# HW1 - Phys 7810-001

due 02/04/21

## Problem 1

[40=2+3+5+5+5+20 pts] The composition law of a Lie group is given by $g(\theta)g(\phi) = g(\xi(\theta,\phi)$ where $\theta = \{\theta^i\}, \phi = \{\phi^i\}$ are n-dimensional parameter vectors and $\xi = \{\xi^i\}$. Show that a) $\xi(\theta, 0) = \xi(0,\theta) = \theta$. b) $\xi(\theta, \xi(\phi,\psi)) = \xi(\xi(\theta,\phi),\psi)$. c) Write

$$g(\phi)g(\theta)g^{-1}(\phi)g^{-1}(\theta) = g(\chi(\theta,\phi)).$$

Show that near the identity element $\chi^i = c^i_{jk}\theta^j\phi^k$. d) By evaluating the commutator $g(\phi)g(\theta)g^{-1}(\phi)g^{-1}(\theta)$ show that the generators satisfy the commutation relations $[X_j, X_k] = ic^l_{jk}X_l$. e) Deduce that $c^l_{jk} = -c^l_{kj}$, and that $c^m_{jk}c^n_{lm} + c^m_{kl}c^n_{jm} + c^m_{lj}c^n_{km} = 0$. f) For a matrix group we define the Cartan-Killing metric on the Lie algebra by $g_{ij} = \mathrm{tr}(X^iX^j)$. i) Show that $c_{ijk} \equiv g_{il}c^l_{jk}$ is totally anti-symmetric in $i, j, k$. ii) If $U = e^{iH}$ is a unitary matrix with $\det U = 1$, show that $\mathrm{tr}H = 0$. g) Let $\psi, \phi, \ldots$ be vectors in the space of n-dimensional column vectors ($\psi = \{\psi_a\}$ etc.) which carry an n-dimensional unitary representation of some Lie group. Suppose the group elements $\{g\}$ of a group of unitary transformations on this vector space are given in some unitary representation by the matrices $D(g)$. i) Show that the totally anti-symmetric tensor $\epsilon_{i_1\ldots i_N} = \pm 1$ (with upper(lower) sign for even(odd) permutations of $1, 2, \ldots, N$) is an invariant of the group $SU(N)$. ii) Suppose the vector $\psi = \{\psi_i,\, i = 1,\ldots,N\}$ is in the fundamental (defining) representation of $SU(N)$. Then the tensor $\psi_{ij}$ transforms as the direct product of $\psi \times \psi \equiv \{\psi_i\psi_j\}$. Define the permutation operator $P$ so that $P\psi_{ij} = \psi_{ji}$. Show that $P$ commutes with the group transformation law. Show that $\psi_{ij}$ is a reducible tensor representation by demonstrating that the symmetric and anti-symmetric combinations $\psi^{\pm}_{ij} \equiv \frac{1}{2}(\psi_{ij} \pm \psi_{ji})$ do not mix under the group transformations.

## Problem 2

[30=5+10+10+5 pts] For the Lie algebra of $SU(N)$ show that a) $C_{ijk}T_jT_k = \frac{i}{2}C_2(G)T_i$ b) Prove the completeness relation (for the generators in the fundamental representation)

$$(T_i)_{\gamma\beta}(T_i)_{\alpha\lambda} = \frac{1}{2}(\delta_{\beta\alpha}\delta_{\gamma\lambda} - \frac{1}{N}\delta_{\beta\gamma}\delta_{\alpha\lambda})$$

c) Show that in any IR $r$ the generators satisfy the relation

$$T_iT_jT_i = \left[C_2(r) - \frac{1}{2}C_2(G)\right]T_j.$$

d) Using this (or otherwise?) show that if the generators in the fundamental are normalized with $C(N) = \frac{1}{2}$ then $C(G) = C_2(G) = N$.

