Quantum Chromodynamics QCD.

This is the theory of Strong interactions.

The constituent fields a

a) Quarks (six flavors) $q_f^i = \{u^i, d^i, \dots t^i\}$ $i = 1, 2, 3$ color index

$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ etc. is a $SU(3)$ triplet

under the color group.

$$\underline{q}_f \to e^{i\theta^\alpha(x)\frac{\lambda^\alpha}{2}} \underline{q}_f \text{ etc.} \qquad \frac{\lambda^\alpha}{2} \text{ generators of } SU(3).$$
$(\alpha = 1, \dots 8)$

b) Gluons - Gauge fields of local $SU(3)$

$$B_\mu^\alpha: \qquad \alpha = 1, \dots, 8. \qquad \underline{B}_\mu = B_\mu^\alpha \frac{\lambda^\alpha}{2}$$

$$\underline{B}_\mu \to U B_\mu U^{-1} - \frac{i}{g} \partial_\mu U U^{-1}.$$

$$U = e^{i\theta^\alpha(x)\frac{\lambda^\alpha}{2}}$$

Note B is flavor neutral.

Q.CD is the local $SU(3)$ gauge invariant theory of these quarks and Gluons,

(λ^α are the Gell-Mann $SU(3)$ matrices normalized s.t $\mathrm{tr}\left(\frac{\lambda^\alpha}{2}\frac{\lambda^\beta}{2}\right) = \frac{1}{2}\delta^{\alpha\beta}$. $\left[\frac{\lambda^\alpha}{2}, \frac{\lambda^\beta}{2}\right] = i f^{\alpha\beta\gamma}\frac{\lambda^\gamma}{2}$.

$$\mathcal{L} = -\frac{1}{4} H^{\alpha}_{\mu\nu} H^{\alpha\,\mu\nu} + \sum_{\text{flavor}} \bar{q}_f \, i \not{D} q_f + \sum_{\text{flav}} m_f \bar{q}_f q_f .$$

Field strength: $H_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig\,[\underline{G}_\mu , \underline{G}_\nu]$.

$$D_\mu q_f = (\partial_\mu - ig G_\mu) q_f$$

There is an additional term consistent with the symmetries namely

$\operatorname{tr} H_{\mu\nu} \tilde{H}^{\mu\nu}$ where $\tilde{H}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} H_{\rho\sigma}$

It may be shown that this can be written as

$$\partial_\mu K^\mu(G)$$

K^μ = (gauge variant) current. For field configurations which vanish at infty this will not give a contribution to action. However there are configuration which do not vanish at infinity (but go to pure gauge terms so the $G_{\mu\nu}$ vanishes at infty) that must be included in path integral. Hence this term must be kept - even though it is not generated in pert. theory. However it violates CP - (strong CP problem) gives contribution to electric dipole moment of neutron

Hence coeff^{nt} must be zero (or nearly zero) naturally.

QCD is part of the standard model of the strong Wk & EM interactions.

QCD is incorporated by

i) adding the free gluon term $-\frac{1}{4} H^{\alpha}_{\mu\nu} H^{\alpha\,\mu\nu}$

ii) changing the covariant derivatives ~~to~~ on quarks to

$$D_\mu q_L = \left(\partial_\mu - i g_1 \frac{Y}{2} B_\mu - i g_2 \frac{\vec{\tau}^a}{2} \cdot \vec{A}^a_\mu - i g_3 \frac{\lambda^\alpha}{2} G^\alpha_\mu\right) q_L.$$

$$D_\mu q_R = \left(\partial_\mu - i g_1 \frac{Y}{2} B_\mu - i g_3 \frac{\lambda^\alpha}{2} G^\alpha_\mu\right) q_R$$

where we've put g_1, g_2, g_3 as the $U(1)$, $SU(2)$ & $SU(3)$ couplings.

