HW2 - Phys 7810-001

due 02/18/21

Problem 1

1) (30 pts) The kinetic term $\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i$ for N real scalar fields is inariant under a symmmetry $\phi_i \to O_{ij}\phi_j$, $i,j = 1,\ldots,N$, where $O^TO = \mathbf{I}$ i.e. the symmetry group is $O(N)$. When N is even this group contains the subgroup $U(N/2) \times U(N/2)$. For $N = 4$ define the complex basis of fields $\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\psi = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$ and construct the 2×2 complex matrix

$$\mathbf{\Phi} = \begin{pmatrix} \varphi & \psi^* \\ \psi & -\varphi^* \end{pmatrix}$$

In terms of this observe that the reality condition on the fields ϕ_i translates to the condition $(i)\bar{\mathbf{\Phi}} \equiv \varepsilon\mathbf{\Phi}^*\varepsilon = \mathbf{\Phi}$ ($\varepsilon = i\sigma_2$) and the $O(N)$ invariant form is $(ii)\boldsymbol{\phi}^T\boldsymbol{\phi} = -2\det\mathbf{\Phi}$. Show that these conditions $(i), (ii)$ are preserved by

$$\begin{aligned} a)\,\mathbf{\Phi} &\to U\mathbf{\Phi}, \\ b)\,\mathbf{\Phi} &\to \mathbf{\Phi}\boldsymbol{V}. \end{aligned}$$

for arbitrary unitary matrices U, V with unit determinant. This shows by explicit construction that $O(4)$ is actually equivalent to $SU(2) \times SU(2)$. Sometimes these are referred to as left/right $SU(2)$'s since the first/second acts by multiplication on the left/right. 2) (10+10 pts) Consider a theory with one Majorana fermion ψ and two real scalar fields φ, χ subject to the transformations

$$\delta\psi \to i\omega\gamma_5\psi,\ \delta\varphi \to 2\omega\chi,\ \delta\chi \to -2\omega\varphi$$

for ω an infinitesimal constant parameter. Write down the most general renormalizable (i.e. with operator dimension less than equal to four) for this set of fields. Identify the vacuum (i.e. potential minimum) field configuration and mass spectrum both in the broken and unbroken phases (i.e. for both choices of sign for the coefficient of the the quadratic term of the potential). Couple a vector field to this system by constructing the appropriate covariant derivatives and find the action for this system that is invariant under local gauge transformations (i.e. ω such that $\partial_\mu \omega \neq 0$). Identify the spectrum in both the broken and the unbroken phases.

Problem 2

(40 pts)

a) If $\mathbf{H}$ is an $SU(2)$ doublet show that so is $\epsilon\mathbf{H}^*$. b) Show the equivalence of the two forms of the standard model kinetic terms. i.e. show that $\overline{(\psi^c)_L}\gamma^\mu D_\mu(\psi^c)_L = \overline{\psi_R}\gamma^\mu D_\mu \psi_R$. c) Derive from the gauge invariant kinetic terms of the Higgs Lagrangian after spontaneous symmetry breakdown, the mass terms for the W and the Z bosons. d) Define the four by four matrix of Higgs fields

$$\mathbf{\Phi} = \frac{1}{\sqrt{2}}(\epsilon\mathbf{H}^*, \mathbf{H}) = \frac{1}{\sqrt{2}}\begin{pmatrix} H^{0*} H^+ \\ -H^- H^0 \end{pmatrix}.$$

Show that we can rewrite the Higgs Lagrangian as

$$\begin{aligned}
\mathcal{L}_{Higgs} &= \mathrm{tr}(D_\mu\mathbf{\Phi})^\dagger(D_\mu\mathbf{\Phi}) - V(\mathbf{\Phi}) \\
V(\mathbf{\Phi}) &= \lambda(\mathrm{tr}\mathbf{\Phi}^\dagger\mathbf{\Phi} - \frac{\mu^2}{2\lambda})^2 \\
D_\mu\mathbf{\Phi} &= \partial_\mu\mathbf{\Phi} + i\frac{g}{2}\sigma.W_\mu\mathbf{\Phi} - i\frac{g'}{2}B_\mu\mathbf{\Phi}\sigma_3
\end{aligned}$$