## Problem 3

[30=5+5+5+5+5+5 pts] a) Starting from the Lorentz algebra and defining $J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}$, $K_i \equiv -M_{0i}$ and $\mathcal{J}^{\pm}_i \equiv \frac{1}{2}(J_i \pm iK_i)$, show that

$$\begin{aligned}
[\mathcal{J}^{\pm}_i, \mathcal{J}^{\pm}_j] &= i\epsilon_{ijk}\mathcal{J}^{\pm}_k, \\
[\mathcal{J}^{\pm}_i, \mathcal{J}^{\mp}_j] &= 0.
\end{aligned}$$

b)Show that if $\chi_L$ is in the $(\frac{1}{2}, 0)$ representation, $\epsilon\chi^*_L$ (here $\epsilon = i\sigma_2$) is in the $(0, \frac{1}{2})$ representation (i.e. it transforms like $\chi_R$). c) Show that $\mathcal{L} = -\frac{1}{2}m\bar{\psi}_M\psi_M = m(\psi^T_L\epsilon\psi_L - \psi^{\dagger}_L\epsilon\psi^*_L)$ where $\psi_M$ is the four component Majorana spinor and $\psi_L$ is a left chiral Weyl spinor. d) Show that $(\psi^c_D)^c = \psi_D$. e) Show that $\mathcal{L} = -m[(\psi^c_D)^T_L C(\psi_D)_L + h.c.)$ is a Dirac mass term and that $\mathcal{L} = -m[\psi^T_{DL}C\psi_{DL} + h.c.)$ is a Majorana mass term. Here $\psi_D$ is a four component Dirac spinor and $C$ is the charge conjugation matrix. .e) Show that if $\psi^c_D = \psi^c_D$ then $\psi_D = \psi_M$. f) Show that $\mathcal{L} = -\frac{1}{2}m(\overline{\psi^c_D})_R\psi_{DL} + h.c.)$ is a Majorana mass.

# HW I (Solutions).

I. $g(\theta)g(\varphi) = g(\xi(\theta,\varphi))$ $\theta = \{\theta^i\}$ $\varphi = \{\varphi^i\}$.

a) $g(0) = 1$ Set $\varphi = 0$ $g(\theta)g(0) = g(\theta) = g(\xi(\theta,0))$

$\Rightarrow \xi(\theta,0) = \theta$ Similarly $(\theta=0)$ $\xi(0,\theta) = \theta$.

b) $g(\theta)(g(\varphi)g(\psi)) = g(\theta)\,g(\xi(\varphi,\psi))$ ~~$= g(\xi(\varphi,\psi))$~~

$= g(\xi(\theta, \xi(\varphi,\psi)))$

Associativity

$= (g(\theta)g(\varphi))g(\psi) = g(\xi(\xi(\theta,\varphi),\psi))$

$\Rightarrow \xi(\theta, \xi(\varphi,\psi)) = \xi(\xi(\theta,\varphi),\psi)$.

For next see foll pages.

A <u>Lie Group</u> is a continuous group with the composition Law defined by a continuously differentiable map. (i.e $\xi$ is continuously differentiable)

<u>def:</u> <u>Generator</u> $X_i(\theta) \equiv i^{-1} g^{-1}(\theta)\, \partial_i g(\theta)$ $i=1,\dots n$.

$g = e^{iX_i\theta^i}$

$g^{-1} = e^{-iX_i\theta^i}$

$\{X_i\}$ generators of Lie Group.

$\delta g = e^{i X_i \delta\theta^i}$

For $\underline{\theta}$ infinitesimal (i.e near the identity)

write $g(\underline{\theta}) = e + i\sum_j \theta^j X_j(0) \Rightarrow g^{-1}(\theta) = e - i\sum_j \theta^j X_j(0)$

(for $SO(2)$ $X = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$) Clearly $X_i = i^{-1} g^{-1} \partial_i g$

Product $g(\underline{\varphi})\, g(\underline{\theta})\, g^{-1}(\underline{\varphi})\, g^{-1}(\underline{\theta})$ — <u>commutator</u> of two elements.

