# Anomalies

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## The $\pi^0 \to 2\gamma$ decay puzzle.

In our pion effective field theory the lowest dimension operator for this process (Imposing Lorentz and gauge invariance)

A - photon field.

$$\mathcal{L}_{\pi\gamma\gamma} = g\,\pi^0\, \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$[g] = M^{-1}$ i.e $g \sim f_\pi^{-1}$

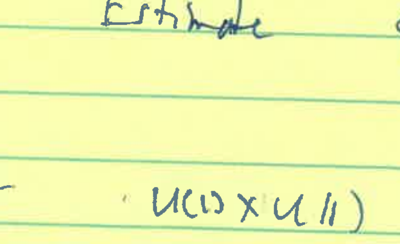
Computing decay width $\Gamma(\pi^0 \to \gamma\gamma) = \frac{m_\pi^3 g^2}{\pi}$ — (A)

Let us try to estimate $g$

EFT by itself will not help here

- need (more) fundamental theory (this is typical of EFT analyses)

Leading contribution



$\pi \sim \bar{q}\gamma_5 q$

Estimate $g \sim \frac{e^2}{16\pi^2 f_\pi} \times ?$ ← cut off in pion theory.

↑ loop factor

$q \to e^{i\alpha\gamma_5\tau_3} q$

But $U(1)\times U(1)$ unbroken by e.m current

$j_{em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d$ (breaks $SU(2)\times SU(2)$)

$= \bar{q}\, Q \gamma_\mu q \quad Q = \frac{B}{2} + \tau_3$ $\quad U_{\tau_3}(1) \times U_B(1)$

But this implies in chiral limit

$m_u = m_d \to 0$ $\quad g \to 0$ i.e $g \propto \frac{m_\pi^2}{M_N^2}$ (or $M_A$) $\cancel{*}$

$\cancel{*}$ (since $\pi^0$ is goldstone of this symmetry)

$[Q_5^0, \mathcal{L}] = 0$

$\Rightarrow \pi^0$ - Goldstone

only derivative couplings unless symm. is broken by $(m_u + m_d)$ $\bar{\psi}\psi$

$10^{11} s^{-1}$

$\Leftrightarrow$ 1 MeV

Thus we expect $g \simeq \frac{e^2}{16\pi^2 f_\pi} \frac{m_\pi^2}{M_N^2}$

So from p-1 - (A) the decay rate would go like $\Gamma(\pi^0 \to \gamma\gamma) \approx \frac{m_\pi^7 \alpha^2}{16\pi^3 f_\pi^2 M_N^4}$

$\simeq 1.9 \times 10^{13} s^{-1}$

While if there is no chiral symmetry (i.e no $\frac{m_\pi^2}{M_N^2}$ fact)

$\Gamma(\pi^0 \to 2\gamma) \simeq \frac{m_\pi^3 \alpha^2}{16\pi^3 f_\pi^2} = 4.4 \times 10^{16} s^{-1}$

Experimentally $\Gamma(\pi^0 \to 2\gamma) = (1.19 \pm 0.08) \times 10^{16} s^{-1}$

clearly there is no chiral symmetry here

What goes wrong?

# Anamalies in Quantum Gauge theories. ①

Symmetry (chiral) transformations leave classical (massless fermion limit) invariant - but may be 'anomalous' in the corresponding Quantum theory. i.e quantum effective action may not be invariant.

Essentially the reason may most transparently be seen in the path integral formulation as arising from non-invariance of (fermionic) measure $[d\psi][d\bar{\psi}]$ under chiral ($\gamma_5$) transformations.

First recall.

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \qquad \bar{\Psi} = (\bar{\psi}_1 \ldots \bar{\psi}_N)$$

$$\int \prod_{i,j=1}^{N} d\psi_i \, d\bar{\psi}_j \; e^{\sum \bar{\psi}_i \underline{A} \psi_k} = \det \underline{A}.$$

$$\left[ \text{Contrast:} \quad \int_{-\infty}^{+\infty} \prod_{i,j}^{N} d\phi_i \, d\phi_j^* \, e^{-\phi^* \underline{A} \phi} = \frac{\pi^N}{\det \underline{A}} \right]$$

$$\psi_i \psi_j = -\psi_j \psi_i \quad \text{etc} \qquad \psi_i^2 = 0 \quad \text{etc.}$$

Also $\int d\psi_i \, \psi_j = \delta_{ij} = \frac{\partial \psi_j}{\partial \psi_i}$

Linear trafⁿ: Under $\psi_i \to J_i^{\ j} \psi_j$ $\quad \int d\psi_i \leftrightarrow \frac{\partial}{\partial \psi_i} \to J^{-1\,i}{}_j \frac{\partial}{\partial \psi_j}$

$$\int \prod_{i=N}^{1} d\psi_i \to \prod \int J^{-1\,1}{}_{j_1} J^{-1\,2}{}_{j_2} \ldots J^{-1\,N}{}_{j_N} \, d\psi^{j_1} \ldots d\psi^{j_N}$$

compare to.