If we put $g_{1,2} = 0$ and drop the $F_{\mu\nu}, G_{\mu\nu}$ terms we have QCD.

The mass terms of course come after spont. symm. breakdown from the Yukawa couplings of the Higgs.

Effective Action for Gauge theory — calculation of QCD β-function.

As discussed at the beginning of the course the β-fn can be calculated from the effective action. This is the simplest way to calculate this in gauge theory (at least to one loop)

It is convenient to use background field method. $A_\mu^\alpha = \bar{A}_\mu^\alpha + A'^\alpha_\mu$ ← "quantum gauge field"

Also change the gauge fixing

ie Take

$$\mathcal{F}^\alpha(A_\mu) = \bar{D}_\mu A'^{\alpha\mu}$$

$\bar{D}_\mu$ is covariant derivative w.r.t "background field" (this is the same as what we called the "classical field" before ϕ_c in ϕ^4 theory.

$$\bar{D}_\mu \Phi = (\partial_\mu - ig\bar{\underline{A}}_\mu)\Phi \qquad \bar{\underline{A}}_\mu = \bar{A}^\alpha_\mu T^\alpha$$

↑ any field

T^α generator of gauge grp in rep Φ.

Background field, gauge transf^{ns}.

$$G: \quad \Phi \to \mathcal{U}(\theta)\Phi \qquad \bar{A}_\mu \to \mathcal{U}\bar{A}_\mu\mathcal{U}^{-1} - \frac{2i}{g}\partial_\mu\mathcal{U}\,\mathcal{U}^{-1}$$

in particular A' the quantum field

transforms covariantly

$$A'_\mu \to U A'_\mu U^{-1}$$

(Of course the full gauge field satisfies the usual transformation law. What we've done is to put the inhomogeneous piece into the background field transformation).

Clearly $\bar{D}_\mu \Phi$ is covariant under ~~backgd~~ gauge transformations in particular

$$\underline{f} = T^\alpha f^\alpha = \bar{D}_\mu \underline{A}'^\mu \to U \bar{D}_\mu A'^\mu U^{-1} = U f U^{-1}$$

So the gauge fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} f^\alpha f^\alpha$$

has backgd gauge invariance.

To get ghost action consider infinitesimal transf^n of the original gauge gp G under which we would have

$$G: \quad \delta \underline{A}'_{\alpha\mu} = i[\theta, A_\mu] + \frac{1}{g}\partial_\mu \theta$$

$$\delta \bar{\underline{A}}_\mu = 0$$ (No transf^n of classical variable).

Defining in adjoint rep ($\underline{\Phi}$ in adjoint)

$$(\bar{D}_\mu \phi)_\alpha = (\partial_\mu 1 - ig \bar{\underline{A}}_\mu)_{\alpha\gamma}\, \phi^\gamma$$

or
$$\bar{D}_\mu \underline{\phi} = \partial_\mu \underline{\phi} - ig [\bar{\underline{A}}_\mu, \underline{\phi}]$$

We have

$$\delta A_\mu' = \frac{1}{g}\left(\bar{D}_\mu \theta - ig\, [\underline{A}_\mu, \underline{\theta}]\right)$$

$i\, C^{\alpha\beta\gamma}\, T^\gamma\, A^\alpha \theta^\beta$

Hence

$$\delta f = \frac{1}{g}\, \bar{D}_\mu \left(\bar{D}^\mu \theta - ig\, [A_\mu', \theta]\right)$$

Alternatively

$$\delta f^\alpha = \frac{1}{g}\, \bar{D}_\mu \left(\bar{D}^\mu \theta^\alpha - g\, C^{\alpha\beta\gamma}\, \theta^\beta A'^\gamma\right)$$

So ghost Action is

$$\mathcal{L}_{gh} = c^{+}_{\alpha}(x)\, \bar{D}_\mu \left(\bar{D}^\mu + g\, C^{\alpha\beta\gamma} A'^\beta\right) c_\alpha$$