Must be a group element $g(\chi)$ $\chi^i = f^i(\theta, \varphi)$

with $f^i(0,\varphi) = f^i(\theta, 0) = 0$

Near identity $\chi^i = A^i + B^i{}_j \theta^j + B'^i{}_j \varphi^j + C^i{}_{jk} \theta^j \varphi^k + C'^i{}_{jk} \theta^j \theta^k$
$+ C''^i{}_{jk} \varphi^j \varphi^k$

From boundary conditions $A = B = B' = C' = C'' = 0$

ie. $\chi^i = C^i{}_{jk}\, \theta^j \varphi^k$

$$g(\varphi)\, g(\theta)\, g^{-1}(\varphi)\, g^{-1}(\theta) = (e + i\varphi^j X_j \cdots)(e + i\theta^k X_k \cdots) \times (e - i\varphi^l X_l \cdots)(e - i\theta^m X_m \cdots)$$

$$= (e + i\varphi^j X_j + i\theta^k X_k - \varphi^j\theta^k X_j X_k + \cdots)(e - i\varphi^j X_j - i\theta^k X_k - \varphi^j\theta^k X_j X_k + \cdots)$$

$$= (e + \varphi^j\theta^k X_j X_k + \varphi^j\theta^k X_k X_j - 2\varphi^j\theta^k X_j X_k \cdots)$$

$$= e + \theta^j\varphi^k [X_j, X_k] + \cdots$$

$$= g(\xi) = e + i c^l{}_{jk}\theta^j\varphi^k X_l$$

So

$$[X_j, X_k] = i \underset{\uparrow}{c^l{}_{jk}} X_l \quad -(A)$$

structure constants.

$$\boxed{c^l{}_{jk} = -c^l{}_{kj}}$$

From the Jacobi identity

$$[X_j, [X_k, X_l]] + [X_k, [X_l, X_j]] + [X_l, [X_j, X_k]] = 0$$

$$c^m{}_{jk} c^n{}_{lm} + c^m{}_{kl} c^n{}_{jm} + c^m{}_{lj} c^n{}_{km} = 0 \quad -(J)$$

The vector space ~~whose basis~~ spanned by $\{X^i\}$ satisfying (A) is called the Lie algebra $h$. of the group $G$.

* The structure constants $c^l{}_{jk}$ specify the Lie group

# SU(n) as a transformation group.

The group of $n \times n$ unitary matrices can be consider as transformation group acting on a $n$-dimensional complex vector space.

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \qquad \psi_i \in \mathbb{C}$$

$$\underline{\psi} \to \underline{\psi}' = \underline{U}\,\psi \qquad U \in SU(n)$$

$$\text{ie } UU^\dagger = U^\dagger U = 1$$

$$\psi'_i = U_{ij}\,\psi_j \qquad \det U = \underline{1}$$

~~The~~ Clearly the scalar product

$$\psi^\dagger \phi = \sum_i \psi_i^* \varphi_i$$

is left invariant by these transformation ($\psi \to U\psi \quad \varphi \to U\varphi$).

The conjugate vector $\psi^\dagger \to \psi'^\dagger = \psi^\dagger U^\dagger$

Useful to write $\psi^i = \psi_i^*$

and $U_i{}^j \equiv U_{ij}$ $\qquad U^i{}_j \equiv U^*_{ij} = U^\dagger_{ji} = U^\dagger{}_j{}^i$

So $\psi_i \to \psi'_i = U_i{}^j \psi_j$

$\psi^i \to \psi'^i = U^i{}_j \psi^j$ .

The $SU(n)$ invt scalar product,

$$\psi^\dagger \phi = \psi^i \phi_i$$

Unitarity $\quad U U^\dagger = U_{ij} U^*_{kj} = U_i{}^j U^k{}_j = \delta_i{}^k$

Similarly $\quad = U^j{}_i U_j{}^k = \delta_i^k$

In this not$^n$ ~~contraction or~~ summation over upper & lower indices

i.e. contracted indices as in 4-vector notation for space-time indices

The vectors $\psi$ give basis for ~~defining~~ the fundamental or defining representation of $SU(n)$.