$\int \prod dx_i$

$\to \int \det J \prod dx_i$

$$= \int J^{-1\,1}{}_{j_1} \ldots J^{-1\,N}{}_{j_N} \, \epsilon^{j_1 \ldots j_N} \, d\psi^1 \ldots d\psi^N$$

$$= \int (\det J)^{-1} \prod d\psi_i \,.$$

Alternatively note. i)

$\int \prod_N^1 d\psi_i \, f = d$,
$$f(\underline{\psi}) = a + \psi_i b_i + \tfrac{1}{2} \psi_{i_1} \psi_{i_2} c_{i_1 i_2} + \ldots + \tfrac{1}{n!} \psi_{i_1} \ldots \psi_{i_N} \times \epsilon^{i_1 \ldots i_N} d.$$

Under $\psi_i \to J_i^{\ j} \psi_j$ $\qquad f(\psi) \to f(\psi') = a + \ldots + \frac{d}{n!} (\det J) \times \epsilon^{i_1 \ldots i_N} \psi_{i_1} \ldots \psi_{i_N}$

$$d = \int \prod d\psi \, f(\psi) \to \int \prod_N^1 d\psi' \, f(\psi') = d$$

$$= \int \prod d\psi' \, \frac{d}{n!} (\det J) \, \epsilon^{i_1 \ldots i_N} \psi_{i_1} \ldots \psi_{i_N}$$

$$= \int \cancel{\prod d\psi} \quad \Rightarrow \quad \int \prod d\psi' = (\det J)^{-1} \int \prod d\psi$$

For a <u>non</u>-chiral transformation –

eg. $g(x) = e^{i\alpha(x) T}$ eg: T generator of U(n) (i.e. including a U(1) part)

[ T corresponds to a purely internal transformation, i.e. no $\gamma_5$ matrix for instance ].

Then $\bar{g} \cdot g = 1$ $(1 \equiv \delta_{nm} \delta^4(x-y))$

For a <u>chiral</u> transformation

$g(x) = e^{i\gamma_5 \alpha(x) T} \Rightarrow \bar{g}(x) = \gamma_4 e^{-i\gamma_5 \alpha T} \gamma_4 = g(x)$.

So $\int [d\psi][d\bar{\psi}] \to \int [\mathrm{Det}\, g]^{-2} [d\psi][d\bar{\psi}]$

Euclidean
$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$
$\gamma_\mu^+ = \gamma_\mu$
$\mathrm{Tr} = \mathrm{tr}_{n} \int d^4x\, M^{nm}_{xx}$
$\gamma_5^+ = \gamma_5$

Infinitesimal chiral transformation

$\underline{g}(x,y) = (\underline{1} + i\alpha(x) \underline{\gamma_5} \underline{T}.) \delta^4(x-y)$.

$\ln \det g = \mathrm{Tr} \ln\{(\underline{1} + i\alpha \gamma_5 \underline{T}.) \delta^4(x-y)\} = \underline{\mathrm{Tr}}(i\alpha \gamma_5 T.)$

$-2 \ln(\mathrm{Det}\, g) = i \int d^4x\, \alpha(x) \mathcal{A}(x)$

$\mathcal{A}(x) \equiv -2\, \mathrm{tr}(\gamma_5 T) \delta^4(x-y)\Big|_{y \to x}$ — (4)

$\downarrow$

$0 \times \infty$ ! undefined.

Same result if we had worked in Minkowski

Now consider linear (local) transf$^n$ (non-singular)

$$\psi(x) \to g(x)\psi(x) \qquad \psi(x) = \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_n\end{pmatrix}.$$
$$\bar\psi(x) \to \bar\psi(x)\bar g(x).$$

Work with Euclidean metric $ds_E^2 = \sum_1^4 \delta_{\mu\nu} dx^\mu dx^\nu$.

$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ $\qquad x^4 \leftrightarrow ix^0 \quad ds^2 = \sum_0^3 \eta_{\mu\nu} dx^\mu dx^\nu$.

$= +dx^{0\,2} - d\vec{x}^2$

$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ $\qquad \gamma_\mu^2 = 1$ $\qquad \gamma_\mu$ eigenvalues $\pm 1$ $\qquad = -ds_E^2$.

$\Rightarrow$ $\gamma_\mu$ all Hermitian. Lorentz $SO(3,1) \to SO(4)$

$\gamma_\mu^\dagger = \gamma_\mu$. $\qquad$ 4D rotations.

$\bar\psi$ is an independent Grassmann variable.

But it must transform as $\psi^\dagger\gamma_4$ $(\psi^\dagger\gamma_0)$

So $\bar g(x) = \gamma_4\, g^\dagger(x)\, \gamma_4$. (or $\gamma_0 g^\dagger \gamma_0$)

Viewing as usual $[d\psi(x)] = \lim_{N\to\infty} \prod_r d\psi(x_r)$

transformation of the measure is

$$\int [d\psi][d\bar\psi] \to \int [\mathrm{Det}\, g]^{-1} [\mathrm{Det}\, \bar g]^{-1} [d\psi][d\bar\psi].$$

where $g$ is a matrix both in internal + spin space as well as in space-time.

i.e. $\quad g_{xn,ym} \equiv g(x)_{nm}\, \delta^4(x-y)$.

