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Chiral symmetry breaking - pions

QCD with light quarks.

Recall $m_u \simeq 2$ MeV $\quad m_d \simeq 5$ MeV $\ll \Lambda_{QCD}$

$\ll T_{string}$

Limit of $m_u, m_d \to 0$

Theory has $SU(2) \times SU(2)$ symmetry.

$\Lambda_{QCD} \simeq 200$ MeV

$T_{string} \simeq 2\pi \Lambda_{QCD} \simeq 1200$ MeV

Generators. $\vec{Q}_\pm = \int d^3x \, j_0^\pm = \int d^3x \, \bar{\Psi} \gamma_\mu \frac{\vec{\tau}}{2} P_\pm \Psi$ (flavor, $i = 1,2,3$)

$m_p \simeq 940$ MeV

i.e. $\vec{j}_\mu = \bar{\psi}_a \gamma_\mu \frac{1}{2}(1 \pm \gamma_5) \frac{\vec{\tau}_{ab}}{2} \psi_b \qquad [\Psi_a] = \begin{pmatrix} u \\ d \end{pmatrix}$

so light quarks $m. \ll T_{str} < m_p$, m_p = proton mass.

color neutral. color indices suppressed

If we also take $m_s \to 0$

$(m_s \simeq 170$ MeV $< T_{string})$

Generators $Q_\pm^\alpha = \int d^3x \, \bar{\Psi}_a \gamma_\mu \frac{1}{2}(1 \pm \gamma_5) \frac{\lambda^\alpha_{ab}}{2} \Psi_b$

flavor SU(3) indices. $a, b = 1, 2, 3$.

$\psi_b = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad SU(3)_{flavor}$ triplet.

<u>NB</u> These are global (approximate) symmetries of QCD - exact in limit $m_u, m_d, (m_s) \to 0$.

Check: $[Q_\pm^\alpha, Q_\pm^\beta] = i f^{\alpha\beta\gamma} Q_\pm^\gamma \qquad [Q_\pm^\alpha, Q_\mp^\beta] = 0$.

(or ϵ if SU(2))

(within std model this ⇒ setting Yukawa ~~Higgs~~ couplings to zero).

In this limit they are exact symmetries of the Lagrangian/Hamiltonian.

However since ∄ parity doubling only the vector symmetries preserve the vacuum. (focus on $SU(2) \times SU(2)$ case)?

$U_V = e^{i\theta^i_V Q^i_V}$

$U_V|0\rangle = |0\rangle$

ie $Q^\alpha_V = Q^{\alpha +} + Q^{\alpha -} = \int d^3x\, \bar{q}_a \gamma_\mu \frac{\tau_{ab}^\alpha}{2} q_b$.

$Q^i_V|0\rangle = 0$

The axial (pseudo-vector) charges.

$$Q_A^\alpha = Q^{\alpha +} - Q^{\alpha -} = \int d^3x\, \bar{q}_a \gamma_\mu \gamma_5 \frac{\tau^\alpha}{2} q_b$$

although commuting with Hamiltonian.

$$[\hat{Q}^\alpha_V, \hat{H}] = [\hat{Q}^\alpha_A, \hat{H}] = 0.$$

$U_A = e^{i\theta^i_A Q^i_A}$ does not leave vacuum invariant.

i.e $U_A|0\rangle \neq |0\rangle$ or $Q_A^\alpha|0\rangle \neq 0$.

This is the phenomenon of spontaneous symmetry breaking (SSB).

We saw this already when discussing the Higgs theory - except there SSB caused by Higgs potential $V(H) = \lambda (H^\dagger H - \mu^2)^2$

$\Rightarrow \langle H_0 \rangle = \mu \neq 0$ violating $SU_W(2) \times U_W(1)$ symmetry of vacuum.

$\Rightarrow$ 3 massless particles (phase of Higgs $\rightarrow$ longitudinal modes of $W^\pm, W^3$ gauge fields. Last semester we discussed the free complex scalar theory without coupling to a gauge field.

$$V(\varphi) = \lambda (\varphi^* \varphi - \mu^2)^2$$

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - V(\varphi)$$

Symm. $\varphi \rightarrow e^{i\theta} \varphi$. $\varphi_0 = \langle \varphi \rangle_0$
leave $\mathcal{L}$ Hamiltonian invariant (not invariant).

