[redacted] (PI) I-1

The loop exp^n and 1PI action

Basic formula (from Phys 7270).

j = external classical source.

$$\langle 0, \text{out} | 0, \text{in} \rangle_j = \langle 0 | T\, e^{i \int j(x) \hat{\varphi}(x)} | 0 \rangle$$

$$= \frac{1}{Z[0]} \int [d\varphi]\, e^{\frac{i}{\hbar} S[\varphi] + i \int \varphi(x) j(x)} \qquad - (1)$$

Recall functional differentiation
$F[\varphi] = \int d^4x\, f(\varphi, \partial\varphi \ldots)$
$\delta F[\varphi] = \int \frac{\partial F}{\partial \varphi} \delta\varphi$

Written explicitly for a scalar field but generalizes to any field theory.

Functional differentiation w.r.t j gives time ordered correlation (Green's) functions.

Separate $\quad S = \underset{\substack{\uparrow \\ \text{free theory}}}{S_0[\varphi]} + \underset{\substack{\uparrow \\ \text{interaction}}}{S_I[\varphi]}$.

$$\int [d\varphi]\, e^{\frac{i}{\hbar} S_0[\varphi] + i \int \varphi(x) j(x)} = e^{-\frac{i\hbar}{2} \int d^4x\, d^4y\, j(x) \Delta_F(x-y) j(y)} \qquad - (2)$$

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon} d^4k \qquad - (3).$$

Thus rewrite (1) as.

$$\langle 0 | T\, e^{i \int j(x) \varphi(x) d^4x} | 0 \rangle = \cancel{e^{\frac{i}{\hbar} S_I\left(-i \frac{\delta}{\delta j}\right)} e^{-\frac{i\hbar}{2} \int d^4x d^4y\, j(x) \Delta_F j(y)}}$$

$$= e^{\frac{i}{\hbar} S_I\left(-i \frac{\delta}{\delta j}\right)} e^{-\frac{i\hbar}{2} \int d^4x\, d^4y\, j(x) \Delta_F(x-y) j(y)}$$

where we used $-i \frac{\delta}{\delta j} \leftrightarrow \varphi$ inside functional integral.

- Generating function for connected Green functions W

$$Z[j] = e^{\frac{i}{\hbar}W[j]} = \int [d\varphi]\, e^{\frac{i}{\hbar}\{S_0[\varphi] + S_I[\varphi]\} + \frac{i}{\hbar}\int j\cdot\varphi\, d^4x}$$

$$= \mathcal{N}\, e^{\frac{i}{\hbar}\int d^4x\, \mathcal{L}_I\left(-i\hbar\frac{\delta}{\delta j(x)}\right)}\, e^{-\frac{i\hbar}{2}\int \frac{j(x)}{\hbar}\Delta_F(x-y)\frac{j(y)}{\hbar}}$$

$$\overset{*}{=} e^{-\frac{i\hbar}{2}\frac{\delta}{\delta\varphi}\cdot\Delta\cdot\frac{\delta}{\delta\varphi}}\, e^{\frac{i}{\hbar}\left[S_I[\varphi] + j\cdot\varphi\right]}\Big|_{\varphi=0}$$

Note: We put $W \to \frac{i}{\hbar}W$ comp to 7270 notes:

So in perturbation expansion propagators occur with a power $\hbar^1$ whilst each vertex (including j-vertex!) has ~~have~~ a power $\hbar^{-1}$.

- Following relation holds between # of Loops L, the # of internal lines I and the number of vertices V

$$L = I - V + 1$$

in a connected diagram

L = # of left over integrations after imposing 4-mom^m conservation at each vertex (+1 to account for overall mom^m conserving delta function.)

-

$*$ For proof of this identity see end of notes Appendix B

Power of $\hbar$ in ~~a~~ L-loop diagram

$$(\hbar)^{I-V} = (\hbar)^{L-1}$$

ie $$\hbar^{-1} W[j] = \hbar^{-1} \sum_{L=0}^{\infty} \hbar^{L} F_L[j]$$

Leading order (classical) term

sum of all tree $(L=0)$ graphs.

Loops give quantum corrections.

Loop expansion $\Leftrightarrow$ semi-classical expansion.

Tree graphs — iterative solution to classical field equations.

Set $\hbar = 1$ Quantum Effective Action

$$Z[J] = e^{iW[J]} = \langle 0 | T\, e^{i\int d^4x\, J(x)\hat{\varphi}(x)} | 0 \rangle$$

P+S 1.3-11.5

$$= \frac{1}{Z[0]} \int [d\varphi]\; e^{iS[\varphi] + i\int d^4x\, J(x)\varphi(x)}$$

$$= e^{i\int d^4x\, \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]}\; e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)}\; \cancel{Z[0]} \quad -(1)$$

NB $W \to iW$ previous notes.

$W[J]$ generating functional for connected Green functions.

$$G_n^c(x_1, \dots x_n) = i^{-n} \frac{\delta^n}{\delta J(x_1) \cdots \delta J(x_n)}\, iW[J]\Big|_{J=0} \quad -(2)$$

$$= \langle 0 | T(\hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n)) | 0 \rangle_c$$

$$= G_n(x_1 \cdots x_n) - \sum G_2(x_1 x_2)\, G_{n-2}(x_3 \cdots x_n) - \text{etc.}$$

eg. $G_4^c(x_1 \cdots x_4) = G(x_1 \cdots x_4) - \sum G(x_1, x_2)\, G(x_3, x_4)$

when $G_1(x) = 0$.