Check invariance $\quad \psi_i \to U_i{}^j \psi_j \quad \psi^i \to U^i{}_j \psi^j$

$$\psi^i \psi_i \to \psi^j U^i{}_j U_i{}^k \psi_k = \psi^j \delta_j{}^k \psi_k$$

$$= \psi^j \psi_j$$

$\psi_i$ — is (basis for) defining (fundamental) rep$^n$. $\quad \underline{n}$

~~Higher~~ $\psi^i$ ... conjugate rep $\quad \underline{n}^*$

Higher rank tensors transform as

$$\psi^{i_1 \dots i_p}_{j_1 \dots j_q} \quad \psi'^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} = \left(U^{i_1}{}_{k_1} U^{i_2}{}_{k_2} \cdots U^{i_p}{}_{k_p}\right)\left(U_{j_1}{}^{l_1} \cdots U_{j_q}{}^{l_q}\right) \psi^{k_1 \dots k_p}_{l_1 \dots l_q}$$

This is (basis for) in general a reducible representation).

$$\cancel{T_{\pm\pm} = \partial_\pm Z_+ \partial_\pm Z_- + \sqrt{\tfrac{|n|}{2}}\, \partial_\pm^2 (Z_+ - Z_-)}$$

(2). Totally anti-symm invt tensor

$$\epsilon_{i_1 \cdots i_n} = \epsilon^{i_1 \cdots i_n} \qquad = 1 \begin{pmatrix} i_1 \cdots i_n \\ 1 \cdots n \end{pmatrix} \text{ even}$$

$$\epsilon'_{i_1 \cdots i_n} = U_{i_1}{}^{j_1} \cdots U_{i_n}{}^{j_n} \epsilon_{j_1 \cdots j_n} \qquad = -1 \text{ odd. perm}$$

$$= (\det U)\, \epsilon_{i_1 \cdots i_n} = \epsilon_{i_1 \cdots i_n} \qquad = 0 \text{ if } i_k = i_\ell \text{ for any } k, \ell.$$

since $\det U = 1$.

Can raise or lower indices using $\epsilon$ tensor.

$$\epsilon^{i_1 i_2 \cdots i_n} \psi_{i_2 \cdots i_n} = \psi^{i_1} \qquad \text{etc}$$

need study tensors with all lower or all upper indices.

Note $\psi = \epsilon^{i_1 \cdots i_n} \psi_{i_1 \cdots i_n}$

is an invariant of $SU(n)$.

These tensors bases for <u>reducible</u> reps

The reason is that permutations of indices (all upper or lower) commute with Group transformation So tensors ~~whose indices~~ whose indices form an ~~rep~~ irreducible rep of the perm$^n$ group transform ~~as~~ amongst themselves under $SU(n)$ transformations. This is because the transformation law ~~in~~ has products of

$U$'s.

Simplest example - 2nd rank tensor.

$$\psi^{ij} \to \psi'^{ij} = U^i{}_k U^j{}_l \psi^{kl}$$

$$\psi'^{ji} = U^j{}_l U^i{}_k \psi^{lk} = U^i{}_k U^j{}_l \psi^{lk}$$

if $\quad P_{12} \psi^{ij} = \psi^{ji}$

$$P_{12} \psi'^{ij} = U^i{}_k U^j{}_l P_{12} \psi^{kl}$$

$P_{12}$ commutes with Group transf$^n$ law.