Phase 1 φ ~~do~~ has no mass term.

$$\varphi = \tfrac{1}{\sqrt{2}}\rho\, e^{i\frac{\pi}{\sqrt{2}\rho_0}} \qquad \partial\varphi\partial\varphi^* = \tfrac{1}{2}\partial\rho\partial\rho + \frac{\rho^2}{\rho_0^2}\partial\pi\partial\pi.$$

$$V = \lambda(\rho^2 - \mu^2)^2 \qquad \langle h\rangle = \rho_0 \equiv \mu.$$

physical spectrum write. $\rho = \rho_0 + \chi$.

Theory is.

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi + \tfrac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2\rho_0^2}(2\rho_0\chi + \chi^2)\partial_\mu\pi\partial^\mu\pi$$
$$+ \lambda\,(\chi^2 + 2\rho_0\chi)^2.$$

π has only derivative interactions!

The original global $U(1)$ symmetry $\varphi \to e^{i\theta}\varphi$

became the symmetry under ~~$\pi \to \sqrt{2}\rho_0\theta$~~.

$$\pi \longrightarrow \pi + \sqrt{2}\,\underbrace{\rho_0\theta}_{\text{constant!}}$$

i.e a shift symmetry.

This is a general result

Nambu
Nambu
+ Jona-Lasinio

Goldstone.
Goldstone
Salam
+ Weinberg

Goldstone - thm.

Spontaneous breaking of a continuous broken symmetry implies the existence of a massless particle.

Suppose Q_A^α is a charge which is one of the generators of the spontaneously broken symmetry

ie In $m_u, m_d \to 0$ limit of QCD this is $Q_A^\alpha = \int d^3x\, \bar{\psi}\gamma_0\gamma_5 \frac{\tau^\alpha}{2}\psi$.

If it is a symmetry generator (i.e. giving an invariance of the Lagrangian / Hamiltonian)

$$i\dot{Q}_A^\alpha = [H, Q_A^\alpha] = 0. \quad \text{— (A)}$$

Suppose $|0\rangle$ is ground state / vacuum with energy E_{vac} (can choose $E_{vac} = 0$ if you like).

i.e $H|0\rangle = E_{vac}|0\rangle$, —

This argument somewhat heuristic since norm of this state is not defined, - not normalizable

But if symmetry is spontaneously broken.

$\Rightarrow$ $Q_A^\alpha |0\rangle \equiv |\pi^\alpha(0)\rangle \neq 0$.

i.e. gives another state $\neq |0\rangle$.

but is degenerate with it since.

$$H|\pi^i(0)\rangle = H\, Q_A^\alpha|0\rangle = Q_A^\alpha H|0\rangle = E_{vac} Q_A^\alpha|0\rangle$$

from p5 (A)

$$= E_{vac}|\pi^i(0)\rangle.$$

— (B).

— i.e vacuum is degenerate.

In fact $|\pi^i(0)\rangle$ is the zero - 3 - momentum-

limit of $|\pi^\alpha(\vec{q})\rangle = \frac{1}{F}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} J^\alpha_{A,0}(\vec{x},0)|0\rangle$

$\uparrow$ normalization constant

$\vec{q}\to 0$ $|\pi^\alpha(0)\rangle = \frac{1}{F} Q_A^\alpha |0\rangle$

state having same energy as

vacuum from eq^n (B) i.e is massless.

Note $\exists$ a Goldstone boson associated with

each "broken" generator.

In QCD there is no scalar potential to generate spontaneous symm breaking. Instead we have an effective scalar $\bar\Psi \Psi$. (dim 3 ! rather than 1)

$= \bar\Psi_L \Psi_R + \bar\Psi_R \Psi_L$.

$\psi_L \to e^{i\vec\theta_L\cdot\vec\tau}\psi_L$
$\psi_R \to e^{i\vec\theta_R\cdot\vec\tau}\psi_R$.

