation relation.

[05 pts] Use the above to conclude that $\theta\mathcal{S}\mathrm{S}_{\mathrm{g.f.}} = 0$. This shows that the gauge fixed action (although it's not gauge invariant any more) is nevertheless BRST invariant.

This is a global (i.e. space time independent since θ is a constant Grassmann number) symmetry of the action. Note that any gauge invariant functional of the physical fields ϕ is automatically BRST invariant i.e.

$$\theta\mathcal{S}\Omega[\phi] = 0.$$

The invariance of the $\mathcal{S}\Psi$term follows from the nil potency of $\mathcal{S}$.

d) [10 pts] Show that the measure in the functional integral (1) is BRST invariant. i.e. if we defined the BRST transformed fields as $\Phi^\theta = \Phi + \theta\mathcal{S}\Phi$ where $\Phi = \{\phi, c, c^+, h)$, show that the Jacobian has determinant unity. i.e.

$$\det \frac{\delta \Phi^\theta}{\delta \Phi} = 1.$$

For the diagonal Jacobian matrix elements we have

$$\frac{\delta c'^\alpha}{\delta c^\delta} = \delta^\alpha_\delta - -\theta c^\gamma c^\alpha{}_{\delta\gamma}, \; \frac{\delta A'^\alpha}{\delta A^\beta} = \delta^\alpha_\beta + \theta c^\alpha{}_{\beta\gamma} c^\gamma.$$

We have assumed that the structure constants are independent of the fields here for simplicity. This is the case both for Y-M and GR.

The off-diagonal terms are proportional to θ. The Jacobian matrix $\mathbf{J}$ is block-diagonal. We have for the A, c submatrix

$$\begin{bmatrix} \mathbf{1} + \theta(-\mathbf{i}\mathbf{T}_\gamma)c^\gamma & O(\theta) \\ O(\theta) & \mathbf{1} + \theta(-\mathbf{i}\mathbf{T}_\gamma)c^\gamma \end{bmatrix}.$$

The c^+, h submatix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Note that the matrix is of the form $\mathbf{1} + O(\theta)$. Since $\theta^2 = 0$ the expansion of the log has only one term - linear in θ. Hence we get

$$\ln \det \mathbf{J} = \mathrm{Tr} \ln \mathbf{J} = 2\mathrm{Tr}\mathbf{T}_\gamma c^\gamma = 0 \Rightarrow \det \mathbf{J} = 1$$

e) [05 pts] Show that the operator $i\mathcal{S}$ is Hermitian (recall that c, c^+ are real Grassmanian fields) when defined with respect to the functional inner product

$$(\Xi', \Xi) = \int [d\Phi] \Xi^{'*}(\Phi)\Xi(\Phi).$$

Hermiticity is the property

$$(i\mathcal{S}\Xi^{'}, \Xi) = \int [d\Phi] \left(i\mathcal{S}\Xi^{'}(\Phi) \right)^* \Xi(\Phi) = \int [d\Phi] \Xi^{'*}(\Phi) i\mathcal{S}\Xi(\Phi) = (\Xi', i\mathcal{S}\Xi).$$

But inside the above functional integral we can do integration by parts. So we can write the the operator on the RHS as

$$-i\left\{c^\alpha K^i_\alpha\frac{\overleftarrow{\delta}}{\delta\phi^i}-\frac{1}{2}c^\beta c^\gamma c^\alpha{}_{\beta\gamma}\frac{\overleftarrow{\delta}}{\delta c^\alpha}-h^\alpha\frac{\overleftarrow{\delta}}{\delta c^{\alpha+}}\right\}$$

$$=-i\left\{\frac{\overleftarrow{\delta}}{\delta\phi^i}c^\alpha K^i_\alpha-\frac{1}{2}\frac{\overleftarrow{\delta}}{\delta c^\alpha}c^\beta c^\gamma c^\alpha{}_{\beta\gamma}-\frac{\overleftarrow{\delta}}{\delta c^{\alpha+}}h^\alpha+c^\alpha K^i{}_{\alpha,i}-c^\beta\delta^\gamma_\alpha c^\alpha{}_{\beta\gamma}\right\}.$$