The action of $SU(2)_L \times U(1)_Y$ on $\mathbf{\Phi}$ is then (with $U_L \varepsilon SU(2)_L$, $SU(2)_L : \mathbf{\Phi} \to U_L\mathbf{\Phi}$, $U(1)_Y \to \mathbf{\Phi}e^{-i\sigma_3\theta/2}$. Check directly the invariance of $\mathcal{L}_{Higgs}$ in the above form under this group action. e) Show that in the limit $g' \to 0$ this Lagrangian has a *global* symmetry $SU(2)_R : \mathbf{\Phi} \to \mathbf{\Phi}U_R^\dagger$, $U_R \varepsilon SU(2)_R$. In other words in this limit the Higgs sector has the approximate accidental *global* symmetry $SU(2)_L \times SU(2)_R : \mathbf{\Phi} \to U_L\mathbf{\Phi}U_R^\dagger$. f) Show that after spontaneous symmetry breakdown this global symmetry broken to $SU(2)_{L+R} : \mathbf{\Phi} \to U_L\mathbf{\Phi}U_L^\dagger$. g)

Show that W_μ^i transforms as a triplet under global $SU(2)_L$ and a singlet under $SU(2)_R$. How does W transform under $SU(2)_{L+R}$? What does this tell us about M_W, and M_Z? Note: This global symmetry is called the "custodial symmetry". g) Show that the t, b Yukawa couplings have a custodial symmetry in the limit $m_t = m_b$.

Problem 3

(40 pts)

a) Show that in the standard model there is no non-trivial CKM like matrix for the lepton sector. Hence deduce that there are two additional $U(1)$ symmetries for the leptons - i.e. a lepton number associated with each generation. b) Let us now extend the standard model to include a (right-handed) Dirac field N_R and add the Yukawa interaction $\Delta\mathcal{L}_{Yukawa} = -f_\nu^{AB}\overline{L_L^A}\epsilon\mathbf{H}^*N_R^i$. How should N_R transform under $SU(2)_L \times U(1)_Y$? What is its lepton number? b) Given that neutrinos actually do have mass one may want to add this field and this term to the Lagrangian. But since neutrino masses are of $O(10^{-3}eV)$ how big can the above Yukawa coupling be? Do you think it is OK to have such a value in your Lagrangian? c) Show that gauge invariance allows a Majorana mass term $-\frac{1}{2}M_R^{AB}(N_R^A)^TCN_R^B + h.c.$. However note that it violates lepton number. d) As an alternative to adding a new field, consider looking at higher dimension operators to generate neutrino masses. So we introduce some high scale M (this could be a scale at which new physics appears). There is then a dimension 5 operator that will contribute to giving a neutrino mass term

$$\mathcal{L}_5 = \frac{c^{AB}}{M}(L_L^A)^T\epsilon\mathbf{H}C\mathbf{H}^T\epsilon L_L^B + h.c.$$

i) Show that $\mathcal{L}_5$ is gauge invariant, and that c^{AB} is a symmetric matrix and that this term violates lepton number. *ii*) Find the effective neutrino mass term coming from the Higgs effect on $\mathcal{L}_5$. Assuming that the dimensionless coupling $c \sim O(1)$ how big must M be in order to generate neutrino masses at the observed values. Can you associate this value with some other physics that you may have heard of? *iii*) Show that the analogue of the CKM matrix in the lepton sector has six physically relevant parameters (remember that c^{AB} is a complex symmetric matrix).

(i)

(iv) $O(2)$ invariance requires that Lagrangian terms must be invariant under such a rotation. This means that ψ and ψ^* must appear an equal number of times in each term, and so forces

$$m_1^2 = m_2^2\,, \quad \lambda_1 = \lambda_2\,, \quad \lambda_{12} = 2\lambda_1\,. \tag{1.8}$$

Denoting $m^2 = m_1^2 = m_2^2$ and $\lambda = \lambda_1 = \lambda_2$, the resulting Lagrangian is

Problem I

$$\begin{aligned} -\mathcal{L} &= \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + m^2(\phi_1^2+\phi_2^2)\right] + \frac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 \\ &= \partial_\mu\psi^*\partial^\mu\psi + m^2\psi^*\psi + \lambda(\psi^*\psi)^2\,. \end{aligned} \tag{1.9}$$

(v) There are two cases, depending on the sign of m^2. If $m^2 > 0$: vacuum is $\phi_1 = \phi_2 = 0$ and both particles have mass m.