Clearly is not a singlet under chiral i.e γ_5- transformations ($\theta_L \neq \theta_R$).

If it develops a vacuum expectation value (VEV)

Preserving Lorentz invariance and parity with a scalar field.

→ i.e $\langle \bar\Psi \Psi \rangle_0 \equiv \langle 0 | \bar\Psi \Psi | 0 \rangle \neq 0$. — (A)

Then clearly chiral symmetry is spontaneously broken.

(A) is necessarily a non-perturbative phenomenon — as is with confinement — both supported by lattice studies.

Let us assume that (based on various non-perturbative - eg Lattice studies) that (in light (u, d) quark QCD)

$$\langle \bar{\psi}\psi \rangle_0 = \langle \bar{u}u + \bar{d}d \rangle_0 \neq 0.$$

$$\Rightarrow \quad SU(2)_+ \times SU(2)_- \rightarrow SU(2)_V$$

$$\frac{1+\gamma_5}{2}\tau^\alpha \qquad \frac{1-\gamma_5}{2}\tau^\alpha \rightarrow 1\,\tau^\alpha \qquad \tau^\alpha = \frac{\sigma^\alpha}{2}$$

~~8 gen~~ 3 generators $Q_A^\alpha = \int d^3x\, \psi^\dagger \gamma_5 \tau^\alpha \psi$

do <u>not</u> annihilate vacuum i.e $Q_A^i |0\rangle \neq 0$

while $Q_V^i |0\rangle = 0$.

$\int \angle G$ ... Goldstone thm $\Rightarrow$ 3 - massless

$\frac{123}{76}$ (i.e derivatively coupled) (pseudo) scalar fields. $\pi^i \rightarrow \pi^\pm = \frac{\pi^1 \mp i\pi^2}{\sqrt{2}}$, $\pi^0 = \pi^3$

Coupling to vacuum thro' j_A^i i.e $\langle \pi^\beta_{(p)} | j_\mu^{A\alpha}(0) |0\rangle = i p_\mu F_\pi \delta^{\alpha\beta}$ $F_\pi \neq 0$ since $Q_A^\alpha |0\rangle \neq 0$.

x. I've used Lorentz invnt norm of states here

. Note there is no state such that $\langle ?| j^V_\mu |0\rangle = f_? \, i p_\mu$

since $Q^V|0\rangle = 0$.

Note: Translations. $e^{i\hat{P}_\mu a^\mu} \varphi(x) e^{-i\hat{P} a} = \varphi(x+a)$.

$$P_\mu |\pi(p)\rangle = p_\mu |\pi(p)\rangle.$$

$$\langle \pi(p)| j^{Ai}_\mu (x) |0\rangle = i p_\mu f_\pi e^{ipx}$$

If we set $x^0 = 0$ $\mu = 0$ and integrate. $\int \frac{d^3x}{(2\pi)^3}$

$$\langle \pi(p)| Q^{Ai} |0\rangle = i p_0 f_\pi \int \frac{d^3x}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} = i p_0 f_\pi \delta^3(\vec{p}).$$

$SU(2) \times SU(2)$ Algebra of charges. $[Q^i_\pm, Q^j_\pm] = i\epsilon^{ijk} Q^k_\pm$

$$[Q^i_\pm, Q^j_\mp] = 0.$$

$$Q_V = Q_+ + Q_- \qquad Q_A = Q_+ - Q_-.$$

$$[Q^i_V, Q^j_V] = i\epsilon^{ijk} Q^k_V.$$

$$[Q^i_A, Q^j_A] = i\epsilon^{ijk} Q^k_A.$$

$$[Q^i_A, Q^j_A] = i\epsilon^{ijk} Q^k_A.$$

Charges & (currents) conserved in $m_u = m_d \to 0$ limit

Note $Q^{\alpha}_{\pm} = \int d^3x\, \bar{\psi}\gamma_0 \frac{1}{2}(1 \pm \gamma_5)\tau^{\alpha}\psi$