$\bar g_{xn,ym} \equiv [\gamma_4\, g^\dagger(x)\, \gamma_4]_{nm}\, \delta^4(x-y)$.

$$\int [d\psi][d\bar\psi] \to \int [d\psi][d\bar\psi]\, e^{i\int d^4x\, \alpha(x)\, \mathcal{Q}(x)}$$

The measure transformation $\Rightarrow$.

effectively

Inside functional integral $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\,\mathcal{Q}(x) - \partial^\mu_\alpha \alpha\, j^5_\mu$ (A)

$j^5_\mu = \bar\psi \gamma_\mu \gamma_5 \psi$

Consider the full functional integral for $\psi$'s + gauge fields (scalars play no role here so ignore!)

$$Z = \underset{\text{all gauge fields}}{\int [dA]}\ \underset{\text{all fermions}}{\int [d\psi][d\bar\psi]}\ e^{i\int \mathcal{L}(x)\, d^4x}$$

$\mathcal{L} = \bar\psi \hat K \psi - \frac{1}{2}\mathrm{tr}\, F^2$

$\hat K_{[A]} = i \not{D} = (i\not\partial + \not{A})$

$$= \int [dA]\, e^{-i\frac{1}{2}\int \mathrm{tr}\, F^2}\, e^{iW[A]}$$

$$e^{iW[A]} = \int [d\psi\, d\bar\psi]\, e^{i\int \bar\psi K_{[A]} \psi\, d^4x}$$

$$= \mathrm{Det}\, i\hat K$$

$\delta\, i\bar\psi \not{D} \psi =$

$\bar\psi\, i\gamma^\mu \partial_\mu\, i\gamma_5 \alpha$

$= -\bar\psi \gamma^\mu \gamma_5 \psi\, \partial_\mu \alpha$

$\bar\psi\, i\not\partial\, \psi$

$\to \bar\psi e^{-i\gamma_5 \alpha}\, i\not\partial\, e^{i\gamma_5\alpha} \psi$

$= \partial_\mu \alpha\, j^5_\mu$

$\gamma^E_\mu{}^\dagger = \gamma_\mu$

The above effect of the measure transformation $\Rightarrow$.

$$iW[A] = \ln \mathrm{Det}\, i\hat K = \mathrm{Tr} \ln i\hat K.$$

$$= \frac{1}{2} \mathrm{Tr} \ln i\hat K^2 + \text{const.}$$

$$iW[A] \to iW[A] + i\int \alpha(x)\, \mathcal{Q}(x)\, d^4x - i\int \partial^\mu \alpha(x)\, j^5_\mu(x).$$

$$\delta \ln \mathrm{Det}\, \hat K = i\,\delta W[A] = i\int d^4x\, [\mathcal{Q}(x) - \partial^\mu j^5_\mu(x)]\,\alpha(x)$$

But $W[A]$ must be independent of $\alpha(x)$

Since $e^{iW[A]} = \int d\psi\, d\bar\psi\; e^{i\int \bar\psi \hat{K} \bar\psi}$

Label change ↓ $= \int d\psi'\, d\bar\psi'\; e^{i\int_x \bar\psi' \hat{K} \bar\psi'}$ from measure. ↓

$j^5_\mu = \bar\psi i\gamma_\mu\gamma_5\psi$

$$= \int [d\psi][d\bar\psi]\; e^{i\int_x \{[\bar\psi(\hat{K})\bar\psi'] + \partial^\mu\alpha\, j^5_\mu + \alpha\mathcal{Q}\}}$$

↑ from locality $\int \alpha(x)$.

So. $\frac{\partial W}{\partial\alpha} = 0 \Rightarrow \partial^\mu j^5_\mu(x) = \mathcal{Q}(x)$. — (A)

↑ But page (5) (A)

Corresponding Minkowski Eqn is $\partial^\mu j^5_\mu(x) = -\mathcal{Q}(x)$

Note $\gamma_5^E = i\gamma_1\gamma_2\gamma_3\gamma_4$

$\gamma_5^{E\dagger} = \gamma_5$

$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

$\gamma_5^\dagger = \gamma_5$

But $\gamma_0 \leftrightarrow i\gamma_4^E$!

is not well-defined.

To properly define it need to start from the regularized determinant.

Proper-time regularization.

Propagator $\hat{K}^{-1} \to \int_\epsilon^\infty ds\, \cancel{K} e^{-s\hat{K}^2} = \frac{e^{-\epsilon\hat{K}^2}}{\hat{K}}$. — (A)

$\epsilon = 1/\Lambda^2$

UV. cutoff corresponding to. eg. $\hat{K}$ - Kinetic operator $\hat{K} = (i\partial\!\!\!/ + A\!\!\!/) = i D\!\!\!\!/$.

$$\hat{K}^2 = -D^2 + \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}$$

$-D^2$ has +ve e.v's.

$D$ - Hermitian.

$$\ln \hat{K}^2_\epsilon = 2 \ln \hat{K}_\epsilon = -\int_\epsilon^\infty ds \, \frac{e^{-s\hat{K}^2}}{s} \quad - (A)$$

$$\left( \text{Diff}^t \text{ w.r.t } K \to 2K_\epsilon^{-1} = \int_\epsilon^\infty 2\hat{K} e^{-s\hat{K}^2} = \frac{e^{-\epsilon K^2}}{K} \right).$$