If $m^2 < 0$: the vacuum is whatever field values minimize the potential. This is not unique but by field redefinition (precisely a rotation by some angle θ) we can take it to be $\phi_1 = \sqrt{-m^2/\lambda}$ and $\phi_2 = 0$. The squared masses for the two particles then are $m_1^2 = -2m^2 > 0$ and $m_2^2 = 0$. With this choice, ϕ_2 is the Goldstone boson.

1.2.2 $N = 4$ case

(i) The most general form for the Lagrangian is

$$-\mathcal{L} = \sum_i \frac{1}{2}\left[(\partial_\mu\phi_i)^2 + m_i^2\phi_i^2\right] + \sum_{i\geq j}\frac{\lambda_{ij}}{4}\phi_i^2\phi_j^2\,. \tag{1.10}$$

(ii) Defining $\sqrt{2}\,\varphi = \phi_1 + i\phi_2$ and $\sqrt{2}\,\psi = \phi_3 + i\phi_4$, we must show that the matrix

$$\Phi \equiv \begin{bmatrix} \varphi & \psi^* \\ \psi & -\varphi^* \end{bmatrix} \tag{1.11}$$

persists in satisfying

$$\bar{\Phi} \equiv \epsilon\Phi^*\epsilon \;=\; \Phi\,, \tag{1.12}$$

$$\det\Phi \;=\; -\varphi^*\varphi - \psi^*\psi = -\frac{1}{2}\phi^T\phi \tag{1.13}$$

under the transformations $\Phi \to U\Phi$ and $\Phi \to \Phi V$, with U, V being unitary matrices having unit determinant. Here $\epsilon = i\sigma_2$.

The relation involving the determinant is trivial since the determinant of a product of matrices is the product of the determinants:

$$\det\,(U\Phi) = (\det U)(\det\Phi) = \det\Phi \tag{1.14}$$

and similarly for ΦV.

For the property involving $\bar{\Phi}$, note first that because $\epsilon = i\sigma_2$, we have $-\epsilon^2 = \mathbf{1}$, so

$$\overline{U\Phi} \equiv \epsilon U^* \Phi^* \epsilon \equiv -\epsilon U^* \epsilon \, \epsilon \Phi^* \epsilon \tag{1.15}$$

and it is sufficient to show that $-\epsilon U^* \epsilon = U$ for $SU(2)$ matrices. This is so because an $SU(2)$ matrix can be written as

$$U = u_0 \mathbf{1} + \vec{u} \cdot i\vec{\sigma} \,, \qquad u_0^2 + \vec{u} \cdot \vec{u} = 1 \tag{1.16}$$

and because the Pauli matrices satisfy

$$\sigma_2 \vec{\sigma}^* \sigma_2 = -\vec{\sigma} \qquad (\text{also}\,,\ \sigma_2 \mathbf{1} \sigma_2 = \mathbf{1})\,. \tag{1.17}$$

Using these with $\epsilon = i\sigma_2$ then gives,

$$-\epsilon U^* \epsilon = \sigma_2 U^* \sigma_2 = \sigma_2 (u_0 \mathbf{1} - \vec{u} \cdot i\vec{\sigma}^*) \sigma_2 = u_0 \mathbf{1} + \vec{u} \cdot i\vec{\sigma} = U\,. \tag{1.18}$$

Therefore

$$\overline{U\Phi} \equiv \epsilon U^* \Phi^* \epsilon = -\epsilon U^* \epsilon \, \epsilon \Phi^* \epsilon = U \epsilon \Phi^* \epsilon = U \bar{\Phi} \tag{1.19}$$

as desired. Exactly the same argument goes through for V.

(iii) Now we consider the case with invariance under only one $SU(2)$, with $\chi \to U\chi$ and $\bar{\chi} \to U\bar{\chi}$, where $\bar{\chi} = \epsilon \chi^*$.