Also

$$F_{\mu\nu} = \partial_\mu [\bar{A}_\nu + A_\nu'] - \partial_\nu (\bar{A}_\mu + A_\mu') - ig [\bar{A}_\mu + A_\mu', \bar{A}_\nu + \ldots$$

$$* \quad = \bar{F}_{\mu\nu} + \bar{D}_\mu A_\nu' - \bar{D}_\nu A_\mu' - ig\, [A_\mu', A_\nu']$$

Fermion ψ — gen in fermion rep.

$$D_\mu \psi = \bar{D}_\mu \psi - ig \underline{A}' \psi$$

$$* \quad \bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu - ig [\bar{\underline{A}}_\mu, \bar{\underline{A}}_\nu]$$

The backgd gauge invariant action is

$$+ig[A'_\mu, A'_\nu] = (-ig)[T^\alpha_\gamma, T^\beta]A'^\alpha_\mu A'^\beta_\nu = -g\,C^{\alpha\beta\gamma}\,T^\gamma A'^\alpha_\mu A'^\beta_\nu$$

$$\mathcal{L} = -\tfrac{1}{4}\left(\bar{F}^\alpha_{\mu\nu} + \bar{D}_\mu A'^\alpha_\nu - \bar{D}_\nu A'^\alpha_\mu + g\,C^{\alpha\beta\gamma} A'^\beta_\mu A'^\gamma_\nu\right)^2$$

$$- \frac{1}{2\xi}\left(\bar{D}_\mu A'^{\alpha\mu}\right)^2$$

$$+ \; c^+_\alpha \left(\bar{D}_\mu \left(\bar{D}^\mu - ig\, T^\beta A'^\beta\right)\right)_{\alpha\gamma} C^\gamma$$

$$+ \; \bar{\Psi}\left(\bar{D}_\mu \Psi - i\,\underline{A}'_\mu \Psi\right) \quad + \mathcal{L}_{\text{matter}}.$$

This is gauge fixed with respect to "quantum" gauge symmetry G but has manifest background gauge invariance (i.e. under $\bar{G}$)

Counter term Lagrangian

The backgd gauge invariance enables us to organize the calculation of counter terms efficiently — They must obey the $\bar{G}$ invariance

$S_{c.t} = \int \mathcal{L}_{c.t}\, d^4x$ the (-ve of) the

divergent parts of $\Gamma_{loop}[\bar{A}, \ldots]$

$\mathcal{L}_{c.t}$ must consist of $\bar{G}$ - gauge invariant functions of $\bar{A}_{\alpha\mu}$ and its derivatives having dimensionality $[M]^D$, $D \leq 4$

Note it must also satisfy ghost number conservation ($c^\alpha \to e^{i\theta} c^\alpha$ $\quad c^{+\alpha} \to c^{+\alpha} e^{-i\epsilon}$)

So (ignoring matter fields for the moment)
in terms of original fields

$$\mathcal{L}_{c.t} = -\frac{1}{4}(Z_A - 1) F^\alpha_{\mu\nu} F^{\alpha\mu\nu} - (Z_c - 1) D_\mu c^\alpha D^\mu c^\alpha$$

Z_A, Z_c are cut-off dependent

renormalisation constant - to be determined from the loop calculation.

i.e.

Original Lag^n $$\mathcal{L} = -\frac{1}{4} F^\alpha_{\mu\nu(0)} F^{\alpha\mu\nu}_{(0)} - D_\mu c^{+\alpha}_{(0)} D^\mu c^\alpha_{(0)}$$

renormalized Lag $\to$ $$= -\frac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} - D_\mu c^{+\alpha} D^\mu c^\alpha$$

counter term Lag $\to$ $$-\frac{1}{4}(Z_A - 1) F^\alpha_{\mu\nu} F^{\alpha\mu\nu} - (Z_c - 1) D_\mu c^{+\alpha} D^\mu c^\alpha$$

i.e. we rescale fields and coupling constant.

$$A^\alpha_\mu \equiv Z_A^{-1/2} A^\alpha_{\mu(0)} \qquad c^\alpha \equiv Z_c^{-1/2} c^\alpha_{(0)}$$

$$g \equiv Z_A^{1/2} g_{(0)}$$

The "bare" or unrenormalized quantities are cutoff (Λ) dependent

NB renormalization of g is determined by the gauge invariance.