Define Effective Action

$$\Gamma[\varphi_c] = W[J] - \int J(x)\varphi_c(x) \quad -(3)$$

where $\varphi_c(x) \equiv \dfrac{\delta W[J]}{\delta J(x)} = \dfrac{\langle 0|\varphi(x)|0\rangle_J}{\langle 0|0\rangle_J} \quad -(4)$

This is a Legendre transform

$$\delta\Gamma[\varphi_c] = \int d^4x \; \frac{\delta\Gamma(\varphi_c)}{\delta\varphi_c}\,\delta\varphi_c$$

$$= \int \cancel{\frac{\delta W}{\delta J}\cdot\delta J} - \cancel{\int \delta J\cdot\varphi_c} - \int J\cdot\delta\varphi_c$$

So $\quad \dfrac{\delta\Gamma}{\delta\varphi_c} = -J$

We will show that Γ is the generating functional for one-particle irreducible graphs.

Γ - quatum effective action

will turn out that $\lim_{\hbar\to 0}\Gamma[\varphi_c] = S[\varphi_c^0]$ for $J\to 0$

when $\dfrac{\delta S[\varphi_c^0]}{\delta \varphi} = 0$

Note that

$$Z[J] = e^{iW[J]} = \lim_{T\to\infty} \langle 0| e^{-iHT} |0\rangle_J$$

Thus we may interpret

$-W[J]/T$ as the vacuum

energy as a function of the

external source

If external source $J = 0$

Effection action satisfies $\frac{\delta\Gamma}{\delta\varphi(x)} = 0$

Solutions of this are the vacuum values of the field (ground state values) $\varphi_c = \langle\varphi\rangle$

vacuum translationally invariant φ_c x independent

so is Γ

$$\Gamma(\varphi_c) = -V_3 T\, U_{eff}(\varphi_c).$$

with $\frac{\partial U_{eff}}{\partial\varphi_c} = 0$ so U_{eff} is ground state energy density of quantum theory

$\delta\Gamma = \int \frac{\delta\Gamma}{\delta\varphi}\delta\varphi$
$= \int \frac{\delta W}{\delta J}\delta J - \int \delta J\,\varphi$
$- \int J.\delta\varphi$
$= - \int J.\delta\varphi.$

Follows that

$$J(x) = - \frac{\delta \Gamma}{\delta \varphi_c(x)} \qquad (5)$$

Γ is the generating functional for 1PI graphs. (one particle irreducible).

Consider

$\Gamma_2(x,y)$

$$\Gamma_2(x,y) \equiv \frac{\delta^2 \Gamma}{\delta\varphi_c(x)\,\delta\varphi_c(y)} = - \frac{\delta J(x)}{\delta \varphi_c(y)}$$

$$\frac{\delta \varphi_c(y)}{\delta J(x)} = \frac{\delta^2 W[J]}{\delta J(x)\,\delta J(y)} = i G_2(x,y)_J \qquad -(6)$$

So

$$\int d^4y\ \Gamma_2(x,y)\, i G_2(y,x')_J = - \int \frac{\delta J(x)}{\delta \varphi_c(y)} \frac{\delta \varphi_c(y)}{\delta J(x')} d^4y \qquad (7)$$

$$= - \delta^4(x - x')$$

$\{$ $J(x) = J[\varphi_c, x]$ $\qquad \delta J(x) = \int \frac{\delta J(x)}{\delta \varphi_c(y)} \delta\varphi_c(y)\, d^4y.$

$\varphi_c(x) = \varphi_c[J, x]$ $\qquad = \int d^4x' \left[\int \frac{\delta J(x)}{\delta \varphi_c(y)} \frac{\delta \varphi_c(y)}{\delta J(x')} d^4y \right] \delta J(x')$

$$\Rightarrow \int d^4y\, \frac{\delta J(x)}{\delta \varphi_c(y)} \frac{\delta \varphi_c(y)}{\delta J(x')} = \delta^4(x - x') \quad \}.$$

$$\Gamma_2(x,y) = \left[+i\, G_2(x,y)_J \right]^{-1}$$

Thus (at $J=0$) $\Gamma_2(x,y)$ is the inverse of the 2-point function $-iG_2(x,x')$

In the free theory $G_2(x,x') = i\Delta_F(x-x')$ is the inverse of the Kinetic Energy operator $\Box + m^2$ (in a scalar field theory)

$$(\Box + m^2)\,\Delta_F = -\delta^4(x-x').$$

Hence in the free theory

$$\Gamma_2^{(0)}(x,y) = -(\Box_x + m^2)\,\delta^4(x-y).$$

[Check $\int d^4y\; \Gamma_2^{(0)}(x,y)(-i)G_2^{(0)}(y,x')$

$$= +\int d^4y\,(-1)(\Box_x + m^2)\,\delta^4(x-y)\;\Delta_F(y-x')$$

$$= -(\Box_x + m^2)\,\Delta_F(x-x') = +\delta^4(x-x')\ ].$$

In the interacting theory $\Gamma^{(2)}(x,y)$ will give ~~the~~ ~~qu~~ ~~contains~~ also the quantum corrections to the free kinetic term.