First, observe that only one of the $SU(2)$-invariant combinations, $\chi^\dagger \chi$, $\bar{\chi}^\dagger \chi$, and $\bar{\chi}^\dagger \bar{\chi}$, is independent because

$$\bar{\chi}^\dagger \bar{\chi} = \chi^\dagger \chi = \frac{\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2}{2} = \chi^\dagger \chi\,, \quad \text{and} \quad \bar{\chi}^\dagger \chi = 0\,. \tag{1.20}$$

Therefore the quadratic parts of the Lagrangian can be written only using χ; the most general form is

$$\partial_\mu \chi^\dagger \partial^\mu \chi + m^2 \chi^\dagger \chi\,. \tag{1.21}$$

Now it remains to compute the most general quartic piece. Besides the obvious term, $\lambda(\chi^\dagger \chi)^2$, one might think of the following alternate terms, involving Pauli matrices, $\vec{\sigma}$:

$$(\chi^\dagger \vec{\sigma} \chi)^2\,, \quad (\bar{\chi}^\dagger \vec{\sigma} \chi)^2\,, \quad (\bar{\chi}^\dagger \vec{\sigma} \chi) \cdot (\chi^\dagger \vec{\sigma} \chi)\,. \tag{1.22}$$

(3)

1.4 Symmetries and Yukawa interactions

The symmetry transformations are

$$\delta\psi = i\omega\gamma_5\psi\,, \quad \delta\varphi = 2\omega\chi\,, \quad \delta\chi = -2\omega\varphi\,, \tag{1.46}$$

for real scalars φ and χ.

1. Problem 1.2 shows how this symmetry constrains the purely scalar part of the Lagrangian, with the allowed purely scalar terms

$$\mathcal{L}_s = -\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi + \partial_\mu\chi\partial^\mu\chi + m^2(\varphi^2+\chi^2)\right] - \frac{\lambda}{4}\left(\varphi^2+\chi^2\right)^2. \tag{1.47}$$

Now, for the terms with fermions. Only Yukawa interactions are allowed by renormalizability. Once a field redefinition is performed to put the kinetic terms into canonical form the most general terms possible then are

$$\mathcal{L}_f = -\frac{1}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi + b_1\overline{\psi}\psi + b_2\overline{\psi}\gamma_5\psi + c_1\varphi\overline{\psi}\psi + c_2\chi\overline{\psi}\psi + c_3\varphi\overline{\psi}\gamma_5\psi + c_4\chi\overline{\psi}\gamma_5\psi\right). \tag{1.48}$$

These transform under the symmetry as

$$\begin{aligned}
\delta(\overline{\psi}\gamma^\mu\partial_\mu\psi) &= 0\,,\\
\delta(\overline{\psi}\psi) &= 2i\omega\overline{\psi}\gamma_5\psi\,,\\
\delta(\overline{\psi}\gamma_5\psi) &= 2i\omega\overline{\psi}\psi\,,\\
\delta(\varphi\overline{\psi}\psi) &= 2\omega\left(\chi\overline{\psi}\psi + i\varphi\overline{\psi}\gamma_5\psi\right),\\
\delta(\chi\overline{\psi}\psi) &= 2\omega\left(-\varphi\overline{\psi}\psi + i\chi\overline{\psi}\gamma_5\psi\right),\\
\delta(\varphi\overline{\psi}\gamma_5\psi) &= 2\omega\left(\chi\overline{\psi}\gamma_5\psi + i\varphi\overline{\psi}\psi\right),\\
\delta(\chi\overline{\psi}\gamma_5\psi) &= 2\omega\left(-\varphi\overline{\psi}\gamma_5\psi + i\chi\overline{\psi}\psi\right).
\end{aligned} \tag{1.49}$$

No term can compensate for the shift in the first two terms, so $b_1 = b_2 = 0$. The remaining terms can cancel in pairs if $c_4 = ic_1$ and $c_3 = -ic_2$. Further, Hermiticity demands that c_1 and c_2 are real, and c_3 and c_4 are imaginary; so this is consistent. Therefore, there are two real parameters for the Yukawa interactions, and the remaining allowed terms are

$$\mathcal{L}_f = -\frac{1}{2}\left[\overline{\psi}\gamma^\mu\partial_\mu\psi + c_1\left(\varphi\bar{\psi}\psi + i\chi\bar{\psi}\gamma_5\psi\right) + c_2\left(\chi\bar{\psi}\psi - i\varphi\bar{\psi}\gamma_5\psi\right)\right]. \tag{1.50}$$