We calculate Z_A to one loop (and thereby the β-fn) by finding the one-loop effective action. Need quadratic piece in A' & c.

$$\left(T^{\gamma\,Adj}\right)_{\alpha\beta} = i C^{\alpha\gamma\beta} = -i C^{\alpha\beta\gamma}$$

$$\mathcal{L}_{quad} = -\frac{1}{4}\left(\bar{D}_\mu A'^{\alpha}_\nu - \bar{D}_\nu A'^{\alpha}_\mu\right)^2 - \frac{g}{2}\bar{F}^{\alpha}_{\mu\nu} C^{\alpha\beta\gamma} A'^{\beta}_\mu A'^{\gamma}_\nu$$

$$- \frac{1}{2\xi}\left(\bar{D}_\mu A'^{\alpha\mu}\right)^2 + c^{+}_\alpha\left(\bar{D}_\mu \bar{D}^\mu c\right)_\alpha$$

$$= \frac{1}{2} A'^{\alpha}_\nu \left[\bar{D}_\rho \bar{D}^\rho \eta^{\nu\mu} - \bar{D}^\mu \bar{D}^\nu\right]_{\alpha\beta} A'^{\beta}_\mu$$

$$+ \frac{g}{2} \bar{F}^{\gamma}_{\mu\nu} C^{\alpha\beta\gamma} A'^{\beta\mu} A'^{\alpha\nu}$$

$$+ \frac{1}{2\xi} A'^{\alpha}_\nu \bar{D}^\nu \bar{D}^\mu A'^{\alpha}_\mu + c^{+}_\alpha\left(\bar{D}_\mu \bar{D}^\mu c\right)_\alpha .$$

Now

$$\left[\bar{D}^\mu, \bar{D}^\nu\right]_{\alpha\beta} = -ig\left(T^\gamma\right)_{\alpha\beta} \bar{F}^{\gamma\mu\nu}$$

$$= -g C^{\alpha\beta\gamma} \bar{F}^{\gamma\mu\nu}$$

Choose gauge parameter $\xi = 1$.

Then $\mathcal{L}_{quad}$ simplifies to

$$\mathcal{L}_{eff,quad} = \frac{1}{2} A'^{\alpha}_{\nu} \left[\bar{D}^{A}_{\delta} \bar{D}^{\delta} \eta^{\nu\mu} + 2g \bar{F}^{\gamma\mu\nu} C^{\gamma\alpha\beta} \right] A'^{\beta}_{\mu}$$

$$+ \; c^{\dagger}_{\alpha} (\bar{D}_{\mu} \bar{D}^{\mu} c)_{\alpha} \quad \left(+ \left(\tfrac{1}{2\xi} - \tfrac{1}{2}\right) \bar{D}^{\nu} \bar{D}^{\mu} \right)$$

So by doing the Gaussian integral we have

$$e^{i\Gamma[\bar{A}]} = \left(\det \Box^{A}\right)^{-1/2} \left(\det \Box^{c}\right)^{1}$$