Let's look at the three point function.

$$\Gamma_3(x_1, x_2, x_3) \equiv \frac{\delta^3 \Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)\,\delta\varphi_c(x_3)}$$

$$= -\frac{\delta^2 J(x_1)}{\delta\varphi_c(x_2)\,\delta\varphi_c(x_3)}$$

Diff^t $\quad \int d^4x_2\, \Gamma_2(x_1,x_2)\, \frac{\delta^2 W[J]}{\delta J(x_1')\,\delta J(x_2)} = -\delta^4(x_1-x_1')$

$$\int d^4x_2\, \Gamma_3(x_1,x_2,x_3)\, \frac{\delta^2 W[J]}{\delta J(x_1')\delta J(x_2)}$$

$\left[\frac{\delta J}{\delta\varphi_c} = \Gamma_2(x,y)\right] \qquad + \int d^4x_2\, d^4x_3'\, \Gamma_2(x_1,x_2)\, \frac{\delta^3 W[J]}{\delta J(x_1')\,\delta J(x_2)\,\delta J(x_3')}\, \Gamma_2(x_3,x_3') = 0.$

$$\Gamma_3(x_1,x_2,x_3) = -\int d^4x_1'\, d^4x_2'\, d^4x_3'\; \Gamma_2(x_1,x_1')\, \Gamma_2(x_2,x_2')\, \Gamma_2(x_3,x_3')\; \frac{\delta^3 W[J]}{\delta J(x_1')\,\delta J(x_2')\,\delta J(x_3')} \qquad -(3)$$

or putting $J=0$.

$$\Gamma_3(x_1,x_2,x_3)\Big|_{J(x)=0} = \int d^4x_1'\, d^4x_2'\, d^4x_3'\, (-iG_2^{-1})(x_1,x_1')(-iG_2^{-1})(x_2,x_2')\,(-iG_2^{-1})(x_3,x_3')\,(-1)\, G_3(x_1,x_2,x_3).$$

$$= i\int d^4x_1'\, d^4x_2'\, d^4x_3'\; G_2^{-1}\, G_2^{-1}\, G_2^{-1}\, G_3.$$

Alternatively we may rewrite (8) as

$$\frac{\delta^3 W[J]}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)} = \int d^4x_1'\, d^4x_2'\, d^4x_3'\; \Gamma_2^{-1}(x_1,x_1')\,\Gamma_2^{-1}(x_2,x_2')\,\Gamma_2^{-1}(x_3,x_3')\; \Gamma_3(x_1',x_2',x_3').$$

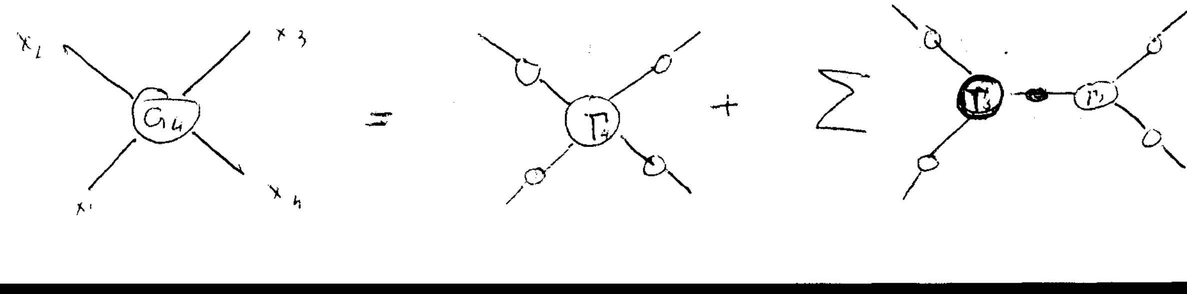
When $J=0$ this gives the connected 3-pt function in terms of the amputated function

$\Gamma^{-1} = iG_2 = i\frac{\delta^2 W}{\delta J\,\delta J}$

Differentiating again w.r.t. $J(x_4)$

$$\frac{\delta^4 W[J]}{\delta J(x_1)\cdots\delta J(x_4)} = \int d^4x_1'\cdots d^4x_4'\; \Gamma_2^{-1}(x_1,x_1')\cdots\Gamma_2^{-1}(x_4,x_4')\; \Gamma_4(x_1'\cdots x_4')$$

$$+\sum\int \Gamma_2^{-1}(x_1,x_1')\,\Gamma_2^{-1}(x_2,x_2')\,\Gamma_3(x_1',x_2',x_5')\,\Gamma^{-1}(x_5',x_5'')\,\Gamma_3(x_5'',x_3',x_4')\,\Gamma_2^{-1}(x_3',x_3)\,\Gamma_2^{-1}(x_4',x_4)$$



Further differentiation gives similar diagrams. General picture is that connected diagrams are tree diagrams whose vertices are Γ_3, Γ_4, etc.

i.e. $\Gamma_{(n)}$ gives all 1PI graphs (one-particle irreducible) graphs with n-legs. (amputated)

General proof.

Effective action reduces to classical action in the limit $\hbar \to 0$ and $\phi_c \to \varphi_c^{(0)}$ (the solution of the classical eq^n of motion.

Classical limit of Γ

$$e^{\frac{i}{\hbar} W[J]} = Z[0]^{-1} \int [d\varphi]\, e^{\frac{i}{\hbar}[S[\varphi] + \int J\cdot\varphi]}$$

$$\xrightarrow{\hbar\to 0} Z[0]^{-1}\, e^{\frac{i}{\hbar}[S[\varphi_0] + \int J\cdot\varphi_0]}$$

where φ_0 sol^n of classical equation $\frac{\delta S}{\delta\varphi} + J = 0$.