Because $\bar{\psi}\psi$ is purely real and $\bar{\psi}\gamma_5\psi$ is purely imaginary, this can be succinctly rewritten in terms of one complex coefficent $c = \sqrt{2}(c_1 - ic_2)$ and the complex field $\Phi \equiv (\varphi + i\chi)/\sqrt{2}$,

$$\mathcal{L}_\psi = -\frac{1}{2}\left[\overline{\psi}\gamma^\mu\partial_\mu\psi - \left(c\Phi\,\overline{\psi}P_L\psi + \text{c.c.}\right)\right], \tag{1.51}$$

where (c.c.) means complex conjugate, and is the same as taking twice the real part of the first term. We see that, in general, Yukawa couplings are complex when written this way.

To obtain the spectrum requires treating separately the cases $m^2 > 0$ and $m^2 < 0$.

If $m^2 > 0$, then the scalar potential is minimized by $\varphi = \chi = 0$, and expanding the lagrangian about this configuration shows that both scalars have mass m and the fermion is massless.

If $m^2 = -\mu^2 < 0$, then the scalar potential is minimized for nonzero fields. By appropriately rotating the fields this can be chosen to be $\varphi^2 = \mu^2/\lambda$ and $\chi = 0$. In this case the squared scalar masses become $m_\varphi^2 = 2\mu^2$ and $m_\chi^2 = 0$, showing that χ is the Goldstone boson. The fermion mass is similarly $m_\psi = c_1\varphi = c_1\mu/\sqrt{\lambda}$.

2. For the gauged version, we must add a gauge kinetic term,

$$\mathcal{L}_g = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.52}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\delta A_\mu = (1/g)\partial_\mu\omega$. All derivatives must also be made covariant derivatives, with

$$\begin{aligned} D_\mu\Phi &= (\partial_\mu + 2igA_\mu)\Phi\,, \\ D_\mu P_L\psi &= (\partial_\mu - igA_\mu)P_L\psi\,, \end{aligned} \tag{1.53}$$

and so therefore,

$$D_\mu P_R\psi = (\partial_\mu + igA_\mu)P_R\psi\,. \tag{1.54}$$

Equivalently $D_\mu\psi = \partial_\mu\psi - igA_\mu\gamma_5\psi$, $D_\mu\varphi = \partial_\mu\varphi - 2gA_\mu\chi$ and $D_\mu\chi = \partial_\mu\chi + 2gA_\mu\varphi$.

If $m^2 > 0$, then the scalar potential is minimized by $\varphi = \chi = 0$, as before. The scalar and fermion masses are as found above, and the gauge boson mass is zero.

If $m^2 = -\mu^2 < 0$, then the scalar potential is minimized by $\varphi^2 = \mu^2/\lambda$ and $\chi = 0$. In this case the scalar and fermion masses are as above, and the gauge boson mass is given by expanding out $-\frac{1}{2}(D_\mu\chi)^2$ to find $m_A = 2g\varphi = 2g\mu/\sqrt{\lambda}$.

HW 2

II b)

$$\overline{(\psi^c)_L}\,\gamma^\mu D_\mu (\psi^c)_L = (P_L C \overline{\psi}^{*T})^\dagger \gamma_0\, \gamma^\mu D_\mu\, P_L (C\overline{\psi}^{*T})$$

$c^\dagger = -c$

$\gamma_0 \psi^*$

$$= (\overline{\psi}^{T\dagger} C^\dagger P_L^\dagger)\gamma_0\, \gamma^\mu D_\mu\, P_L C \overline{\psi}^{*T}$$

$$= -(\psi^\dagger \gamma_0)^{T\dagger} C P_L\, \gamma_0 \gamma^\mu D_\mu\, P_L C\, \overline{\psi}^T$$

$$= -\psi^T \gamma_0 C\, P_L\, \gamma_0 \gamma^\mu D_\mu\, P_L C\, \overline{\psi}^T$$