Where the operator matrices are

$$\left(\Box^{A}\right)_{(\alpha\nu x),(\beta\mu y)} \equiv \left(\bar{D}_{\delta} \bar{D}^{\delta} \otimes \eta - 2ig \bar{F} \right)^{\alpha\beta}_{\mu\nu} \delta^{4}(x-y)$$

$$(\eta)_{\mu\nu} = \eta_{\mu\nu} \qquad (\bar{F})^{\alpha\beta}_{\mu\nu} = (T^{\gamma Adj})_{\alpha\beta} \bar{F}^{\gamma}_{\mu\nu} = -i C^{\alpha\beta\gamma} F^{\gamma}_{\mu\nu}$$

$$\left(\Box^{c} \right)_{(\alpha x),(\beta y)} = \left(\bar{D}_{\rho} \bar{D}^{\rho} \right)_{\alpha\beta} \delta^{4}(x-y)$$

We calculate these determinants for constant $\bar{A}$. This is sufficient to extract Z_A. Then exactly as in the ϕ^4 case we go to momentum space to evaluate

$\mathrm{Tr} = \int_x \mathrm{tr}\,(x,x)$

$$\ln \det \Box = \mathrm{Tr} \ln \Box = \delta^{4}(0) \int d^4q \; \mathrm{tr} \ln \Box(q) = \frac{V}{(2\pi)^4} \int d^4q \; \mathrm{tr} \ln \Box_q$$

$$\text{So} \qquad i\Gamma[\bar{A}] = \frac{V}{(2\pi)^4} \int d^4q \; \mathrm{tr} \left[-\frac{1}{2} \ln \Box^{A}(q) + \ln \Box^{c}(q) \right]$$

where $\Box_q$ is obtained from box by putting $\partial_\mu \to -iq_\mu$

$$\square^A(q)_{(\alpha\nu),(\beta\mu)} = \left(\bar{D}_g(q)\,\bar{D}^g(q) \otimes \underline{\eta} + 2ig\bar{F}\right)^{\alpha\beta}_{\mu\nu}$$

$$\square^C(q)_{\alpha\beta} = \left(\bar{D}_g(q)\,\bar{D}^g(q)\right)^{\alpha\beta}$$

$$\bar{D}_g(q) = \left(-iq_g \underset{\substack{\uparrow \\ \text{unit matrix}}}{\mathbb{1}} - ig\underline{\bar{A}}_g\right)$$

(see p(

$$\square^A(q) = \left[-\left(q\mathbb{1} + g\underline{\bar{A}}_g\right)^2 \otimes \eta - 2ig\,F\right]$$

$(\eta \cdot \eta)_{\mu\nu} = \eta_\mu{}^\lambda \eta_{\lambda\nu} = \eta_{\mu\nu}$ etc.

$$\mathrm{tr}\ln \square^A(q) = \mathrm{tr}\ln(-q^2) + \mathrm{tr}\ln\left[\mathbb{1} + \frac{1}{q^2}\left(2qgA + g^2\underline{A}^2\right)\otimes\eta + 2ig\,\underline{F}\right]$$

$$= \mathrm{tr}\ln(-q^2) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,\mathrm{tr}\left(\frac{2qgA}{q^2}\eta + g^2\frac{\underline{A}^2}{q^2}\eta + \frac{2igF}{q^2}\right)^n$$

To calculate Z_A we only need the coeff of $\bar{A}^4$

Terms $O(\bar{A}^4)$

$$\left[\mathrm{tr}\ln\square^A(q)\right]_{\bar{A}^4} = \mathrm{tr}\left[-\frac{g^4}{4}\left(\frac{2q.\bar{A}}{q^2}\right)^4 \otimes\eta + \frac{1}{3}\cdot 3g^2\frac{(2q.\bar{A})^2}{(q^2)^3}\otimes\eta\left(g\underline{\bar{A}}^2\otimes\eta + 2igF\right) - \frac{1}{2}\frac{1}{(q^2)^2}\left(g^2\underline{\bar{A}}^2\otimes\eta + 2igF\right)^2\right]$$

$$= \mathrm{tr}\left[-\frac{g^4}{4}\left(\frac{2q.\bar{A}}{q^2}\right)^4\otimes\eta + g^4\frac{(2q.\bar{A})^2\bar{A}^2}{(q^2)^3}\otimes\eta - \frac{1}{2}\frac{g^4}{(q^2)^2}\underline{\bar{A}}^4\otimes\eta + \left(\frac{2}{(q^2)^2}\right)g^2\underline{\bar{F}}^2\right]$$