So comparing $O(\hbar^{-1})$ terms

$$\lim_{\hbar\to 0} W[J] = \left[S[\varphi_0] + \int J\,\varphi_0\, d^4x\right]\Big|_{\frac{\delta S}{\delta\varphi}+J=0}$$

But $W[J] = \left[\Gamma[\varphi] + \int J\,\varphi\right]\Big|_{\frac{\delta\Gamma}{\delta\varphi}+J=0}$.

This is true for an arbitrary source J

$\frac{\delta\Gamma}{\delta\varphi}+J=0$ solved by $\varphi=\varphi_c$

So $\varphi_c \xrightarrow{\hbar\to 0} \varphi_0$ and $\Gamma[\varphi] \xrightarrow{\hbar\to 0} S[\varphi]$.

$\Gamma[\varphi_c]$ as generating functional of 1PI graphs

g some loop counting parameter

Consider $\int[d\varphi]\, e^{\frac{i}{g}[\Gamma[\varphi_c] + \int J\varphi\, d^4x]} \xrightarrow{g\to 0} e^{\frac{i}{g}[\Gamma+\int J\varphi]}\Big|_{\varphi_c}$

where $\left[\frac{\delta\Gamma}{\delta\varphi} + J\right]_{\varphi=\varphi_c} = 0$.

But at this point $\Gamma + \int J\varphi = W[J]$

So $\int[d\varphi]\, e^{\frac{i}{g}[\Gamma + \int J\varphi]} \xrightarrow{g\to 0} e^{\frac{i}{g}W[J]}$

But we have already shown that $\hbar \to 0$ in path integral gives the sum of all tree graphs.

Thus we have proved the following

Theorem $W[J]$ is the sum of tree graphs for the vacuum to vacuum amplitude evaluated using the action $\Gamma[\varphi] + \int j(x)\varphi(x) d^4x$.

But $W[j]$ is the sum of all connected graphs using $S[\varphi] + \int j\varphi$

Hence $\Gamma[\varphi_c] + \int j\varphi$ gives all 1PI graphs coming from the action $S[\varphi] + \int j\varphi$.

Discussion

Gaussian Integrals.

$$I[a] = \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

Generalize to n variables

$A = [a_{ij}]$

$$I[\underline{A}] = \int_{-\infty}^{+\infty} dx_1 \cdots dx_n\, e^{-\underline{x}^T \underline{A}\, \underline{x}} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A = R^T D R \qquad R R^T = R^T R = 1 \quad A^T = A$$

R – orthogonal matrix

$$D = \mathrm{diag}\,(d_1 \ldots d_n).$$

$$I[A] = \int dx_1 \ldots dx_n\, e^{-\underline{x}^T \underline{R}^T \underline{D}\, \underline{R} x} = \int dy_1 \ldots dy_n\, e^{-\underline{Y}^T \underline{D} Y}$$

$$\underline{y} = R\, x. \qquad \text{Jacobian} \quad J = \det\left|\frac{\partial y_i}{\partial x_j}\right| = \det R_{ij} = 1$$

$$I[A] = \prod_{i=1}^{n} \int dy_i\, e^{-y_i d_i y_i} = \pi^{n/2} (d_1 \cdots d_n)^{-1/2}$$

or $$I[A] = \pi^{n/2} [\det A]^{-1/2}.$$

provided A is a positive definite matrix (i.e. $d_i > 0$)

Similarly if z_i are n complex variables and C is a Hermitian +ve definite matrix.

One more useful expression

$$I[\underline{A}, \underline{j}] = \int_{-\infty}^{+\infty} \prod_{i=1}^{n} dx_i\, e^{-\underline{x}^T \underline{A}\, \underline{x} + \underline{j}^T x} \qquad \underline{j} = \begin{pmatrix} j_1 \\ \vdots \\ j_n \end{pmatrix}$$

$$A^T = A$$

This may be brought to the standard form of a Gaussian Integral by change of variable

$$\underline{x}^T \underline{A}\, x - j^T \underline{x} = (x - \tfrac{1}{2}A^{-1}j)^T A\,(x - \tfrac{1}{2}A^{-1}j) - \tfrac{1}{4} j^T A^{-1} j.$$

put $\quad x' = x - \frac{1}{2}A^{-1}j\,, \qquad I[A,j] = e^{-\frac{1}{4}j^T A^{-1} j} \prod_{i=1}^{n}\int dx_i'\, e^{-x'^T A x'}$.

Since measure is translation invariant $\qquad dx = dx'$.

ie $\quad I[A,j] = e^{-\frac{1}{4}j^T A^{-1} j}\, \pi^{n/2} \det{}^{-1/2} A.$

Evaluating Gaussian functional integrals in field theory.

Also using $\quad A = R^T D R \qquad R^T R = 1$

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$\det A = \det R^T \det D \det R = \det R^T R \, \det D = 1.\prod_{r=1}^{n} d_r$$

$$\ln\det A = \ln \prod_{r=1}^{n} d_r = \sum_{r=1}^{n} \ln d_r = \operatorname{Tr}\ln D$$

$$\operatorname{Tr}\ln A = \operatorname{Tr}\ln(R^T D R) = \operatorname{Tr}\{R^T(\ln D)R\}$$

$$= \operatorname{Tr}\{R R^T \ln D\} = \operatorname{Tr}\ln D$$

So $\quad \ln\det A = \operatorname{Tr}\ln A.$

Gaussian integration . in free field theory
(quadratic term in exp^n).

$$S_2[\varphi] = \int \varphi(x)\, K(x,y)\, \varphi(y) = \int \varphi \hat{K} \varphi.$$

eg. Klein Gordon $\quad K(x,y) = D_{xy} \equiv (-\Box_E + m^2)\,\delta^4(x-y)$

$$(-\Box_E + m^2)\,u_n(x) = \int d^4y\, \hat{K}(x,y)\, u_n(x) = \lambda_n u_n(x)$$

u_n set of orthonormal eigenfunctions
(Work in a box so discrete set!).