Define Symm & Anti-symm tensors.

$$\psi_+^{ij} = \frac{1}{2}(\psi^{ij} + \psi^{ji}) \qquad \psi_-^{ij} = \frac{1}{2}(\psi^{ij} - \psi^{ji})$$

$$P_{12} \psi_+^{ij} = + \psi_+^{ij} \qquad P_{12} \psi_-^{ij} = - \psi_-^{ij}$$

Clearly $\psi_+$, $\psi_-$ do not mix under SU(n) transformation

$$\psi_\pm^{ij} \to \psi_\pm'^{ij} = U^i{}_k U^j{}_l \psi_\pm^{kl}$$

ie 2nd rank tensor fives reducible rep

$\psi_\pm^{ij}$ are the (basis for) irr. rep.

Young tableaux - give a soln to the general problem of finding IR's of $S_n$. Cheng & Li p104-…

# Problem Solution I

$\psi_B^* \epsilon^{AB} \psi_A^*$

(11)

~~9a~~

III

$\psi_A \to M_A{}^B \psi_B.$

$\psi_A \epsilon^{AB} \psi_B.$

$\to M_A{}^C \psi_C \epsilon^{AB} M_B{}^D \psi_D.$

$= M_A{}^C M_B{}^D \epsilon^{AB} \psi_C \psi_D.$

$= \det M \, \epsilon^{CD} \psi_C \psi_D = \psi_C \epsilon^{CD} \psi_D$

Since $\det M = 1$

$$\chi_L \to e^{i \frac{\vec{\sigma} \cdot \vec{\theta}}{2}} \chi_L$$

$$\chi_L \to e^{-\frac{1}{2} \vec{\sigma} \cdot \vec{\eta}} \chi_L.$$

$\sigma_2^* = -\sigma_2$ $\qquad$ $\sigma_2 \sigma_1^* \sigma_2 = -\sigma_1.$

$\sigma_2 \sigma_3^* \sigma_2 = -\sigma_3.$

$\sigma_2 \sigma_i^* \sigma_2 = -\sigma_i.$ $\qquad$ $\sigma_2 \sigma_2^* \sigma_2 = -\sigma_2.$

$\sigma_2 \sigma_i^* = -\sigma_i \sigma_2$

$$J: \quad i\sigma_2 \chi_L^* \to i\sigma_2 e^{-i \frac{\sigma_i^*}{2} \theta^i} \chi_L^*$$

$$= e^{+i \frac{\sigma_i}{2} \theta^i} (i\sigma_2) \chi_L^*.$$

$$K: \quad i\sigma_2 \chi_L^* \to i\sigma_2 e^{\frac{\sigma_i^*}{2} \eta^i} \chi_L^*$$

$$= e^{-\frac{\sigma_i}{2} \eta^i} \chi_L$$

II

i). $\mathrm{tr}\,(T_i^{(r)} T_j^{(r)}) = C(r)\,\delta_{ij}$ ; $T^{2\,(r)} = \sum_{i=1}^{\dim G} T_i T_i = C_2(r)\, I$ — (A)

for $r = G$. $\quad i\,(T_j^{(G)})_{ik} = C_{ijk}$. — adjoint rep.

$$\Rightarrow \quad C_{ijk} C_{ljk} = C_2(G)\,\delta_{il}. \quad -\ (B)$$

Taking the trace of the second relation in (A) and using the 1st

we get $\quad d(r)\, C_2(r) = |G|\, C(r) \quad -\ (B) \qquad |G| = \dim G.$

Normalize generators of $S(N)$ in fundamental rep ($r = N$) by

$$C(N) = \frac{1}{2} \;\Rightarrow\; C_2(N) = \frac{(N^2-1)}{N}\cdot\frac{1}{2}$$

a) $\quad C_{ijk} T_j T_k = \frac{1}{2} C_{ijk} [T_j, T_k] = \frac{1}{2} C_{ijk}\, i C_{jkl} T_l.$

$$= \frac{i}{2} C_2(G)\, T_i. \qquad \text{from (B)}.$$

b) Any $N \times N$ matrix can be expanded in terms of the $N^2-1$ generators and the unit matrix $1$ ($a$ linear independent matrices (or tho norm'd) if $1 \to \frac{1}{N}$)