~~Define~~ $= \int d^3x\; j^{\pm}_0(\vec{x}, t).$

Recall $\partial j^{\pm} = 0 \Rightarrow \dot{Q}^{i\pm} = 0.$

$j^{\pm\alpha}_{\mu} = \bar{\psi}\gamma_{\mu}\frac{1}{2}(1 \pm \gamma_5)\tau^{\alpha}\psi.$ $\partial^{\mu} j^{\pm i}_{\mu} = 0$ when $m_u, m_d = 0$

$j^{iV}_{\mu} = \bar{\psi}\gamma_{\mu}\tau^i\psi$, $J^{iA}_{\mu} = \bar{\psi}\gamma_{\mu}\gamma_5\tau^i\psi.$

$\psi \Leftrightarrow q.$

From equal-time commutators $\{\psi_{(x)}, \psi^{\dagger}_{(0)}\}_{x_0=0} = \delta^3(x)$ etc

Obtain corresponding current algebra.

So $[j^{V\alpha}_0(\vec{x}, t), j^{V\beta}_{\mu}(\vec{y}, t)] = i\epsilon^{\alpha\beta\gamma} j^{\gamma}_{\mu}(\vec{y}, t)\,\delta^3(\vec{x} - \vec{y})$

$[j^{A\alpha}_0(\vec{x}, t), j^{A\beta}_{\mu}(\vec{y}, t)] = i\epsilon^{\alpha\beta\gamma} j^{V\gamma}_{\mu}(\vec{y}, t)\,\delta^3(\vec{x} - \vec{y}).$

$[j^{V\alpha}_0(\vec{x}, t), j^{A}_{\mu}(\vec{y}, t)] = i\epsilon^{\alpha\beta\gamma} j^{A}_{\mu}(\vec{y}, t)\,\delta^3(\vec{x} - \vec{y}).$

Recall full symmetry of 2-flavor QCD is actually (in $m_u = m_d = 0$ limit).

$SU_+(2) \times SU_-(2) \times U_V(1) \times U_A(1).$

$U_V(1)$: Baryon # $U_A(1)$: anomalous.

Focus on $SU(2) \times SU(2).$

$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0$

$\downarrow$ $SU(2)$ Isospin

$\uparrow$ Implies mb 8-Goldstone bosons not 9!

Mass term $\mathcal{L} \ni -m_u \bar{u}u - m_d \bar{d}d$.

$$= -(m_u - m_d)(\bar{u}u + \bar{d}d) - (m_u + m_d)(\bar{u}u - \bar{d}d)$$

cause <u>explicit</u> breaking of $SU(2)_A \times SU(2)$.

$$\partial^\mu j_\mu^{i\,V}(x) = (m_u - m_d)(\bar{\psi} \tau^i \psi). \quad \to m_u = m_d \to 0$$

$$\partial^\mu j_\mu^{i\,A}(x) = (m_u + m_d)(\bar{\psi} \tau^i \gamma_5 \cancel{\psi} + \cancel{\bar{\psi} \tau^i \gamma_5 \psi})$$

$m_u = m_d \quad \to \quad SU(2)_V$ sym.

$SU(2) \times SU(2)$ in $m_u = m_d \to 0$

EFT of $SU(2)\times SU(2)$

symmetric theory of pions

- low energy replacement ~~of~~ of QCD.
- Account for confinement and χSB.
- Construct from known "facts" abt & low energy QCD. * $SU(2)\times SU(2) \to SU(2)?$

i.e. χSB * "Goldstone" pions.

Define $\Sigma = [\Sigma_{ij}]$ $i,j = 1,2$. is 2×2 matrix of fields.

Transf^m under $SU(2)\times SU(2)$.

$$\Sigma \to g_L \Sigma g_R^\dagger, \quad \Sigma^\dagger \to g_R \Sigma g_L^\dagger \quad \text{(A)}$$

$SU(2)\times SU(2)$

EFT which is $SU(2)\times SU(2)$ symmetric and keeping only dim 4 or less (ie renormalizable) operators is -

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\{\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\} + m^2 \mathrm{tr}(\Sigma\Sigma^\dagger) - \frac{\lambda}{4}\mathrm{tr}\{\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger\}$$

As in Higgs model we chose a $-ve$ mass^2 term to generate SSB.