Inner product $\quad (u_n, u_m) \equiv \int u_n(x)\, u_m(x)\, d^4x = \delta_{nm}$

Expand

$$\varphi(x) = \sum_n a_n u_n(x)$$

$$\int \varphi(x)\, \hat{K}\, \varphi(x) = \sum \lambda_n a_n^2 . \quad \lambda_n > 0 \text{ if } \hat{K} \text{ +ve definite.}$$

Functional measure $\quad \int [d\varphi] = \prod \int da_n.$

[metric. $\quad \|\delta\varphi\|^2 = \sum_{n,m} \int d^4x\, \delta a_n\, \delta a_m\, u_n u_m = \sum_n (\delta a_n)^2.$].

$$\int [d\varphi]\, e^{-S_2[\varphi]} = \int \prod \left[da_n\, e^{-\lambda_n a_n^2} \right]$$

$$= \prod \sqrt{\frac{\pi}{\lambda_n}} = \prod \pi^{n/2} \prod \lambda_n^{-1/2} = \prod \pi^{n/2} \det{}^{-1/2} \hat{K}$$

$$-\Gamma_1[\varphi_c] = -\frac{1}{2} \ln\det[D] = -\frac{1}{2} \mathrm{Tr} \ln[D]$$

up to irrelevant const.

Abstract Basis states $|n\rangle$ $\qquad \hat{K}\,|n\rangle = \lambda_n |n\rangle$

$\langle n|m\rangle = \delta_{nm} \qquad \sum_n |n\rangle\langle n| = 1$

$u_n(x) = \langle x|n\rangle \qquad \langle x|y\rangle = \delta^4(x-y) \qquad \int d^4x\, |x\rangle\langle x| = 1$

$$\mathrm{Tr}\,\hat{K} = \sum_n \langle n|\hat{K}|n\rangle = \sum_n \int d^4x\, \langle n|x\rangle\langle x|\hat{K}|n\rangle$$

$$= \int d^4x \sum_n \langle x|\hat{K}|n\rangle\langle n|x\rangle = \int d^4x\, \langle x|\hat{K}|x\rangle$$

$$\boxed{\langle x|p\rangle = \frac{e^{ip\cdot x}}{(\sqrt{2\pi})^4}} \qquad = \int d^4p\, \langle p|\hat{K}|p\rangle$$

$$\langle y|\hat{K}|x\rangle = \left(-\Box_E + m^2\right)\delta^4(x-y).$$

Note completeness relation $\sum |n\rangle\langle n| = 1$

$$\Rightarrow \quad \sum_n u_n(x)\, u_n(y) = \delta^4(x-y).$$

Inverse:

$$K^{-1}(x,y) \equiv G(x,y) = \sum_n \lambda_n^{-1} u_n(x) u_n(y)$$

Check $\hat{K}\cdot G(x,y) = \sum_n \lambda_n^{-1}\, \hat{K}\, u_n(x)\, u_n(y)$

$$= \sum_n u_n(x) u_n(y) = \delta^4(x-y).$$

since $\hat{K}\, u_n(x) = \lambda\, u_n(x)$.

Note inverse defined only if none of eigenvalues are zero $\lambda_n \neq 0 \;\forall n$.

To evaluate Γ go to Euclidean space. We could have started with the Euclidean formulation

$$e^{-W[J]} = \int [d\varphi]\, e^{-[S_E - \int J\phi]}$$

$$W[J] = \Gamma[\varphi_c] - \int J\varphi_c$$

and got

$$\Gamma_1[\varphi_c] = -\ln \int [d\phi]\, e^{-S[\varphi_c+\varphi']}\Big|_{1PI}.$$

$$S_2[\varphi] = \frac{1}{2}\int \varphi'(x)\left(-\Box_E + m^2\right)\delta^4(x-y)\phi'(y)\, d^4_{x_E} d^4_{y_E}$$

$\mu, \nu = 1,..,4$
$\tau \equiv x^4$

$$\Box_E = \partial_\tau^2 + \vec{\nabla}^2 = \delta^{\mu\nu}\partial_\mu\partial_\nu \qquad d^4x_E = d\tau\, d^3x = d^3x\, dx^4.$$