Note $\quad \mathrm{Tr}\, T_i = 0 \qquad \mathrm{Tr}\, 1 = N.$ (in fundamental),

i.e $\quad \underline{U} = a_i T_i + u\, 1_{(N\times N)} \qquad u = \mathrm{Tr}\, U / N$

$\mathrm{tr}\, T_j U = a_i \mathrm{tr}(T_j T_i) = \frac{1}{2} a_i \qquad$ from 1st eq$^n$ (A).

$$\underline{u} = a_i \underline{T}_i + u \frac{\mathbb{1}}{N} \qquad \mathrm{Tr}\,\mathbb{1} u = u.$$

$$\mathrm{Tr}\, T_j u = a_i \,\mathrm{Tr}\, T_j T_i = a_j \cdot \frac{1}{2}$$

$$a_i = \mathrm{Tr}\, T_j u$$

$$\underline{u} = \left(2\,\mathrm{Tr}\, u T_i\right) T_i + \mathrm{Tr}\left(\mathrm{Tr}\, u\, \mathbb{1}\right) \frac{\mathbb{1}}{N}$$

$$u_{\alpha\beta} = 2\, u_{\beta\gamma} (T_i)_{\gamma\beta} (T_i)_{\alpha\delta} + u_{\alpha\alpha}\, \delta_{\alpha\delta}$$

$$\Rightarrow \left(u_{\beta\gamma}\right)\left(\delta^{\beta}_{\alpha}\delta^{\gamma}_{\delta} - 2\,(T_i)_{\gamma\beta}(T_i)_{\alpha\delta} - \frac{1}{N}\delta^{\beta\gamma}\delta_{\alpha\delta}\right) = 0$$

$$(T_i)_{\gamma\beta}(T_i)_{\alpha\delta} = \frac{1}{2}\left(\delta_{\beta\alpha}\delta_{\gamma\delta} - \frac{1}{N}\delta_{\beta\gamma}\delta_{\alpha\delta}\right).$$

$$[T_i, T_j] = i f_{ijk} T_k.$$

$$T^i T^j T^i = T^i [T^j, T^i] + T^i T^i T^j$$

$$= i T^i f^{jik} T_k + C_2(r) \cdot \mathbb{1}\, T^j$$

$$T^i_{(G)jk} \overset{?}{=} \pm i f_{ijk} \qquad \sum_i T_i^{(r)} T_i^{(r)} = C_2(r) \cdot \mathbb{1}$$

$$r = G \qquad f^{acd} f^{bcd} = C_2(G)$$

$$T^i T^j T^i = f^{jik} T_i T_k + C_2(r) T^j$$

$$= i f^{jik} \frac{1}{2} [T_i, T_k] + C_2(r) T^j$$

$$= i f^{jik} \frac{i}{2} f_{ikl} T^l + C_2(r) T^j$$

$$= \left[ -\frac{1}{2} C_2(G) + C_2(r) \right] T^j$$

$$T^{i(F)}_{\alpha\beta} T^{i(F)}_{\gamma\delta} = \frac{1}{2} \left( \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} \right)$$

$$T^i T^j T^i = \left[ -\frac{1}{2} C_2(G) + C_2(F) \right] T^j$$

$$T^i_{\alpha\beta} T^j_{\beta\gamma} T^i_{\gamma\delta} = \frac{1}{2} \left( \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} \right) T^j_{\beta\gamma}.$$

$$= \frac{1}{2} \left( \cancel{0} - \frac{1}{N} T^j_{\alpha\delta} \right) = \left( -\frac{1}{2} C_2(G) + C_2(F) \right) T^j_{\alpha\delta}.$$

$$-\frac{1}{2} C_2(G) + \frac{N^2-1}{2N} = -\frac{1}{2N} \Rightarrow C_2(G) = N$$