Polar split.
$$\Sigma = \frac{S}{\sqrt{2}} \mathbb{1}\, e^{\frac{2i}{f_\pi} \pi^\alpha \tau^\alpha}$$
unit matrix

$\vec{\tau} = \frac{\vec{\sigma}}{2}$

$\mathrm{tr}\, \tau^\alpha \tau^\beta = \frac{1}{2}\delta^{\alpha\beta}$.

$$\partial_\mu \Sigma = \left(\frac{1}{\sqrt{2}} \partial_\mu S \mathbb{1} + \frac{S}{\sqrt{2}} \frac{2i}{f_\pi} \partial_\mu \pi^\alpha \tau^\alpha\right) e^{\frac{2i}{f_\pi} \vec{\pi}\cdot\vec{\tau}},$$

$$\mathrm{tr}\left(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger\right) = \frac{1}{2}(\partial_\mu S)^2 \cdot 2 + \frac{S^2}{\cancel{2}} \frac{\cancel{4}}{f_\pi^2} (\partial_\mu \pi^\alpha)^2 \cdot \cancel{\frac{1}{2}}.$$

$$S = v + \sigma(x) \qquad \partial_\mu S = \partial_\mu \sigma.$$

$S^2 = (f_\pi + \sigma)^2 = f_\pi^2 + 2 f_\pi \sigma + \sigma^2$.

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} \partial_\mu \pi^\alpha \partial^\mu \pi^\alpha + \frac{\sigma}{f_\pi} \partial_\mu \pi^\alpha \partial^\mu \pi^\alpha$$
$$+ \frac{\sigma^2}{2 f_\pi^2} \partial_\mu \pi^\alpha \partial^\mu \pi^\alpha + V(\sigma)$$

potential independent of π

Note π has only derivative couplings.

$\frac{F}{2} \to \frac{f}{\sqrt{2}}$

Global symmetry. $\Sigma \to g_L \Sigma g_R^\dagger$

becomes with $g_L = \exp(i\theta_L^\alpha \tau^\alpha)$, $g_R = \exp(i\theta_R^\alpha \tau^\alpha)$

and infinitesimal $\theta_{L,R}^\alpha$

$\sigma \to \sigma$.

$$\pi^\alpha \to \pi^\alpha + \frac{f_\pi}{2}(\theta_L^\alpha - \theta_R^\alpha) - \frac{1}{2}\epsilon^{\alpha\beta\gamma}(\theta_L^\beta + \theta_R^\beta)\pi^\gamma$$

$$+ O(\theta^2) \quad - (A)$$

$\Rightarrow$ no mass term

$\theta_L^\alpha = \theta_R^\alpha$ is unbroken $SU_V(2)$ symmetry.

leaves $\langle \Sigma_{ij} \rangle$ invariant. $\langle \Sigma_{ij} \rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

~~The~~ σ – heavy field, invariant under symmetry. – decouple!

$m \to \infty$ with $f_\pi = v = \frac{2m}{\sqrt{\lambda}}$ – fixed.

E of M for σ.

$$(\Box - m^2)\sigma = \frac{\sqrt{2}}{f}(\partial\pi)^2 + O(\lambda\sigma, \text{etc}).$$

finite

$\to 0$ in this limit set $\sigma = \sigma_0 \to 0$.

constant

σ

$\to 0$

Non-linear ~~σ~~ sigma model.

- $SU(2)\times SU(2)$ realized non-linearly.

Introduce $SU(2)$ matrix

$$U(x) = \exp\frac{2i}{f_\pi}(\vec{\pi}, \vec{\tau}) = \exp\frac{2i}{f_\pi}\begin{pmatrix}\pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0\end{pmatrix}$$

$$\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2) \qquad \pi^0 = \pi^3.$$

π's transform under $U \to g_L U g_R^\dagger$ — (A)

according to p 14 - A.