So upto an irrelevant (infinite!) constant (A)

gives a regularization of the ~~p~~det

compatible with that of propagator

$$\text{So } \ln \det K_\epsilon = \frac{1}{2} \text{Tr} \ln K^2_\epsilon = -\frac{1}{2} \text{Tr} \int_\epsilon^\infty \frac{ds}{s} e^{-s\hat{K}^2}$$

$$\text{So } \delta \ln \det K_\epsilon = \frac{-1}{2} \text{Tr} \int_\epsilon 2\delta \underline{K} \hat{K} \, e^{-s\hat{K}^2} ds = \text{Tr} \, \delta \hat{\underline{K}} \hat{K}^{-1} e^{-\epsilon \hat{K}^2}.$$

$$\text{So } \int d^4x \, \alpha(x) \langle \partial_\mu j_5^\mu(x) \rangle = \lim_{\epsilon \to 0} \int d^4x \, \text{tr} \left\{ -2\alpha(x) \gamma_5 \underline{I} e^{-\epsilon \hat{K}^2} \delta(x-y) \right\}_{y \to x}$$

$$\mathcal{A}(x) = \lim_{\epsilon \to 0} \text{tr} \left\{ -2 \cancel{\alpha(x)} \gamma_5 \underline{I} e^{-\epsilon \hat{K}^2} \delta^4(x-y) \right|_{y \to x}$$

$$= \lim_{\epsilon \to 0} \text{tr} \left\{ -2 \gamma_5 \underline{I} e^{-\epsilon \hat{K}^2} \int \frac{e^{ip\cdot(x-y)}}{(2\pi)^4} d^4p \right|_{y \to x}.$$

Note $\quad H(x,y;s) \equiv e^{-s\hat{K}_x^2} \delta^4(x-y)$ is

called the heat kernel - Sol$^n$ of $-\hat{K}_x^2 H(x,y;s) = \frac{\partial H(x,y;s)}{\partial s}$

the ~~Heat~~ eq$^n$ with no source. It has an ~~a~~ expansion in terms of gauge invariant ~~a~~ terms which can be used to work out anomalies -

So ~~a~~ regulated anomaly is (replacing p4 (A))

~~So $\mathcal{A}(x) =$~~

$\epsilon \equiv \frac{1}{\Lambda^2}$

$$\mathcal{A}(x) = -2\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left\{\gamma_5 I\, e^{-\frac{\not{D}_x^2}{\Lambda^2}}\right\} e^{ip\cdot(x-y)}\Big|_{y\to x}$$

product rule ↓ *

$$= -2\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left\{\gamma_5 I\, e^{+(\not{D}_x + i\not{p})^2/\Lambda^2} \cdot 1\right\}$$

$p \to \Lambda p$

$$= -2\Lambda^4 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\, \gamma_5 I\, e^{+\left(\frac{\not{D}_x}{\Lambda} + i\not{p}\right)^2} \quad - (A)$$

In general Heat Kernal expn can be used but here.
use Weinberg argument.

* nb when $\partial_x$ is on extreme right it gives zero in expn of exponential. Recall $\not{D}_x = i\partial_\mu + A$, $\partial A \neq 0$

in expanding a exponent.

$$= \left(\frac{\not{D}_x}{\Lambda} + i\not{p}\right)^2 = p^2 - \frac{ip\cdot D_x}{\Lambda} - \left(\frac{\not{D}_x}{\Lambda}\right)^2$$

Define $\quad f(s) \equiv e^{-s}$ $\quad f(0)=1 \quad f(\infty)=0$

$\lim_{s\to\infty} s f'(s) \to 0$

Then exponential fn. in (A) is $f\left[\left(i\not{p} + \frac{\not{D}_x}{\Lambda}\right)^2\right]$

$$= f\left[p^2 - \frac{ip\cdot D_x}{\Lambda} - \frac{\not{D}_x^2}{\Lambda^2}\right]$$

$$= f(p^2) + f'(p^2)\left(-\frac{ip D_x}{\Lambda} - \frac{\not{D}_x^2}{\Lambda^2}\right) + \frac{f''(p^2)}{2!}\left(-\frac{ip D_x}{\Lambda} - \frac{\not{D}_x^2}{\Lambda^2}\right)^2$$

$$+ \cdots$$

For Dirac trace of $(\gamma_5 \times \cdots)$ to be non-vanishing need at least 4 powers of $\gamma$'s ie need $\not{D}_x^4$
higher ~~order~~ #'s of $\gamma$ have $\Lambda^{-4-n}$ $n>0$ -which vanish.

So. $\mathcal{A}(x) = -\int \frac{d^4p}{(2\pi)^4} f''(p^2)\, \mathrm{tr}\left(\gamma_5 T \not{D}_x^{\,4}\right)$.

– independent of $\Lambda$ !

$$\int d^4p\, f''(p^2) = 2\pi^2 \int_0^\infty p^3 dp\, f''(p^2) = \pi^2 \int_0^\infty p^2 dp^2\, f''(p^2)$$

$f(s) = e^{-s}$

$$= -\pi^2 \int_0^\infty dp^2\, f'(p^2) = \pi^2 \qquad (f(0)=1)$$

$$\not{D}_x^{\,2} = D_{x\mu}\gamma^\mu\gamma^\nu D_{x\nu} = D_{x\mu}\left\{\frac12 \underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2\delta^{\mu\nu}} + \frac12[\gamma^\mu,\gamma^\nu]\right\} D_{x\nu}$$

$[D_\mu, D_\nu] = i F_{\mu\nu}$.