But the last two terms are zero since $K^i_{\alpha,i}=0$ from the gauge invariance of the measure over the physical fields (use invariance of metric $||\delta\phi||^2$ or the explicit expression for K in a gauge theory for example) and since $c^\alpha{}_{\beta\alpha}=0$. Hence the result follows.

f) [05 pts] The above holds for any gauge theory from Y-M to gravity and supergravity. In other words if one has a set of symmetries $\vec{K}_\alpha$ under which the action is invariant and which form a closed algebra, the gauge fixed action is of the form (10). Define the BRST charge operator $\hat{Q}$ acting on a Dirac ket or bra state as $<\Phi|\hat{Q}|\Xi>=i\mathcal{S}\Xi(\Phi)$ where $|\Phi>$ is an eigenstate of the set of field operators $\hat{\Phi}$, in other words $\Xi(\Phi)=<\Phi|\Xi>$ is the representation in the field space basis of the ket vector $|\Xi>$. Clearly the nilpotency of $\mathcal{S}$ implies that $\hat{Q}^2=0$. Physical states may now be defined as states in the BRST cohomology, i.e. if $|\Xi>$ is a physical state $Q|\hat{\Xi>}=0$ i.e. it is BRST closed. Note that this is an equivalence class i.e. you can add any BRST exact state (i.e. a state $\hat{Q}|\mathrm{any}>$ to a physical state and it still satisfies the above condition). Show that the S-matrix for the scattering of physical states, is independent of the gauge fixing, i.e.

$$\delta_\Psi<\alpha;\mathrm{out}|\beta;in>=0.$$

We have (either from the path integral representation of the matrix element or from Schwinger's action principle

$$\delta_\Psi<\alpha;\mathrm{out}|\beta;in>=<\alpha;\mathrm{out}|\int\mathcal{S}\delta\Psi|\beta;in>=-i<\alpha;\mathrm{out}|\left[\hat{Q},\delta\Psi\right]|\beta;in>=0$$

if $|\alpha>,|\beta>$ are phsical states.

g) To see the meaning of the physical state condition one could consider its effect on states in a gauge theory. See for example the discussion on pp 33-34 in Weinberg Quantum Theory of Fields vol 2 or on pp 453-455 in Srednicki QFT. One can also show using BRST methods that the quantum effective action in background gauge invariant gauges (see Lec5) not only satisfies background gauge invariance but also is gauge fixing independent at its extrema i.e solutions of $\delta\Gamma/\delta\phi = 0$ are independent of ξ.

Problem 2 [20 pts]

In class we derived the form

$$U_n^{(1)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{tr}e^{-\mathbf{V}''(\phi_c)s}\frac{1}{(4\pi s)^{n/2}}$$

for the one-loop contribution to the effective potential in n dimensions. a) [10 pts] By rewriting the integral in terms of a Gamma function show that in the limit $\epsilon \equiv 4-n \to 0$ this expression takes the form

$$\lim_{\epsilon\to 0}U_{4-\epsilon}^{(1)} = -\frac{1}{2}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln\frac{\mathbf{V}''}{\mu^2} + \frac{3}{2} + O(\epsilon)\right).$$

where μ is an arbitrary scale factor (called the renormalization scale). In the so-called $\overline{\mathrm{MS}}$ subtraction scheme, one adds the counter term

$$\delta S = \frac{1}{2}\int d^4x\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi)\right),$$

to the original action (with couplings defined at the mass scale μ) so that we have for the one-loop corrected quantum effective action (1PI action to one-loop)

$$\begin{aligned}\Gamma_{\mathrm{1PI}} &\simeq S_{\mathrm{classical}(\mu)} + \lim_{\epsilon\to 0}\left(\delta S + \Gamma_{4-\epsilon}^{(1)}\right) \\ &= S_{\mathrm{cl}(\mu)} + \frac{1}{2}\int d^4x\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\ln\frac{\mathbf{V}''}{\mu^2} - \frac{3}{2}\right) \qquad (12)\end{aligned}$$

b) [10 pts] Consider a scalar field theory with two fields, one a light field with mass m and another with a heavy field with mass M. The potential for the theory is

$$V(\phi,\Phi;\mu) = \Lambda_{\mathrm{cc}}(\mu) + \frac{1}{2}m^2(\mu)\phi^2 + \frac{1}{2}M^2(\mu)\Phi^2 + \frac{\lambda(\mu)}{4!}\phi^4 + \frac{\eta(\mu)}{4}\phi^2\Phi^2 + \frac{\sigma(\mu)}{4!}\Phi^4 + \ldots$$