The linear terms in F vanish since $\mathrm{tr}\,\eta F = 0$

For constant $\bar{A}$

$$\bar{F}_{\mu\nu} = -ig[\bar{A}_\mu, \bar{A}_\nu]$$

$$\mathrm{tr}_{(\mu,\nu)}\ \bar{F}_{\mu\nu}\bar{F}^{\mu\nu} = -g^2 \mathrm{tr}\ [\bar{A}_\mu, \bar{A}_\nu][\bar{A}^\mu \bar{A}^\nu]$$

$$= -g^2 \mathrm{tr}\left\{(\bar{A}_\mu \bar{A}_\nu - \bar{A}_\nu \bar{A}_\mu)(\bar{A}^\mu \bar{A}^\nu - \bar{A}^\nu \bar{A}^\mu)\right\}$$

Drop bars!

$$= -g^2 \mathrm{tr}\left\{A_\mu A_\nu A^\mu A^\nu + A_\nu A_\mu A^\nu A^\mu - A_\mu A_\nu A^\nu A^\mu - A_\nu A_\mu A^\mu A^\nu\right\}$$

$$= -g^2\left[2\,\mathrm{tr}(A_\mu A_\nu A^\mu A^\nu) - 2\,\mathrm{tr}(A^2)^2\right]$$

$(q.A)^2 = q_\mu q_\nu A^\mu A^\nu = \frac{q^2}{4} A^2$

Inside integral (symmetric integration - consequence of Lorentz inv)

$$(q.A)^4 = q_\mu q_\nu q_\lambda q_\sigma A^\mu A^\nu A^\lambda A^\sigma$$

To fix coeff take $\eta^{\mu\nu}\eta^{\lambda\sigma}$ trace

$$= \frac{q^4}{24}(\eta_{\mu\nu}\eta_{\lambda\sigma} + \eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}) A^\mu A^\nu A^\lambda A^\sigma$$

$$= \frac{q^4}{24}\left((A^2)^2 + A_\mu A_\nu A^\mu A^\nu + A_\mu A_\nu A^\nu A^\mu\right)$$

$-\frac{1}{3} - \frac{1}{2} = -\frac{5}{6}$

$$-\frac{1}{4}\mathrm{tr}\left(\frac{(2q.A)^4}{(q^2)^4}\right) = -\frac{4}{24}\mathrm{tr}\left(2(A^2)^2 + A_\mu A_\nu A^\mu A^\nu\right)/q^4$$

$$\mathrm{tr}\,\frac{(2q.A)^2 A^2}{(q^2)^3} = \frac{1}{\cancel{4}}\,\mathrm{tr}\,\frac{\cancel{4}\, q^2 A^2 A^2}{(q^2)^3} = \mathrm{tr}\, A^4 / q^4$$

$$g^4\,\mathrm{tr}\left[-\frac{1}{4}\left(\frac{2q.A}{q^2}\right)^4 \otimes \eta + \frac{(2q.A)^2 A^2}{(q^2)^3} \otimes \eta - \frac{1}{2}\frac{A^4}{q^4} \otimes \eta\right]$$

$$= \frac{4g^4}{q^4}\mathrm{tr}\left[-\frac{1}{3}A^4 - \frac{1}{6}A_\mu A_\nu A^\mu A^\nu + A^4 - \frac{1}{2}A^4\right]$$

same trace over η

$$= \frac{4g^4}{q^4}\frac{1}{6}\mathrm{tr}\left[A^4 - A_\mu A_\nu A^\mu A^\nu\right] = \frac{1}{3}\frac{g^2}{q^4}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$$