$$= D^2 - \frac{i}{2}\sigma^{\mu\nu}\frac12[D_\mu, D_\nu].$$

$$= D^2 + \frac12 \sigma^{\mu\nu} F_{\mu\nu}$$

$$\mathrm{tr}_{\text{Dirac}}\, \gamma_5 \sigma^{\mu\nu}\sigma^{\lambda\sigma} = +4\, \epsilon_E^{\mu\nu\lambda\sigma}$$

$$\Rightarrow \mathcal{A}(x) = -\frac{\pi^2}{16\pi^4}\frac14 \mathrm{tr}\left(\gamma_5 T \cdot \sigma^{\mu\nu}\sigma^{\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}\right)$$

$$= -\frac{1}{16\pi^2}\mathrm{tr}_{\text{gauge gp}}\left(T\, T_i\, T_j\right) \epsilon^{\mu\nu\lambda\sigma} F^i_{\mu\nu} F^j_{\lambda\sigma}.$$

↑ check sign

$\gamma_5^{(E)} = (\gamma_1\gamma_2\gamma_3\gamma_4)^E$

$$\partial_\mu j_5^\mu(x) = \frac{1}{32\pi^2}\mathrm{tr}\left(T\, \{T_i, T_j\}\right) \epsilon^{\mu\nu\lambda\sigma} F^i_{\mu\nu} F^j_{\lambda\sigma}$$

$\gamma_5^{(M)} = i\gamma_0\gamma_1\gamma_2\gamma_3$

$$T_i T_j = \frac12 c_{ijk} T_k + \frac12 \{T_i, T_j\}$$

$\Leftrightarrow$

Since $c_{ijk} F^i_{\mu\nu} F^j_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = + c_{jik} F^j_{\mu\nu} F^i_{\lambda\sigma} \epsilon^{\lambda\sigma\mu\nu}$

$$= -c_{ijk} F^j_{\mu\nu} F^i_{\lambda\sigma} \epsilon^{\cdots}$$

$$= -c_{ijk} F^i_{\mu\nu} F^j_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = 0.$$

In Minkowski

$$\text{So } \gamma \; \langle \partial^\mu j^5_\mu(x) \rangle = \cancel{\phi} \; \mathcal{A}(x)$$

— (A)

check, sign. factor $i$ from gaussian integration and $i$ from spinor trace

$$= -\frac{1}{32\pi^2} \operatorname{tr} \underbrace{\left[ T \{ T_i, T_j \} \right]}_{\text{Anomaly coefficient}} \epsilon^{\mu\nu\lambda\sigma} F^i_{\mu\nu} F^j_{\lambda\sigma}$$

Compare with W- 22.2.45

Back to $\pi^0 \to \gamma\gamma$ decay.

$U(1)$ $SU(2)_V$
$Q = \frac{B}{2} + \tau_3 = \begin{pmatrix} 1/6 + 1/2 & 0 \\ 0 & 1/6 - 1/2 \end{pmatrix}$

Recall $j^5_\mu = \bar{q} \gamma_\mu \gamma_5 T q$. $T = \tau_3 = \frac{\sigma_3}{2}$ to have quantum #'s of $\pi^0$.

$T_i$, $T_j$ correspond to the charge matrix $Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$ in $q = \begin{pmatrix} u \\ d \end{pmatrix}$

basis. Also if there are $N_c$ colours then the tr will imply a ~~sum~~ trace over unit $N_c \times N_c$ matrix.

$$\text{So } \operatorname{tr} \tau_3 Q^2 = N_c \cdot \left(\frac{2}{3}\right)^2 \cdot \left(+\frac{1}{2}\right) + N_c \left(\frac{1}{3}\right)^2 \cdot \left(-\frac{1}{2}\right)$$

$\frac{4}{9} - \frac{1}{9} = \frac{1}{3}$

$$= \frac{N_c}{2} \cdot \frac{1}{3}.$$

$$\text{So } \mathcal{A}(x) = \frac{N_c e^2}{96\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}.$$

What does this imply for our argument on pp -1 to 0 on the coupling $g$?

$j^{A\lambda}_\mu \leftrightarrow f_\pi \partial_\mu \pi$ - p. 17

The $U_A(1)$ ~~co~~ transformation corresponds to choosing $\theta_L^3 = -\theta_R^3 \equiv \theta$, so.

$$\pi^0 \to \pi^0 + f_\pi \theta . \qquad \pi^0 \equiv \pi^3$$

This implies that pion Lagrangian (i.e EFT replacing low energy QCD) must contain a term that replaces the anomalous transformation of the QCD measure. (see page 5 (A) with $\alpha(x) = \theta$ (constant).

i.e we have a term.

$$\frac{\pi^0 \mathcal{L}(x)}{f_\pi} = \cdot \frac{N_c e^2}{96\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \pi^0(x)$$

$\Rightarrow$ (see p-1) $\qquad g = \dfrac{N_c e^2}{96\pi^2 f_\pi}$.

$\alpha = \dfrac{e^2}{4\pi}$ $\Rightarrow$ $\Gamma(\pi^0 \to 2\gamma) = \dfrac{N_c^2 \alpha m_\pi^3}{576\pi^3 f_\pi^2} = \left(\dfrac{N_c}{3}\right)^2 \times 1.11 \times 10^{16}\, s^{-1}$

Observed $\Gamma(\pi^0 \to 2\gamma) = (1.19 \pm (0.08)) \times 10^{16} s^{-1}$ - agrees iff $N_c = 3$!