II

# Sol I

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Majorana mass term: $\bar{\psi}_M \psi_M = (\psi_L^\dagger, \psi_L^T \epsilon^+)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_L \\ \epsilon\psi_L^*\end{pmatrix}$

$\psi_M = \begin{pmatrix}\psi_L \\ \epsilon\psi_L^*\end{pmatrix}$

$$= (\psi_L^\dagger, -\psi_L^T\epsilon)\begin{pmatrix}\epsilon\psi_L^* \\ \psi_L\end{pmatrix}$$

$$= -\psi_L^T\epsilon\psi_L + \psi_L^\dagger\epsilon\psi_L^*$$

$$\psi_D = \begin{pmatrix}\psi_L \\ \epsilon\chi_L^*\end{pmatrix}$$

Dirac Mass term (Alt rep). $\Delta\mathcal{L} = -m(\psi_D^C)_L^T C\psi_L + h.c.$

$$(\psi_D^C)_L^T = (\chi_L^T, \psi_L^\dagger\epsilon^T)\underbrace{\tfrac{1}{2}(1-\gamma_5)}_{P_L}$$

$P_L = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$$= (\chi_L^T, 0)$$

$$\Delta\mathcal{L} = -m(\chi_L^T, 0)\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\psi_L \\ 0\end{pmatrix} + h.c.$$

$$= m\chi_L^T\epsilon\psi_L + h.c. \quad = \text{Dirac Ma}$$

$$= -m\bar{\psi}\psi \quad \text{(as shown in notes)}.$$

$$\Delta\mathcal{L} = -\tfrac{1}{2}m(\psi_{DL}^T C\psi_{DL} + h.c.) = -\tfrac{1}{2}m\left((\psi_L^T, 0)\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\psi_L \\ 0\end{pmatrix} + h.c.\right)$$

$$= \frac{m}{2}(\psi_L^T\epsilon\psi_L + h.c.) = \text{Majorana mass.}$$

II $\psi_D^C = \psi_D$ Then $\psi_D = \psi_M$.

$$\psi_D^C = \begin{pmatrix}\chi_L \\ \epsilon\psi_L^*\end{pmatrix} = \psi_D = \begin{pmatrix}\psi_L \\ \epsilon\chi_L^*\end{pmatrix} \Rightarrow \chi_L = \psi_L \Rightarrow \psi_D = \psi_M$$

IIa

$\sigma^\mu = (1, \sigma^i)$

$\bar\sigma^\mu = (1, -\sigma^i)$

$\sigma_2 \sigma^{1,3} \sigma_2 = -\sigma^{1,3} = -\sigma^{1,3T}$

$\sigma_2 \sigma^2 \sigma^2 = \sigma^2 = +\sigma^{2T} = \bar\sigma^{2T}$

$\epsilon = i\sigma_2$

$\epsilon^\dagger = -i\sigma_2 = \epsilon^T = -\epsilon$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_2^T = -\sigma_2$

$\sigma_1^T = \sigma_1$

$\sigma_3^T = \sigma_3$

$$C \gamma^\mu C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

$$= \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^\mu \sigma_2 \\ -i\bar\sigma^\mu \sigma_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_2 \sigma^\mu \sigma_2 \\ \sigma_2 \bar\sigma^\mu \sigma_2 & 0 \end{pmatrix} = + \begin{pmatrix} 0 & \bar\sigma^{\mu T} \\ \sigma^{\mu T} & 0 \end{pmatrix}$$

$$= + \gamma^{\mu T}. \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

$$\epsilon^T = -\epsilon \quad \Rightarrow \quad C^T = -C.$$

$$\epsilon^2 = -1 \quad \Rightarrow \quad C^2 = -1, \quad C = -C^{-1}$$

$$C^\dagger = \begin{pmatrix} -\epsilon^\dagger & \\ & \epsilon^\dagger \end{pmatrix} = \begin{pmatrix} +\epsilon & \\ & -\epsilon \end{pmatrix} = -C.$$