The β-functions are obtained by demanding that U is independent of the arbitrary scale μ. Derive the flow equations for the cosmological constant and the purely ϕ dependent couplings:

$$\mu\frac{d\Lambda_{\mathrm{cc}}}{d\mu} = \frac{1}{32\pi^2}\left(m^4 + M^4\right) \qquad (13)$$

$$\mu\frac{dm^2}{d\mu} = \frac{1}{16\pi^2}\left(\lambda m^2 + \eta M^2\right) \qquad (14)$$

$$\mu\frac{d\lambda(\mu)}{d\mu} = \frac{3}{16\pi^2}\left(\left(\lambda^2 + \eta^2\right)\right) \qquad (15)$$

From an effective field theory point of view if $M \gg m$ we would have expected that effect of the heavy field to drop out of the light field beta functions in the regime $\mu \ll M$. However we see in the above that this does not happen. Can you explain this?

In dimensional regularization one may start with the D dimensional version of the proper time representation eqn. (**??**) but without the cut-off in the s-integral, i.e.

$$\begin{aligned}
\Gamma_D^{(1)} &= -\frac{1}{2}\int_0^\infty \frac{ds}{s}\int d^Dx\sqrt{g}\mathrm{tr}e^{-\mathbf{V}''(\phi_c)s}\frac{1}{(4\pi s)^{D/2}} \\
&= -\frac{1}{2}\int d^Dx\sqrt{g}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{ds}{s}s^{-D/2}\mathrm{tr}e^{-sV''} = -\frac{1}{2}\int d^Dx\sqrt{g}\frac{1}{(4\pi)^{D/2}}\mathrm{tr}\left(\mathbf{V}''(\phi\right. \\
&= -\frac{1}{2}\int d^Dx\sqrt{g}\frac{1}{(4\pi)^{D/2}}\mathrm{tr}\left(\mathbf{V}''(\phi)\right)^{D/2}\Gamma(-D/2) \\
&= -\frac{1}{2}\int d^Dx\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\left(\frac{\Gamma(2-D/2)}{D/2(D/2-1)}\frac{(\mu^2)^{(D/2-2)}}{(4\pi)^{D/2-2}}e^{\left(\frac{D}{2}-2\right)\ln\frac{V''}{\mu^2}}\right)
\end{aligned}$$

In the above we've introduced and arbitrary mass scale μ and used the integral representation for the Gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. Writing $D = 4-\epsilon$ and expanding in a Laurent series in ϵ we get (see for example Peskin and Schroeder [**?**] eqns. (11.77,78))

$$\Gamma_{4-\epsilon}^{(1)} = -\frac{1}{2}\int d^4x\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)-\ln\frac{\mathbf{V}''}{\mu^2}+\frac{3}{2}+O(\epsilon)\right),$$

where μ is an arbitrary scale factor. In $\overline{\mathrm{MS}}$ one adds the counter term

$$\delta S = \frac{1}{2}\int d^4x\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)\right),$$

to the original action (with couplings defined at the mass scale μ) so that we have for the one-loop corrected quantum effective action (1PI action to one-loop)

$$\begin{aligned}
\Gamma_{1\mathrm{PI}} &\simeq S_{\mathrm{cl}(\mu)} + \lim_{\epsilon\to 0}\left(\delta S + \Gamma_{4-\epsilon}^{(1)}\right) \\
&= S_{\mathrm{cl}(\mu)} + \frac{1}{2}\int d^4x\sqrt{g}\mathrm{tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}\frac{1}{2}\left(\ln\frac{\mathbf{V}''}{\mu^2}-\frac{3}{2}\right) \qquad (16)
\end{aligned}$$