Note $\operatorname{tr} \underline{F}^2 = -\operatorname{tr} F_{\mu\nu} F^{\mu\nu}$

So

$$\int d^4q \left[\operatorname{tr} \ln \Box^A(q)\right]_{A^4} = g^2 \int \frac{d^4q}{(q^2)^2} \left(\frac{1}{3} - 2\right) \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

The ghost calculation is exactly the same except that $\otimes\eta$ factor and the term F^2 are missing so must divide by 4 and ~~ignore~~ remove the (-2) term. i.e.

$$\int d^4q \left[\operatorname{tr} \ln \Box^C(q)\right]_{A^4} = g^2 \int \frac{d^4q}{(q^2)^2} \frac{1}{4} \cdot \frac{1}{3} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

$$\int d^4q \operatorname{tr} \left[-\frac{1}{2} \ln \Box^A(q) + \ln \Box^C(q)\right]$$

$$= g^2 \int \frac{d^4q}{(q^2)^2} \left(+\frac{5}{6} + \frac{1}{12}\right) \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

$(T^\gamma)_{\alpha\beta} = -iC^{\alpha\beta\gamma}$

$$= g^2 \frac{11}{12} \int \frac{d^4q}{(q^2)^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}.$$

$$\operatorname{tr} F_{\mu\nu} F^{\mu\nu} = \operatorname{tr} T^\gamma_{Ad} T^\delta_{Ad} F^\gamma_{\mu\nu} F^{\delta\,\mu\nu} = C^{\alpha\beta\gamma} C^{\alpha\beta\delta} F^\gamma_{\mu\nu} F^{\delta\,\mu\nu}$$

For simple groups

$$\operatorname{tr} T^\gamma T^\delta = C_R \delta_{\gamma\delta}$$

Dynkin # of rep R.

Once the structure consts fixed, eg by normalizing generators in fundamental in our case $\operatorname{tr} t^\alpha t^\beta = \frac{1}{2}$, C_R has a definite value in each rep.

Note $C_R > 0$ for compact gp.

eg. $SU(2) \qquad t^\alpha = \frac{\tau^\alpha}{2} \qquad \mathrm{tr}\, t^\alpha t^\beta = \frac{1}{2}\delta^{\alpha\beta}$

$$c^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma} \quad (\alpha,\beta,\gamma = 1,2,3)$$

$$\mathrm{tr}\, T^\gamma_{Ad} T^\delta_{Ad} = \epsilon^{\alpha\beta\gamma}\epsilon^{\alpha\beta\delta} = 2\,\delta^{\gamma\delta}$$

in $SU(N) \qquad C_{Adj} = N$

Thus

$$i\Big[\Gamma_{1\text{-loop}}[A]\Big]_{A^4} = \frac{V}{(2\pi)^4}\left(\frac{11}{12}g^2\right)\int \frac{d^4q}{(q^2)^2}\; \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$$

$q^2 \to q^2 + i\epsilon$

$$= \frac{11}{12} g^2 C_A F^\alpha_{\mu\nu}F^{\alpha\mu\nu} \frac{V}{(2\pi)^4}\int \frac{d^4q}{(q^2+i\epsilon)^2}$$

NB This is completely equivalent to procedure we adopted before:

Defining the integral by going to Euclidean space (Wick rotation) and imposing a ultraviolet cut-off Λ and renormalization scale μ we have

$q_0 = i q_4$

$$\int \frac{d^4q}{(q^2+i\epsilon)^2} = 2\pi^2 i \int_\mu^\Lambda \frac{|q|^3 d|q|}{|q|^4} = 2\pi^2 i \ln\frac{\Lambda}{\mu}$$

$$\Big[\Gamma_{1\text{-loop}}[A]\Big]_{\bar{A}^4} = \frac{11}{24\pi^2} C_A g^2 \ln\frac{\Lambda}{\mu}\; \frac{1}{4}\int_4 d^4x\; \bar{F}^\alpha_{\mu\nu}\bar{F}^{\alpha\mu\nu}$$