$$(\psi_D{}^c)^c = \psi_D$$

$$(\psi_D{}^c)^c = C\gamma^0 \psi^{c*} = C\gamma^0 C^* \gamma^0 \psi$$

$$= C\gamma^0 C \gamma^0 \psi = -C\gamma^0 C^{-1} \gamma^0 \psi$$

$$= +\gamma^{0T} \gamma^0 \psi = \gamma^{0 2} \psi = \psi$$

III



a) $\Delta\mathcal{L} = -m\bar{\psi}_D\psi_D = -\frac{1}{2}m\left(\bar{\psi}_D\psi_D + \bar{\psi}_D^c\psi_D^c\right)$.

$\psi^C = C\gamma^0\psi^*$ $\qquad \psi^{\cancel{*}\dagger} = \psi^T\gamma^0 C^\dagger$

$\bar{\psi}^C\psi^C = \psi^T\gamma^0 C^{\dagger}\gamma^0 C\gamma^0\psi^*$ $\cancel{\psi^T\gamma^0}$

$= \psi^T\gamma^0(-\gamma^0)\gamma^0\psi^* = -\psi^T\gamma^0\psi^*$

$= -\left(\psi^T\gamma^0\psi^*\right)^T = \psi^\dagger\gamma^0\psi$

( Alt
$-\psi^T\gamma^0\psi^* = -\psi_\alpha\gamma^{0\alpha\beta}\psi^*_\beta$

$\gamma^{0T} = \gamma^0$ $\qquad = \psi^*_\beta\gamma^{0\beta\alpha}\psi_\alpha = \bar{\psi}\psi.$ ).

so $\bar{\psi}_D^c\psi_D^c = \bar{\psi}_D\psi_D$.

Show $\quad -m\bar{\psi}_D\psi_D = -\frac{m}{2}\left(\bar{\psi}_M^1\psi_M^1 + \bar{\psi}_M^2\psi_M^2\right)$.

Write $\quad \psi_M^1 = \frac{1}{\sqrt{2}}\left(\psi_D + \psi_D^C\right)$.

$\psi_M^2 = \frac{1}{\sqrt{2}}\left(\psi_D - \psi_D^C\right)$.

$(\psi_M^1)^c = \frac{1}{\sqrt{2}}\left(\psi_D^c + (\psi_D^c)^c\right) = \psi_M^1$

$(\psi_M^2)^c = \frac{1}{\sqrt{2}}\left(\psi_D^c - (\psi_D^c)^c\right) = -\psi_M^2$.

# IV

Note.
$$\psi_D = \begin{pmatrix} \psi_L \\ \epsilon \chi_L^* \end{pmatrix} \qquad \psi_D^c = \begin{pmatrix} \chi_L \\ \epsilon \psi_L^* \end{pmatrix}.$$

$$\psi_M^1 = \begin{pmatrix} \lambda_L \\ \epsilon \lambda_L^* \end{pmatrix} \qquad \psi_M^2 = \begin{pmatrix} \tilde{\lambda}_L \\ -\epsilon \tilde{\lambda}_L^* \end{pmatrix}$$

$$\lambda_L = \psi_L + \chi_L \qquad \tilde{\lambda}_L = \psi_L - \chi_L$$

Show $\Delta \mathcal{L} = -\frac{1}{2} m \left( \overline{(\psi_D^c)_R}\, \psi_{DL} + h.c. \right)$

is a majorana mass.

$$\overline{(\psi_D^c)_R} = \psi_{DR}^{c\dagger} \gamma_0 = (0, \psi_L^T \epsilon^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= (-\psi_L^T \epsilon, 0)$$

$$\psi_{DL} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

$$\overline{(\psi_D^c)_R}\, \psi_{DL} = -\psi_L^T \epsilon\, \psi_L.$$

So $\Delta L = \frac{1}{2} m \left( \psi_L^T \epsilon \psi_L + h.c. \right)$