First and cleanest evidence for colors!

# Strong CP violation.

Classical QCD invariant under (in particular $\gamma_5 \to \mathbb{1}$) transformation i.e $\psi(x) \to e^{i\gamma_5 \alpha(x)}\psi(x)$.

(i.e. $I \to \mathbb{1}$ case see p 9 $\alpha = \ldots$)

From previous discussion we saw that the measure is not invariant

instead.

Back to canonical norm $A \to gA$

$$\int[d\psi][d\bar\psi] \to \int[d\psi][d\bar\psi]\, e^{i\alpha(x)\int \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}}$$

We can do independent (chiral) rotations on (say) $u$ and $d$ quarks $u(x) \to e^{i\gamma_5\theta_u}u(x)$

$d \to e^{i\gamma_5\theta_d}d(x) \Rightarrow [d\psi][d\bar\psi] \to e^{i(\theta_u+\theta_d)\int\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}}$.

This would $\Rightarrow$ ~~CP~~ term $\Delta h = \theta_{QCD}\int\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$.

Can be removed $\theta_{QCD} \to \underbrace{\theta_{QCD} + \theta_u + \theta_d}_{=0}$

i.e. by choosing $\theta_u + \theta_d = -\theta_{QCD}$.

However the mass terms break the chiral symmetry!

$$\mathcal{L}_m = -m\,\bar{q}\left(\frac{1+\gamma_5}{2}\right)q - m_u\,\bar{u}\left(\frac{1+\gamma_5}{2}\right)u - m_d\,\bar{d}\left(\frac{1+\gamma_5}{2}\right)d + (H.c.) \quad -(A)$$

In general these parameter $m_u, m_d$ could be complex in which case they would break P and CP. Consider a field redefinition.

$$u \to u' = e^{i\gamma_5\theta_u}u \qquad d \to d' = e^{i\gamma_5\theta_d} \quad (B)$$

we have

$$m_u \to e^{i\theta_u}m_u \qquad m_d \to e^{i\theta_d}m_d \quad -(C)$$

$q = \begin{pmatrix} u \\ d \end{pmatrix}$

$$\Rightarrow \int [du][d\bar{u}][dd][d\bar{d}]\ \exp i\int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{q}\not{D}q + (\underbrace{\theta}_{QCD} + \theta_u + \theta_d)\frac{1}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

Clearly one can use (B) to remove phases from $m_u, m_d$ according to (C) but then cannot choose them to cancel the $\theta$-term - unless of course either $m_u$ (or $m_d$) $= 0$ in which case $\theta_u$ (or $\theta_d$) is arbitrary can can be set equal to $-\theta$ thereby eliminating $\theta_{QCD}$

However $m_u, m_d \neq 0$ ($m_u \simeq 2\,\mathrm{MeV}$ $m_d \simeq 5\,\mathrm{MeV}$)

So need alternative way of eliminating $\theta_{QCD}$.

$\theta_{QCD} \Rightarrow \not{CP}$ ($\Rightarrow \not{T}$) $\Rightarrow$ edm for neutron.

Note: if $m_u = |m_u| e^{-i\theta_u}$ $m_d = |m_d| e^{-i\theta_d}$

Then physical ~~CP~~ angle is $\Theta^{phys}_{QCD} = \theta_{QCD} + \theta_u + \theta_d$

$= \theta_{QCD} + i \ln \det \mathcal{M}$.

$= \theta_{QCD} \cancel{+} \arg[\det \mathcal{M}]$.

This is a (Chiral) invariant and is of physical Significance.

However $O(1)$ values of this angle would conflict with expt — in particular EDM of neutron.

In fact (using EFT calculation) it can be shown that ~~(after rotating away $\theta_{QCD}$ get $\theta^{phys} = \arg(\det \mathcal{M})$)~~

$$d_N = (5.12 \times 10^{-16} e.\mathrm{cm})\bar{\theta}$$

Expt $|d_N| < 2.9 \times 10^{-26} e \Rightarrow \bar{\theta} < 10^{-10}$.

Back to ~~general~~ Anomaly discussion.

If we couple in the weak interaction there are additional "$\theta$" terms i.e CP violating terms. So in all we have

With canonically normalized gauge fields

$$\Delta\mathcal{L}_{CP} = \theta_{QCD}\frac{g_s^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F^a_{\mu\nu}F^a_{\lambda\sigma} + \theta_2\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}\underbrace{W^a_{\mu\nu}W^a_{\lambda\sigma}}_{SU(2)} + \theta_1\frac{g'^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}\underbrace{B_{\mu\nu}B_{\lambda\sigma}}_{U(1)}$$

As in pure Q.C.D. case we ~~I~~ can remove the $\theta_2$, $\theta_1$ term by chiral rotation of the left handed quarks $\Rightarrow$ phase to Yukawa couplings. However $\theta_2$ ~~this~~ can be removed by a chiral rotation on the right handed quarks (which are uncharged under $SU(2)$ $\Rightarrow$ no effect on $\theta_2$). ~~Similarly we~~ Also we can do a $U(1)_Y$ rotation on $\nu$'s to remove $\theta_1$-term since either $\nu_R$ ∄ or if we have one it is uncharged under

Thus only $\cancel{CP}$ effect is from

$$\theta_{QCD}^{phys} = \theta_{QCD} - \arg[\det m] ,$$

However this cannot be seen in perturbation theory. . . .