Most general invariant Lagrangian

(now we don't restrict ourselves to dimension 4 operators).

$$\mathcal{L} = \frac{f_\pi^2}{4}\mathrm{tr}\{(D_\mu U)(D^\mu U)^\dagger\} + L_1 \mathrm{tr}\{(D_\mu U)(D^\mu U)^\dagger\}^2$$
$$+ L_2\, \mathrm{tr}\{(D_\mu U)(D_\nu U)^\dagger\}\,\mathrm{tr}\{(D^\nu U)^\dagger (D^\mu U)\}$$
$$+ L_3\, \mathrm{tr}\{(D_\mu U)(D^\mu U)^\dagger (D_\nu U)(D^\nu U)^\dagger\}$$
$$+ \cdots\cdots + \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)_{e.m.}$$

D_μ could be covariant derivative w.r.t. E+M. ~~$SU(2)\times U(1)$~~ $_W$ ~~⊕~~ $SU_c(3)$ color has "integral" "neutral under $SU(3)$".

Clearly ∃ an infinite set of terms

This is an EFT valid up to some cutoff - What is the cutoff?

Expand leading terms.

$$\frac{f_\pi^2}{4} \operatorname{tr}\{(D_\mu U)(D^\mu U)^\dagger\} = \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + (D_\mu \pi^+)(D_\mu \pi^+)^\dagger$$

$$+ \frac{1}{2f_\pi^2}\left[-\frac{1}{3}\pi^0\pi^0 D_\mu \pi^+ D^\mu \pi^- + \dots\right] \leftarrow \text{dim } 6$$

$F \to \frac{1}{5}f$

$$+ \frac{1}{4f_\pi^4}\left[\frac{1}{18}(\pi^-\pi^+)^2 D_\mu \pi^0 D^\mu \pi^0 + \dots\right]$$

$\uparrow$ dim 8.

Higher dimension operators suppressed by powers of $\frac{1}{f_\pi}$. So cutoff is f_π.

EFT valid for scales $E \ll f_\pi$.

- it is a low energy effective theory.

It contains all the low energy interactions of π's (i.e. the Goldstone bosons of $SU(2) \times SU(2) \to SU(2)$

Pion decay $\pi^{\pm} \to \mu^{\pm} \nu_\mu$

Recall. $\langle 0 | J_\mu^{A\alpha}(x) | \pi^\beta(p) \rangle = i p_\mu f_\pi e^{-ip\cdot x} \delta^{\alpha\beta}$ — (A)

In QCD $J_\mu^{A\alpha} = \bar\psi \gamma_\mu \gamma_5 \tau^\alpha \psi$.

Here $\to f_\pi \partial_\mu \pi$.

where $\langle 0 | \pi^\alpha(x) | \pi^\beta(p) \rangle = e^{-ip\cdot x} \delta^{\alpha\beta}$.

which then $\Rightarrow$ (A) above.

At low energies weak interactions described by Fermi theory.

(Obtained from std model by decoupling EFT $W^{\pm}$, Z (ie $m_W^{\pm}, m_Z \to \infty$)

$$\mathcal{L}_{Fermi} = \frac{G_F}{\sqrt{2}} (J_\mu^L)^\dagger (J^{\mu L}).$$

$$J_\mu^L = \bar\psi_u \gamma^\mu (1-\gamma_5) \psi_d + \bar\psi_{(\nu_\mu)} (1-\gamma^5) \psi_{(\mu)}$$

+ 4 - quark terms → non-leptonic Wk int

2 - quark 2 leptons → semi leptonic wk

4 - lepton. — lepton decays (eg $\mu \to e \nu\bar\nu$)

check factors of $\sqrt{2}$

Matrix element for $\pi^+ \to \mu^+ \nu_\mu$.

$$\mathcal{M}(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F}{\sqrt{2}} f_\pi p^\mu_{\langle 0|} \bar{\psi}_{\nu_\mu} \gamma^\mu (1-\gamma^5) \psi_\mu |\mu^+ \nu\rangle$$

From this get

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{G_F^2 f_\pi^2}{4\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2.$$

after integrating over two-body leptonic phase space.