By Gaussian integration (upto irrelevant constants)

$$\Gamma_1[\varphi_c] = +\frac{1}{2}\ln \mathrm{Det}\left[D_{xy}\right]$$

$$D_{xy} = \left(-\Box_E^x + m^2(\varphi_c)\right)\delta^4(x-y). = D_{yx}.$$

For any symmetric (or Hermitian) matrix M

$$\ln \det M = \mathrm{Tr} \ln M$$

So $$\Gamma_1[\varphi_c] = +\frac{1}{2}\mathrm{Tr}\ln\left[D_{xy}\right]$$

~~Dot~~ Note that D_{xy} is a +ve definite operator (matrix). So e.v's $\lambda_n > 0$

$$\int d^4x\, d^4y\; \varphi(x) D_{xy} \varphi(y) = \int d^4x \left[\partial_\mu\varphi \partial_\nu\varphi\, \delta^{\mu\nu} + \frac{m^2}{2}(\varphi_c)\varphi^2 \right] > 0$$

$$D_{xy} = \left(-\Box_E + m^2(\varphi_c)\right)\delta^4(x-y).$$

$\langle x | \hat{p}_\mu | y \rangle = -\frac{i\partial}{\partial x}\delta(x-y)$

$$= \langle x | \left(\hat{p}^2 + m^2(\varphi_c) \right) | y \rangle$$

$$\mathrm{Tr}\ln[D_{xy}] = \int d^4x\, \langle x | \ln\left(\hat{p}^2 + m^2(\varphi_c)\right) | x \rangle.$$

$$= \int d^4p\, \langle p | \ln\left(\hat{p}^2 + m^2(\varphi_c)\right) | p \rangle$$

$$= \int d^4p\, \ln\left(p^2 + m^2(\varphi_c)\right) \langle p | p \rangle.$$

$$= \frac{V_4}{(2\pi)^4} \int d^4p\, \ln\left(p^2 + m^2(\varphi_c)\right).$$

$$\langle p | p \rangle = \delta^4(0) = \lim_{q\to 0} \int d^4x\, \frac{e^{iqx}}{(2\pi)^4} = \frac{V_4}{(2\pi)^4}.$$

Can make this completely rigorous by working in a box with periodic boundary conditions.

More generally suppose the (Euclidean) Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi\,\delta^{\mu\nu} + V(\varphi)$$

(so $V(\varphi) = \Lambda_0 + \varphi' V'(\varphi_c) + \frac{\varphi'^2}{2} V''(\varphi_c) + \ldots$)

$\uparrow$
classical c.c.

$m^2(\varphi_c) \to V''(\varphi_c)$.

Again linear terms do not contribute to 1PI graphs and to 1-loop only the φ'^2 term will contribute.

$$-\Gamma_1[\varphi_c] = \ln\int e^{-S_2}[d\varphi'] = \ln e^{-\int\left[\frac{1}{2}(\partial\varphi')^2 + \frac{\varphi'^2}{2}V''(\varphi_c)\right]}$$

$$= \ln e^{-\int d^4x_E\,\varphi'[-\Box_E + V''(\varphi_c)]\varphi'}$$

$$\Gamma_1[\varphi_c] = \frac{1}{2}\frac{V_4}{(2\pi)^4}\int d^4p_E \ln(p_E^2 + V''(\varphi_c)).$$

Diagramatic Evaluation of Γ.

Expand Γ in powers of φ_c

(analog of exp of $W[J]$ in powers of J).

$$\Gamma[\varphi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \Gamma^{(n)}(x_1, \ldots x_n) \varphi_c(x_1) \ldots \varphi_c(x_n). \quad -① $$

(If $\varphi[J]\big|_{J=0} \neq 0$ shift $\varphi_c \to \varphi_c - \varphi_c[J=0]$

i.e expand around $\varphi_c[J=0]$).

$\Gamma^{(n)}(x_1 \ldots x_n)$ sum of 1PI graphs with n external lines.

Alternative expansion (in derivatives).

$$\Gamma[\varphi_c] = \int d^4x \, [\, \mathcal{U}(\varphi_c) + \tfrac{1}{2}(\partial_\mu \varphi_c)^2 Z[\varphi_c] + \ldots] \quad -②.$$

↑ effective potential.

Introduce Fourier transform.

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \ldots + k_n) e^{i(k_1x_1 + \ldots + k_nx_n)} \tilde{\Gamma}^{(n)}(k_1, \ldots, k_n).$$

$$\Gamma[\varphi_c] = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} \int d^4x \, e^{i(\sum_{r=1}^n k_r)\cdot x} e^{i\sum_{r=1}^n k_n x_n}$$

$$[\tilde{\Gamma}^{(n)}(0, \ldots 0) \varphi_c(x_1) \ldots \varphi_c(x_n) + O(k_i)]$$

$$= \int d^4x \Big\{ \sum_n \frac{1}{n!} \tilde{\Gamma}^{(n)}(0, \ldots 0)[\varphi_c(x)]^n + O(\partial\varphi_c) \Big\}.$$

So $\mathcal{U}(\varphi_c) = \sum_n \frac{1}{n!} \tilde{\Gamma}(0, \ldots 0)[\varphi_c(x)]^n$ – i.e sum of diagrams with zero ext mom^m.

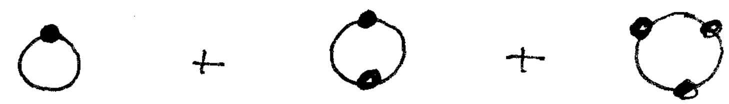
To $O(\hbar)$ $U(\varphi_c)$ is sum of all 1-loop graphs. These come from the quadratic terms in the expansion

$\varphi=\varphi_c+\varphi'$ $$S_E=\int d^4x\left(\tfrac{1}{2}\partial_\mu\varphi'\partial_\nu\varphi'\delta^{\mu\nu}+V''(\varphi_c)\frac{\varphi'^2}{2!}+\dots\right).$$

treat second term as interaction

Feyman rule $\quad$ —— $\rightarrow$ $1/p^2$

$\bullet$ $\rightarrow$ $-V''$

$U_1(\varphi_c)$ = 

1PI functions with n-vertices

$$U_1^{(n)}(\varphi_c)\,\cancel{\tilde{\Gamma}^{(2n)}(0,\dots,0)}=-\frac{1}{2n}\int\frac{d^4p}{(2\pi)^4}\left[\left(-V''(\varphi_c)\right)\cdot\frac{1}{p^2}\right]^n$$

Symmetry factor - reflection and rotation not a new diagram.