$\gamma_0 C = C\gamma_0$

$C P_L = P_L C$

$$= -\psi^T P_R\, \gamma^{\mu T} D_\mu\, P_L\, \overline{\psi}^T$$

$$= -(P_R\psi)^T_a\, (\gamma^{\mu T})_{ab} D_\mu\, \overline{\psi_R}^T{}_b$$

$$= \overline{\psi_R}\, \gamma^\mu \overleftarrow{D}_\mu\, \psi_R.$$

(addition

– sign from interchange of ψ's.

a) $\underline{H}$ SU(2) doublet

i.e. $\underline{H} \to e^{i\frac{\vec\sigma}{2}\cdot\vec\theta}\, \underline{H}$.

$\epsilon^2 = -1$

$$\underline{H}^* \to e^{-i\frac{\vec\sigma^*}{2}\cdot\vec\theta}\, H^*$$

$\epsilon = i\sigma^2$.

$\epsilon^{-1} = -\epsilon$.

$\epsilon = i\sigma^2$

$$\epsilon\, \underline{H}^* \to \epsilon\, e^{-i\frac{\vec\sigma^*}{2}\cdot\vec\theta}\, H^* = \epsilon\, e^{-i\frac{\vec\sigma^*}{2}\cdot\vec\theta}\, \epsilon^{-1}\epsilon H^*$$

$\epsilon\,\sigma^{1,3}\epsilon^{-1}$

$= +\epsilon\sigma^{1,3}\epsilon^{-1}$

$= -\sigma^{1,3} = -\sigma^{1,3\,*}$

$\epsilon\,\sigma^2\epsilon^{-1} = \sigma^2$

$= -\sigma^{2*}$

$$= e^{-\frac{i}{2}\,\epsilon\vec\sigma^*\epsilon^{-1}\cdot\vec\theta}\, \epsilon H^* = e^{+\frac{i}{2}\vec\sigma\cdot\vec\theta}\, \epsilon H^*$$

i.e ϵH^* transform like H.

$\psi^C = C\bar{\psi}^T$ $\quad (\psi^C)_L = C\gamma^0 \psi_R^*$

$$\overline{\psi^C}_L = (C\gamma^0\psi_R^*)^\dagger \gamma_0$$

$$= \psi_R^T \gamma^0 (-C)$$

c) See lecture notes Lec 2 p 4E.

d) $$\Phi = \frac{1}{\sqrt{2}}(\epsilon \underline{H}^*, \underline{H}) = \frac{1}{\sqrt{2}}\begin{pmatrix} H^{0*} & H^+ \\ -H^- & H^0 \end{pmatrix}.$$

$$\Phi^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \underline{H}^T(\epsilon\epsilon) \\ H^\dagger \end{pmatrix}$$

$$\mathrm{tr}\, \Phi^\dagger \Phi = \frac{1}{2}\begin{pmatrix} -\underline{H}^T\epsilon \\ H^\dagger \end{pmatrix}(\epsilon \underline{H}^*, \underline{H}) = H^\dagger H.$$

$$\Rightarrow \quad V(\Phi) = \lambda\left(\mathrm{tr}\,\Phi^\dagger\Phi - \frac{\mu^2}{2\lambda}\right)^2 = \lambda\left(H^\dagger H - \frac{\mu^2}{2\lambda}\right)^2.$$

$D_\mu = \partial_\mu - ig\frac{\vec{\sigma}}{2}\cdot\vec{W} - ig'\frac{1}{2}B_\mu$

$$D_\mu \Phi = (D_\mu(\epsilon H^*), D_\mu \underline{H}) = (\epsilon(D_\mu H)^*, D_\mu H)$$

where we used $\sigma\epsilon = -\epsilon\sigma^T = -\epsilon\sigma^*$.