In our toy model the classical potential is,

$$V(\phi,\Phi;\mu)=\Lambda_{\rm cc}(\mu)+\frac{1}{2}m^2(\mu)\phi^2+\frac{1}{2}M^2(\mu)\Phi^2+\frac{\lambda(\mu)}{4!}\phi^4+\frac{\eta(\mu)}{4}\phi^2\Phi^2+\dots$$

and the field dependent mass matrix is

$$\mathbf{V}''=\begin{bmatrix} m^2+\frac{\lambda}{2}\phi^2+\frac{\eta}{2}\Phi^2 & \eta\phi\Phi \\ \eta\phi\Phi & M^2+\frac{\eta}{2}\phi^2\end{bmatrix}. \tag{17}$$

The β-functions are obtained by demanding that $\Gamma_{\rm 1PI}$ is independent of the arbitrary scale μ. Writing out this equation for the effective potential we get

$$\begin{aligned}0=\mu\frac{d}{d\mu}V_{\rm 1PI}&=\mu\frac{d\Lambda_{\rm cc}}{d\mu}+\frac{1}{2}\mu\frac{dm^2}{d\mu}\phi^2+\frac{1}{4!}\mu\frac{d\lambda(\mu)}{d\mu}\phi^4+\dots\\ &-\frac{1}{2}{\rm tr}\frac{(\mathbf{V}'')^2}{(4\pi)^2}.\end{aligned} \tag{18}$$

$${\rm tr}\left[\mathbf{V}''\right]^2=\left(m^4+M^4\right)+\left(\lambda m^2+\eta M^2\right)\phi^2+\frac{1}{4}\left(\lambda^2+\eta^2\right)\phi^4+\dots \tag{19}$$

Hence we may read off the flow equations for the couplings:

$$\mu\frac{d\Lambda_{\rm cc}}{d\mu}=\frac{1}{32\pi^2}\left(m^4+M^4\right) \tag{20}$$

$$\mu\frac{dm^2}{d\mu}=\frac{1}{16\pi^2}\left(\lambda m^2+\eta M^2\right) \tag{21}$$

$$\mu\frac{d\lambda(\mu)}{d\mu}=\frac{3}{16\pi^2}\left(\left(\lambda^2+\eta^2\right)\right) \tag{22}$$

Let us focus on the cosmological constant. Integrating the first equation between $M_{\rm KK}$ and cosmological scales $\mu\ll m$ we get (after generalizing to include also fermions and gauge bosons - not required)

$$\Lambda_{\rm cc}(\mu\ll m)=\Lambda_{\rm cc}\left(M_{\rm KK}\right)+\frac{1}{64\pi^2}{\rm Str}\left(\mathbf{m}^4+\mathbf{M}^4\right)\ln\left(\frac{\mu^2}{M_{\rm KK}^2}\right) \tag{23}$$

There is no decoupling of high mass states from the low mass beta function eqns since in obtaining these eqns we integrated over all scales.

The following three problems will not be graded. I'm including them since you might find it useful to try them out. The first two are essentially from Sredicicki and the last one is from Peskin and Schroeder. I will discuss a generalized version of the last one when we discuss anomalies.

Problem 3

[30 pts] In class we discussed the quantization of Non-Abelian gauge theories and in particular derived the Feynman rules for a class of Lorentz invariant gauges. Here you should proceed in the same manner (i.e. using the Faddeev-Popov (FP) method) to quantize in a class of non-covariant gauges - defined by the gauge fixing condition $n^\mu A_\mu = 0$ where n^μ is a fixed but arbitrary 4-vector (for example in the so-called axial gauge a.k.a. Arnowitt-Finkler gauge $n^\mu = \eta^{\mu 3}$). Derive the Feynman rules in this gauge (you only need to work out those which are different from what we had for the covariant gauges in class). Show that the FP ghosts are non propagating and that only physical degrees of freedom of the gauge field propagate. (Although the gauge fixing is non-covariant the S-matrix is still Lorentz invariant. In this sense it is similar to the Coulomb gauge in QED.