So $$U(\varphi_c)=V(\varphi_c)+\int\frac{d^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\cdot\left(\frac{V''(\varphi_c)}{p^2}\right)^n$$

$$=V(\varphi_c)+\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln\left(p^2+V''(\varphi_c)\right)-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\ln p^2$$

So $\Gamma_1[\varphi_c] = +\frac{1}{2}\frac{V_4}{(2\pi)^4}\int d^4p_E \ln\left(p_E^2 + V''(\varphi_c)\right)$

Define effective potential

by writing $\Gamma[\varphi_c] \doteq +V_4\, U(\varphi_c)$

(for constant φ_c). ↑ effective potential.

So to one loop

$$U(\varphi_c) = \cancel{\Lambda_0} + \frac{1}{2}m^2\varphi_c^2 + \frac{\lambda}{4!}\varphi_c^4 + \frac{1}{2}\int\frac{d^4p_E}{(2\pi)^4}\ln\left(p_E^2 + V''(\varphi_c)\right) + \cdots$$

$$= \underbrace{\qquad\qquad}_{V(\varphi_c)} + \Lambda_0$$

where we've absorbed all the constants into Λ_0.

$\int d^4p \to \int^{\Lambda} d^4p$ The integral is obviously divergent.

Need a cutoff (physical momentum scale up to which the field theory is valid.

Since Γ is dimensionless $U \sim [M]^4$ $\quad (V_4 = [L]^4 = [M]^{-4})$

Cut off dependent terms $\sim m_{\varphi_c}^{m}\Lambda^n$ $\qquad \cancel{d+}\, m+n=4$.

i.e. $a_0\Lambda^4 + a_1\Lambda^2 m^2(\varphi_c) + a_2 m^4(\varphi_c)\ln\frac{\Lambda}{m(\varphi_c)} + O(\Lambda^{-k})_{k>0}$

Appendix B

A useful identity

$$F\left(\frac{\partial}{i\partial x}\right)G(x) = G\left(\frac{\partial}{i\partial y}\right)F(y)\,e^{ixy}\Big|_{y=0}.$$

where F, G are two smooth fns

In field theory notes (P12) we used the fnl integral version of this.

To prove this use fourier transforms.

$$LHS = F\left(\frac{\partial}{i\partial x}\right)\int dp\, e^{ip\cdot x}\,\tilde{G}(p)$$

$$= \int dp\; F(p)\,\tilde{G}(p)\,e^{ipx}$$

$$RHS = G\left(\frac{\partial}{i\partial y}\right)\int dq\;\tilde{F}(q)\,e^{iqy}\,e^{ixy}\Big|_{y=0}$$

$$= \int dq\;\tilde{F}(q)\;G(q+x)$$

$$= \int dq\;\tilde{F}(q)\int \tilde{G}(p)\,e^{ip(q+x)}\,dp.$$

$$= \int dp\,F(p)\;\tilde{G}(p)\,e^{ip.x} = LHS.$$

Appendix C

Evaluating $\Gamma[\varphi_c]$. - Heat Kernel - dimreg.

$$\boxed{\Gamma[\varphi_c] = W[j] - j \cdot \varphi_c}.$$

$j = j[\varphi_c]$ by solving $\boxed{\frac{\delta W}{\delta j} = \langle\varphi\rangle_0 \equiv \varphi_c}$

$$\Rightarrow \quad \delta\Gamma / \delta\varphi_c = -j$$

But

$$e^{iW[j]} = \int [d\varphi]\, e^{iS[\varphi] + ij \cdot \varphi}$$

$$\Rightarrow \quad e^{i\Gamma[\varphi_c]} = e^{i(W[j[\varphi_c]] - j \cdot \varphi_c)}$$

$$= \int [d\varphi]\, e^{iS[\varphi] + ij[\varphi_c] (\varphi - \varphi_c)} \Big|_{j = -\frac{\delta\Gamma}{\delta\varphi_c}}$$

put $\varphi = \varphi_c + \varphi'$, $[d\varphi] = [d\varphi']$

$$= \int [d\varphi'] e^{iS[\varphi_c + \varphi'] + ij[\varphi_c] \cdot \varphi'} \Big|_{j = -\frac{\delta\Gamma}{\delta\varphi_c}}$$

Assuming trivial measure on field space

$$= \int [d\varphi'] \exp i \Big[S[\varphi_c] + \frac{1}{2}\varphi' \cdot K[\varphi_c] \cdot \varphi' + S_I[\varphi_c, \varphi'] + i\, \bar{j}[\varphi_c] \cdot \varphi' \Big] \Big|_{\bar{j} = -\frac{\delta\Gamma}{\delta\varphi_c} + \frac{\delta S}{\delta\varphi_c}}$$

$\boxed{S_I \text{ at least cubic in } \varphi'}$:

Define $\boxed{K[\varphi_c] = \frac{\delta^2 S[\varphi_c]}{\delta\varphi_c \delta\varphi_c}}.$