QCD anomaly eqn for $J_\mu^5$ was

$$\partial^\mu J_\mu^5 = \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}^a F_{\lambda\sigma}^a$$

$$= \frac{g_s^2}{8\pi^2} \partial^\mu K_\mu .$$

Where $K_\mu = \epsilon_{\mu\nu\lambda\sigma}\left(A_a^\nu F_a^{\lambda\sigma} - \frac{g}{3} f^{abc} A_a^\nu A^\lambda A^\sigma\right).$

$su_c(3)$ index

· It is important to note the $K_\mu$ is Not gauge invariant — It is called the Chern-Simons current. Its integral $\int d\sigma^\mu K_\mu$ over a 3-surface is (locally) gauge invariant - called CS action.

Aside: In corresponding case involving photons.

Non-gauge invariance $\Rightarrow$ $\partial^\mu(\underbrace{J^5_\mu - K_\mu}_{\tilde{J}^5_\mu}) = 0$

cannot be used to ~~dis~~ discuss $\pi^0$-decay.

Since argument depended on gauge invariance.

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- Vertex ~~is~~ of the form $\theta . F\tilde{F}$

$= \theta . \partial_\mu K^\mu$

will not contribute in perturbation theory — $\int_V \partial_\mu K^\mu d^4x = \int_{\partial V} d\sigma^\mu K_\mu$.

Boundary term

At "infinity" field strengths $\to 0$

ie $A_\mu \to g^{-1}\partial_\mu g$ . pure gauge

Strictly speaking this is a Euclidean argument $V \to \mathbb{R}^4$, $\partial V \to S^3_\infty$

So $\int_V \partial_\mu K^\mu d^4x \propto \mathrm{Tr}\int_{\partial V} d\sigma_\mu \epsilon^{\mu\nu\lambda\sigma} g^{-1}\partial_\nu g g^{-1}\partial_\lambda g . g^{-1}\partial_\sigma g$.

Consider $SU(2) \subset SU(3)_c$

$\Pi_3(SU_2) = \Pi_3(S_3) = \mathbb{Z}$. | $\propto n \in \mathbb{Z}$

winding # transformation.

$\therefore$ $n \neq 0$ only for non-contractible gauge

Leads to the notion of instantons.

- non-trivial ~~gau~~ finite action classical self dual solutions. $F = \tilde{F}$.

classified by winding #.

$$\int_{V_4} F_{\mu\nu} F^{\mu\nu} = \int_{V_4} F_{\mu\nu} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}.$$

$$\sim \pm \int_{V_4} \partial_\mu K_\mu = \int_{S^3_\infty} d\sigma^\mu K_\mu.$$

We will not discuss these any further but their existence is evidence that $F\tilde{F}$ terms have physical significance.

$\Downarrow$

(non-perturbative)

Discuss effects in pion EFT at low energies: with $U = \exp\{i\pi^\alpha \tau^\alpha / F_\pi\}$

$$\mathcal{L}_{EFT} = \frac{f_\pi^2}{4} \mathrm{Tr}\{D_\mu U D^\mu U^\dagger\} + \frac{v^3}{2} \mathrm{Tr}(MU + M^\dagger U^\dagger) \quad \text{— (A)}$$

where $v^3 = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$, $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

— or generally the quark mass matrix of QCD.

As we saw earlier we can (in QCD) ~~cat~~ remove $\bar{\theta}^{phys}_{QCD}$ from $F\tilde{F}$ term and put it entirely in $\underline{M} \longrightarrow \underline{M} e^{i\bar{\theta}^{phys}_{QCD}}$ — (B)

$\Rightarrow$ contribution to Hamiltonian is $\bar{\theta}^{phys}$ dependent

$$\Delta E(\theta) = -v^3(m_u + m_d)\cos\bar{\theta}^{phys} = -f_\pi^2 m_\pi^2 \cos\bar{\theta}^{phys}. \quad \text{— (C)}$$

Different "$\theta$" vacua have different energy.

If $\bar{\theta}^{phys}$ were a dynamical (field) variable — then could justify $\bar{\theta}^{phys} \to 0$ a lowest energy configuration (really $\bar{\theta}^{phys}_{expt} < 10^{-11}$!)

The dynamical Solution originally introduced via
"Peccei-Quinn" Symmetry $\Rightarrow$ Axions
$\Downarrow$
(Additional anomalous $U(1)_{PQ}$ symmetry - ~~...~~
& Spontaneously broken $\Rightarrow$.
(pseudo) Goldstone bose $\rightarrow$ Axion $a$).
explicitly broken by anomalies. Weinberg-Wilczek

In the EFT we don't need PQ*

Just postulate a light pseudo-scalar "axion"

$a(x)$ coupling to $\pi$'s ~~via~~ via

$$\underline{M} \rightarrow \underline{M}\, e^{ia(x)/f_a}$$

$\uparrow$ dimensionfull parameter needs to be determined by expt.

add and kinetic term

$\Delta\mathcal{L} \sim + \frac{1}{2}\partial_\mu a\, \partial^\mu a$

$\uparrow$ canonically normalized.