Hence $$\mathrm{tr}(D_\mu\Phi)^\dagger(D_\mu\Phi) = (D_\mu H)^\dagger D^\mu H.$$

$\Phi \to U_L \Phi \qquad \Phi^\dagger \Phi \to \Phi^\dagger U_L^\dagger U_L \Phi = \Phi^\dagger \Phi$

$D_\mu \Phi \to U_L D_\mu \Phi \qquad (D_\mu \Phi)^\dagger D_\mu \Phi \to (D_\mu \Phi)^\dagger U_L^\dagger U_L D^\mu \Phi$

$= (D_\mu \Phi)^\dagger (D^\mu \Phi)$

$U(1)_Y: \quad \Phi \to \Phi e^{-i\sigma_3 \theta/2}$

$\mathrm{tr}\, \Phi^\dagger \Phi \to \mathrm{tr}\, e^{+i\sigma_3\theta/2} \Phi^\dagger \Phi e^{-i\sigma_3 \theta/2}$

cyclicity: $= \mathrm{tr}\, \Phi^\dagger \Phi$.

$\mathrm{Tr}\, (D_\mu \Phi)^\dagger (D^\mu \Phi) \to \mathrm{tr}\left(e^{i\sigma_3\theta/2} (D_\mu \Phi)^\dagger (D^\mu \Phi) e^{-i\sigma_3\theta/2}\right)$

cyclicity: $= \mathrm{tr}\, (D_\mu \Phi)^\dagger (D^\mu \Phi)$ ↑ commutes with σ_3.

e) $g' \to 0$ limit. $\Phi \to U_L \Phi U_R^\dagger$.

$U_{L,R}$ - global.

$\mathrm{tr}\, (D_\mu \Phi)^\dagger (D^\mu \Phi) \to \mathrm{tr}\, U_R (D_\mu \Phi)^\dagger U_L^\dagger U_L (D^\mu \Phi) U_R^\dagger$

$= \mathrm{tr}\, (D_\mu \Phi)^\dagger (D^\mu \Phi)$. $g' \to 0$ → now no σ_3 term!

f) Spontaneous symm. breakdown

$\langle \Phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \qquad U_L \langle \Phi \rangle \neq \langle \Phi \rangle$

$\langle \Phi \rangle U_R^\dagger \neq \langle \Phi \rangle$

But $U_L \langle \Phi \rangle_0 U_L^\dagger = \frac{v}{\sqrt{2}} U_L I U_L^\dagger = \frac{v}{\sqrt{2}} I = \langle \Phi \rangle_0$.

So $SU(2)_L \times SU(2)_R \to SU(2)_{L+R}$.

Under $SU(2)_L$ gauge transf^n.

$$\underline{W}_\mu \equiv \frac{\vec{\sigma}}{2}\cdot\vec{W}_\mu \to U_L \frac{\vec{\sigma}}{2}\cdot\vec{W}\cdot U_L^\dagger + \frac{i}{g}(\partial_\mu U_L)U_L^\dagger$$

For const U_L (global) 2^{nd} term absent and $U_L \frac{\vec{\sigma}}{2}\cdot\vec{W}\cdot U_L^\dagger = e^{\frac{i\vec{\sigma}}{2}\cdot\vec{\theta}} \frac{\vec{\sigma}\cdot\vec{W}}{2} e^{-\frac{i\vec{\sigma}}{2}\cdot\vec{\theta}}$

$$\Rightarrow W^i \to W^i + \epsilon^{ijk}\theta^j W^k + \ldots$$

i.e transforms in adjoint rep. i.e a triplet

~~It is~~ $\underline{W}_\mu$ is a singlet under $SU(2)_R$ since U_R acts on the ~~left~~ right on $\underline{\Phi}$. ($\underline{\Phi} \to \underline{\Phi}\, U_R$ – so $\underline{W}$ does not need to transform to maintain invariance).

Hence $\underline{W}$ transforms as a triplet under $SU(2)_{L+R}$ also.

This tells us that (in the limit $g'\to 0$)

$M_W = M_Z$ (Z is just W^0 here!)

(9) (15)

9) If $m_t = m_b$ then 3^{rd} generation

Yukawa is (since $f_t = f_b = f$)

$$\mathcal{L}^3_{Yukawa} = -f\,\bar{Q}_L\,\Phi\,Q_R + h.c.$$

$$Q_R \equiv \begin{pmatrix} t_R \\ b_R \end{pmatrix}.$$

$SU(2)_L$: $Q_L \to U_L Q_L$

Let $Q_R \to U_R Q_R$ under $SU(2)_R$.