Problem 4

(40 pts)

Wilson line:

Consider the Wilson line integral i.e. the path ordered line integral

$$\mathbf{W}_P(x_1, x_0) = P \exp\left\{ i g_S \int dt \frac{dx^\mu}{dt} \mathbf{A}_\mu(x(t)\right\}. \tag{24}$$

Here $\mathbf{A}_\mu = A^i_\mu T_i$ (with T_i the generators of some gauge group in some representation R), and path ordering implies that in the expansion of the exponential the products of $A's$ is ordered in the t - i.e. it is "time ordered" in t which parametrizes some curve in space time. Find the solution to the "parallel propagation" equation

$$\mathbf{D}_t \psi(x(t) = \frac{dx^\mu}{dt} \mathbf{D}_\mu \psi(x(t)$$

where $\mathbf{D}_\mu$ is the covariant derivative in the representation R under which ψ transforms. Hence show that the Wilson line transforms under gauge transformation $\mathbf{A} \to \mathbf{A}^{\mathbf{g}}$ as

$$\mathbf{W}_P(x_1, x_0) \to g(x_1) \mathbf{W}_P g^{-1}(x_0)$$

where $g(x)$ is the gauge group element in the representation R at the point x. Deduce that the Wilson loop integral

$$\mathbf{W}_C = \mathrm{tr} P \exp\left\{ i g_S \oint_C dt \frac{dx^\mu}{dt} \mathbf{A}_\mu(x(t)) \right\}$$

where the integral is taken over a closed path is gauge invariant.

For $U(1)$ gauge theory show that the vacuum expectation value of the Wilson loop is

$$< 0|\mathbf{W}_C|0 >= \exp\left[i g_s^2 \oint_C dx^\mu \oint_C dy^\nu \Delta_{\mu\nu}(x-y) \right]$$

where $\Delta_{\mu\nu}(x-y) = \eta_{\mu\nu}/\left[4\pi^2(x-y)^2\right]$ is the propagator for the gauge field A_μ in Feynman gauge. The integral is divergent because of the singularity as $x \to y$. By cutting the integral off at a length scale a show that the integral takes the form

$$< 0|\mathbf{W}_C|0 >= \exp\left[-\frac{c g_s^2}{a} L \right]$$

where L is the length of C.

By taking C to be a circle of radius $R = L/2\pi$ evaluate the constant c. Remember that the integral should be cut off when $|x-y| < a$.

Problem 5

[30 pts]

The (Abelian) anomaly equation is

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}.$$

a) By integrating this over space time show that

$$\Delta N_R - \Delta N_L = -\frac{g^2}{2\pi^2} \int d^4x \mathbf{E}.\mathbf{B}. \tag{25}$$

Here $\Delta N_{R,L}$ is the change in the number of (right/left handed) fermions in a time interval. b) Show that the Hamiltonian for massless fermions may be written in the form

$$H = \int d^3x \left[\psi_R^\dagger(-i\sigma.\mathbf{D})\psi_R - \psi_L^\dagger(-i\sigma.\mathbf{D})\psi_L\right],$$

with $D^i = \nabla^i - ieA^i$. c) Consider the eigenvalue problme for ψ_R i.e. the eqn. $-i\sigma.\mathbf{D})\psi_R = E\psi_R$. Choose the E-M potential to be $A^\mu = (0, 0, Bx^1, A)$ with A, B being constants. Then the eigen vectors can be written as

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i\left(k_2x^2+k_3x^3\right)}.$$

Show that the functions ϕ_i obey the harmonic oscillator equation. d) I the system of fermions is in a box with side of length L and periodic boundary conditions, the momenta $k_{2,}k_3$ will be quantized $k_i = 2\pi n_i/L$. From the eqn derived in part c) show that the condition that the center of the oscillation is inside the box leads to the condition $k_2 < gBL$ and that each energy level has a degeneracy $eL^2B/2\pi$. e) Consider the effect of changing the background by $\Delta A = 2\pi/gL$. Show that the vacuum loses right-handed fermions and by repeating the analysis for left-handed fermions that the vacuum gains the same number of left handed fermions and that the net change is in accord with (25).