The counter-term must cancel this. So

$$Z_A - 1 = \frac{11}{24\pi^2} C_A g^2 \ln\frac{\Lambda}{\mu}$$

$$Z_A = 1 - b g^2 \ln\frac{\Lambda}{\mu} \qquad b \equiv -\frac{11}{24\pi^2} C_A$$

Using the relation $g(\mu) = Z_A^{1/2}\, g_{(0)}(\Lambda)$

we have

$$g(\mu) = \left(1 - b g^2 \ln \frac{\Lambda}{\mu}\right)^{1/2} g_0(\Lambda)$$

So to leading order

$$\beta_g \equiv \mu \frac{dg}{d\mu} = + \frac{b}{2} g^3 = - \frac{11}{48\pi^2} C_A g^3 < 0$$

for $g > 0$

or $$\mu \frac{dg^2}{d\mu} = b g^4 = - \frac{11}{24\pi^2} C_A g^4 < 0.$$

wow

This is to be compared with QED

$$Z_3 = 1 - \frac{e^2}{6\pi^2} \ln(\Lambda/\mu) \Rightarrow \beta_e = + \frac{e^3}{12\pi^2} > 0$$

$e = Z_3^{1/2} e_0$ or $\lambda \varphi^4$

NB It can be shown that (see eg Cheng+Li p280) only Y-M fields have -ve β fn near origin in g.

$$\beta_\lambda = + \frac{3}{16\pi^2} \lambda^2(\mu) > 0.$$

It is easy to include the contribution of fermions and scalars. $\Gamma[\bar{A}]$ will now include two more factors

$$i\,\Gamma[\bar{A}] = \left(-\tfrac{1}{2} \operatorname{tr} \ln \Box_A\right) + \operatorname{tr} \ln \Box_C + \operatorname{tr} \ln(\not{D} + M) - \tfrac{1}{2} \operatorname{tr} \ln(\Box_\phi - m^2)$$

The fermion contribution can be evaluated from comparison with QED, where gauge gp is U(1) with $t = 1$

$$Z_A^\psi = - \frac{g^2}{6\pi^2} C_\psi$$

$C_\psi \delta_{\gamma\delta} = \operatorname{tr} T_\psi^\gamma T_\psi^\delta$ is Dynkin index in Ferm. rep

So $$Z_A = 1 - bg^2 \ln \frac{\Lambda}{\mu}.$$

SU(N) - n flavours

$$b = -\frac{11}{24\pi^2} C_A + \frac{n C_\psi}{6\pi^2} = -\frac{1}{24\pi^2}(11N - 2n)$$

if there are n SU(N) multiplets of fermions

In QCD $\quad (G = SU(3) \quad T_\chi = \frac{\lambda}{2} \Rightarrow C_A = 3$
$C_\psi = \frac{1}{2}\ ) \quad n = \#$ flavours

$$b = -\frac{1}{24\pi^2}(33 - 2n)$$

$b < 0$ ie asymptotic freedom if
# of flavours ≤ 16.

The solution to the β-fn eq^n is

$$\frac{1}{g^2(\mu)} = -b \ln\mu + \text{const}.$$

or $$g^2(\mu) = \frac{1}{-b\ln(\mu/\Lambda_{QCD})}$$ $\quad \Lambda_{QCD}$ - integration const

For large scales $(\mu \gg \Lambda_{QCD})$
$g^2 \to 0$ - Asymptotic freedom

Infrared slavery $\longrightarrow$ At small scales g^2 becomes large.
Of course this perturbative calculation
breaks down before $\mu \simeq \Lambda$ is approached.
This is the "explanation" of why we do not
see quarks as free particles (confinement) and why we detect
them in deep inelastic scattering.