But $\Phi \to U_L \Phi U_R^\dagger$ under $SU(2)_L \times SU(2)_R$

So if $f_t = f_b$ then $\mathcal{L}^3_{Yukawa}$ is invariant.

VII a) See notes Lec 2.

b) $\Delta \mathcal{L}_{\text{Yukawa}} = - f_2^{AB} \overline{L_L^A} \epsilon H^* N_R^B$

L_L and ϵH^* both doublets and

$\overline{L_L^A} \epsilon H^*$ is an $SU(2)_L$ invariant

so N_R is a singlet.

Also L_L, H have equal and opposite

Y so N_R - must have $Y = 0$.

i.e N_R is neutral under $SU(2) \times U(1)$

To conserve lepton # $\ell = +1$ for N_R.

c) $m_\nu \lesssim 10^{-3}\,eV$ $V \simeq 250\ GeV$
$= 2 \times 10^9\ eV$.

so $f_2 \sim O(10^{-12})$

unnaturally small! - technically natural though - fermion mass

Nothing fundamentally wrong but

generally ~~one~~ should avoid such small parameters without an explanation of their size!

c) $\Delta \mathcal{L}_M = -\frac{1}{2} M_R^{AB} (N_R^A)^T C N_R^B + h.c.$

N_R - is a gauge singlet under $SU(2)_L \times U(1)$

So this is gauge invariant

But N_R has $l = +1$ so term has $l = 2$

and violates lepton #.

d). $\mathcal{L}_5 = \frac{c^{AB}}{M} \left[(L_L^A)^T \epsilon H \right] C \left[H^T \epsilon (L_L^B) \right] + h.c.$

i) $(L_L^{AT} \epsilon H)$ is an $SU(2)_L \times U(1)$ invariant

(see (b)). So this term is

gauge invariant.

ii) Also this term $\mathcal{L}_5$ has

$l = 2$ so violates lepton #.

Interchanging (and taking the transpose of) each

factor in [ ] brackets gives a − sign from

fermion interchange cancelled by − sign

coming from ~~int~~ anti-symm. of C

so $c^{AB} = c^{BA}$.

12 (8)

Neutrino mass

ii) $m_\nu \sim \frac{C}{M} \cdot \frac{v^2}{2} \cdot \approx O(1) \times \left(\frac{100\,GeV}{M}\right) 100\,GeV$

$\simeq 10^{-3}\,eV = 10^{-12}\,GeV.$

$\Rightarrow M \sim 10^{+16}\,GeV.$

iii) $C^{AB} = C^{BA} \Rightarrow 2 \times \frac{3\times 4}{2} = 2\times 6$

C^{AB} - complex symm. real parameters

so # of physically relevant parameters is

$$\underset{\substack{\uparrow \\ \text{from } e\text{-Yukawa}}}{2\times 3^2} + \underset{\substack{\uparrow \\ C_{AB}}}{2\times 6} - \underset{\substack{\uparrow \\ 2.\ U(3) \\ \text{symmetries} \\ \text{of matter Lagrangian}}}{2\times 3^2} = 12$$

$-\ 6 -$ e-type and ν-type masses

$\Rightarrow 6$ - parameters of MNS matrix

$= 3$ angles $+ 3$ - phases

Aside: $\nu_L \to O\,\nu_L \quad O^T O = 1$ to diagonalize C^{AB}

$e_L \to U_L e_L$ to diagonalize ξ_L as before.

$M_{\cancel{MNS}} = \quad \bar{e}_L^A \nu_L^A \to \bar{e}_L^A (U_L^\dagger O)_{AB} \nu_L^B$

$O O^T = 1$

O - complex

$M_{MNS} = U_L^\dagger O$ (If parameters real this is orthogonal $\Rightarrow$ 3 angles)

$U \to 3^2 = 9$ parameters

$O \to 2 \times \frac{3 \times 2}{2} = 6$ parameters.

However $e_L^A \to U_L^{AB} e_L^B$ is a symmetry of matter action so 6 parameters left

= 3 angles + 3 phases.

To put it yet another way in the absence of the c_{AB} (dimension 5) term there is no analog of CKM matrix as argued before

So only new parameteres are from diagonalizing h_5 i.e from O which has 6-parameters - 3 (if O is real orthogonal) + 3 phases.