Lifetime $\tau = \Gamma^{-1} = 2.6 \times 10^{-8}$ s, $m_\pi \simeq 139.5$ MeV, $m_\mu = 106$ MeV

$$G_F = 1.16 \times 10^{-5} \text{ GeV}^{-2}.$$

$$\to f_\pi = 92 \text{ MeV}.$$

This fixes all parameters in chiral Lagrangian - to leading order.

L_i on the other hand are dimensionless constants - cannot be fixed a priori - in principle can be determined by QCD calculations.

Turn on (light) quark masses in QCD.

$\Delta\mathcal{L} = -\bar{\psi} \underline{M} \psi$

$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

$\Delta\mathcal{L} = -m_u \bar{u} u - m_d \bar{d} d . \qquad \psi = \begin{pmatrix} u \\ d \end{pmatrix}$

$= -(m_u + m_d)\bar{\psi} \mathbb{1} \psi - (m_u - m_d)\bar{\psi}\sigma_3 \psi .$

chiral (γ_5) transf^{ns}: broken by both terms.

Vector: transfns $\psi \to e^{i\vec{\theta}\cdot\vec{\tau}}\psi$ · leave

$\Delta\mathcal{L}$ invariant if $m_u = m_d$. — isospin symmetry.

Spurion analysis — suppose $\underline{M}$ is

a (non-dynamical — i.e no kinetic term) field

transforming as $M \to g_R M g_L^{+}$. — (A)

Then $\Delta\mathcal{L} = \bar{\psi}_R M \psi_L + \bar{\psi}_L M^{+} \psi_R .$

($\psi_R \to g_R \psi_R \quad \psi_L \to g_L \psi_L$) — is (B)

invariant.

Thus we can write down mass terms

in chiral Lagrangian by simply adding

all terms which include M

~~but~~ and are invariant under

p 14 – (A) (which reflects the way they appear in QCD)

and p.15 (A'):

Thus we replace the QCD mass

term by.

$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$

$U = e^{\frac{2i}{f_\pi}\vec{\pi}\cdot\vec{\tau}}$

$$\mathcal{L}_M = \frac{V^3}{2} \mathrm{tr}\left(MU + M^+ U^+\right).$$

$$= V^3 (m_u + m_d) - \frac{V^3}{2 f_\pi^2} (m_u + m_d) (\pi_0^2 + \pi_1^2 + \pi_2^2) + \dots$$

Match vacuum energy to QCD ~~check~~ $\Rightarrow V \simeq \Lambda_{QCD}$

~~Compare~~ so ~~that~~ $V^3 \simeq \langle \bar{u} u \rangle = \langle \bar{d} d \rangle$ (dim ^{n} 3)

vac. $SU(2)$ inv.^{nt} upto $\mathcal{O}(m_u - m_d)$

~~agrees~~ to agree with $\langle \Delta h_{mass} \rangle$ in QCD.

This $\Rightarrow$. $$m_\pi^2 = \frac{V^3 (m_u + m_d)}{2 f_\pi^2}.$$

i.e 3 – pion masses are equal

(even ~~if m_u~~ though $m_u \neq m_d$!).

Va $\Lambda_{QCD} \simeq 250$ MeV, $f_\pi = 92$ MeV $m_\pi = 140$ MeV $\Rightarrow m_u + m_d \approx 0(10)$ MeV.

Notable points:

1) Isospin symmetry not so much because $m_u \simeq m_d$ (in fact $m_d \simeq 2 m_u$!) but because both are small $m_u, m_d \ll \Lambda_{QCD}$

2). $U_A(1)$ broken by anomalies

do not expect a goldstone boson i.e. no light singlet pseudoscalar.

3) $SU(2) \times SU(2)$ symmetry can be used to construct an EFT valid upto scale $\sim f_\pi$.

- Can be used to discuss pion physics.
- Also couple to nucleons

~~4)~~

4) Above (point 3) technique can be generalized see for example Weinberg Vol II Sec ^{n} 19.6.