Then p.19 (C) becomes ~~an~~ a potential

for the "axion" i.e.

$$V(a) = \Delta E(\theta + a) = -S_\pi^2 m_\pi^2 \cos\left(\bar\theta + \frac{a}{f_a}\right) \quad \text{(A)}$$

* Actually in full QCD too one can just had $+\frac{1}{2}\partial a \partial a + \frac{1}{32\pi^2}\frac{a(x)}{f_a} F^a_{\mu\nu} \epsilon^{\mu\nu\lambda\sigma} F^a_{\lambda\sigma}$

Now the dynamics of a Solves the fine-tuning (ie why is naturally $O(1)$ $\bar{\theta}$ actually $< 10^{-11}$) since

the axion field - as the universe cools - will settle at the minimum of its potential $V'(\theta) = 0$ at $\bar{\theta} + a_0 = 0$

ie $a_0 = -\theta^{phy}$. ie. effective

$\cancel{CP}$ parameter is $\theta_{eff} = \bar{\theta} + \frac{a}{f}_0 = 0$. (A)

(which is the total phase of the mass matrix)

$f_a$ - Axion decay constant -- analog of $f_\pi$

The quantized excitations around $a = a_0$ are (pseudo-scalar) axion particles.

- Could be a dark matter candidate.

From p20 eq$^n$ (A) $m_a^2 = \frac{f_\pi^2 m_\pi^2}{f_a^2}$ - (B)

* Two bounds on size of $f_a$.

Astrophysical - Red giant stars - stability

$\langle a \rangle = a_0 = f_a \bar{\theta}$

$\Rightarrow f_a > 10^{10}$ GeV. - lower bnd

Cosmological : upper bound $f_a < 10^{12}$ GeV

( Axion density cannot exceed "critical density"
- otherwise universe "overclosed"
- will recollapse ~~on~~ a very short time scale. )

* How short?

This implies axion is very weakly coupled

~~to~~ and extremely light

$$10^{-4} eV < m_a < 10^{-2} eV$$

* An alternative "solution" to the strong CP problem is to simply set $\bar{\theta} = 0$. A non-zero value will arise from ~~the~~ CP violation in weak interactions but at ~~a~~ some high order $O(g_{wk}^4)$ and ~~so~~ is small enough.

This is a "technical" fine-tuning only!

Note that axion has a shift symmetry broken only by anomalies. $a \to a + \text{const}$.

This means that the additional (axionic) terms in QCD are. (keeping only dim $\leq 4$ terms)

$$\Delta \mathcal{L} = -\frac{1}{2}\partial_\mu a \partial^\mu a - \frac{i\lambda_u}{f_a}\partial_\mu a\, \bar{u}\gamma_5\gamma^\mu u$$

$$- \frac{i\lambda_d}{f_a}\partial_\mu a\, \bar{d}\gamma_5\gamma^\mu d + \frac{1}{32\pi^2}\left(\bar{\theta} + \frac{a}{f_a}\right) \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

— derivative couplings to quarks — used to compute axionic processes.

A more accurate calculation takes into account mixing between $\pi^0$ and $a' = a - a_0$ gives,

$$m_a^2 = \frac{f_\pi^2}{f_a^2} \frac{m_u m_d}{(m_u+m_d)^2} m_\pi^2$$

(See Weinberg Vol II eq 23.6.26.

# Internal Anomalies of Std Model.

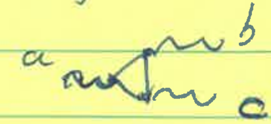
Consistency of QFT

$\Rightarrow$ Gauge invariance needs to be preserved $\Rightarrow$ no anomalies associated with currents coupling to gauge fields!

From p(10) eq$^n$ (A) Consistency requires that anomaly coefficients (defined in first $= d_{abc}$)

For $A_{abc} \equiv \mathrm{tr}\left(T_a^R \{T_b^R, T_c^R\}\right)$ should be zero for all internal (gauged) symmetries.

Consider 1st anomaly.




• $U_Y(1)^3$ all 3 generators correspond to $U_Y(1)$.

Coupling matrix is $Y_L \frac{1}{2}(1-\gamma_5) + Y_R \frac{1}{2}(1+\gamma_5)$ *

$\uparrow$ Left handed $\qquad \uparrow$ Right handed

weak hypercharge.

* note that non-chiral gauge theory will be always non-anomalous.

Effectively (ie coeff of $\gamma_5$) T's

come with opposite signs for left and right handed couplings.

$$\Rightarrow \partial_\mu j^\mu_Y = \frac{g'^2}{32\pi^2}\left(\sum_{\text{left}} Y_L^3 - \sum_{\text{right}} Y_R^3\right) \epsilon^{\mu\nu\lambda\sigma} B_{\mu\nu} B_{\lambda\sigma}.$$

$$A_{YYY} = 2Y_L^3 - Y_e^3 - Y_\nu^3 + 3\left(2Y_Q^3 - Y_u^3 - Y_d^3\right)$$

lepton left doublet ($Y_L$); colors (3); left quark doublet ($Y_Q$)

$Y_L = -\frac{1}{2}$, $Y_e = -1$ $Y_\nu = 0$ $Y_Q = \frac{1}{6}$. $Y_u = \frac{2}{3}$, $Y_d = -\frac{1}{3}$.

Check that $A_{YYY} = 0$.

Similarly all other potential anomalies vanish in the Std model including,

$\therefore$ ~~including~~ potential gravitational* anomalies.


See eg. Table 30.1 of Schwartz.

* This is a non-trivial check on any gauge field theory!