In general $K_{(i,x);(j,y)} \equiv \delta^2 S[\varphi_c] / \delta\varphi_c^i(x)\, \delta\varphi_c^j(y)$ -

P26

use $\delta / i\delta J \Longleftrightarrow \varphi'$

$$e^{i\Gamma[\varphi_c]} = e^{iS[\varphi_c]} e^{-\frac{1}{2}\mathrm{Tr}\ln K[\varphi_c]} \times \underbrace{e^{i S_I[\varphi_c, \frac{\delta}{i\delta J}]} e^{-\frac{i}{2} \bar{J} \cdot K[\varphi_c]^{-1} \cdot \bar{J}}}_{e^{-\frac{i}{2}\frac{\delta}{\delta\varphi_c'} \cdot K^{-1} \cdot \frac{\delta}{\delta\varphi_c'}} e^{i S_I[\varphi_c, \varphi_c']} \times e^{i \bar{J}\cdot\varphi_c'}\Big|_{\varphi_c'=0}} \Bigg|_{\bar{J} = -\Gamma_{,\varphi_c} + S_{,\varphi_c}}$$

$$\Gamma_{,\varphi_c} = \frac{\delta\Gamma}{\delta\varphi_c}$$

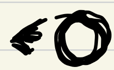
use identity on p24

So we have, an all orders expression.

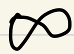
$$\Gamma[\varphi_c] = S[\varphi_c] + \frac{i}{2}\mathrm{Tr}\ln K[\varphi_c] + i^{-1}\ln\Big\{\exp\Big(-\frac{i}{2}\frac{\delta}{\delta\varphi_c'}\cdot K^{-1}\cdot\frac{\delta}{\delta\varphi_c'}\Big) \times \exp\Big(i S_I(\varphi_c; \varphi_c') + (-\Gamma_{,\varphi_c} + S_{,\varphi_c})\cdot\varphi_c'\Big)\Big\}_{\varphi_c'=0}$$

However we know that Γ computes 1PI diagrams (see p12, p13).

A vertex involving a single φ_c' will necessarily give 1P reducible diagram. So effectively, we may write (for purpose of computing $\Gamma[\varphi_c]$ in pert. th.

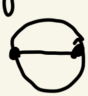
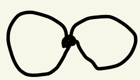
$$\Gamma[\varphi_c] = S[\varphi_c] + \frac{i}{2}\mathrm{Tr}\ln K[\varphi_c] \leftarrow$$ 

classical one-loop.

$$+\, i^{-1}\ln\left\{\exp\left(-\frac{i}{2}\frac{\delta}{\delta\varphi_c'}\cdot K^{-1}\cdot\frac{\delta}{\delta\varphi_c'}\right) \times \exp\left(i S_I(\varphi_c;\varphi_c')\right)\right\}_{1PI,\ \varphi_c'=0} \leftarrow$$   + ...

≥ 2 - loop terms.

Here the instruction on the RHS means that in computing it only 1PI diagrams should be kept.

2-loop terms  +  Lines are propagator $K^{-1}[\varphi_c]$

This expression is formal - so need to regularize the divergences which arise

Consider the representation for the propagator.

$$\underline{K}^{-1}[\varphi_c; x, y] = -i\langle x| \int_0^\infty ds\, e^{is\hat{\underline{K}}[\varphi_c]} |y\rangle \quad \text{- (A)}$$

$\hat{K} \to \hat{K} + i\epsilon$ gives zero as $s \to \infty$.

$\langle x|y\rangle = \delta(x-y)$; $-i\partial_\mu \leftrightarrow \hat{p}_\mu$; $\hat{x}^\mu|x\rangle = x^\mu|x\rangle$ etc.

Suppose we wish to calculate the effective potential - i.e the no-derivative part of $\Gamma[\varphi_c]$.

Note: $\Gamma[\varphi_c]$ may be expanded

either as $\quad \Gamma[\varphi_c] = \sum_n \frac{1}{n!}\int_{x_1}\cdots\int_{x_n}\Gamma(x_1,\ldots,x_n)\,\varphi_c(x_1)\ldots\varphi_c(x_n)$

or as $\quad \Gamma[\varphi_c] = \int d^4x \left[-U(\varphi_c) + Z[\varphi_c](\partial\varphi_c)^2 \right.$ + higher derivative terms$\left.\right]$

$U(\varphi_c)$ - Effective potential.

See (P6).

To compute $U(\varphi_c)$ we may set φ_c to a constant.

The "Heat Kernel"*

$$H(is\,|\,x,y) = \langle x|\, e^{is(\hat{p}^2 - V''(\varphi_c))}\,|y\rangle$$

constant φ_c $\quad = e^{is V''(\varphi_c)} \langle x|\, e^{is\hat{p}^2}\,|y\rangle$

Insert $\int d^4p\, |p\rangle\langle p| = 1$. $\quad \langle x|p\rangle = \frac{e^{ix\cdot p}}{(2\pi)^2}$

So

$$H(is\,|\,x,y) = e^{-isV''(\varphi_c)} \int \frac{d^4p}{(2\pi)^4}\, e^{is p^2 + i(x-y)\cdot p}$$

$\boxed{\text{In } n \text{ dims } (2\pi)^4 \to (2\pi)^n}$

$$= \frac{e^{-isV''(\varphi_c)}}{16\pi^2 i s^2}\, e^{-i(x-y)^2/4s}$$

$\boxed{\text{In } n\text{-dims } s^2 \to s^{n/2}}$

Hence we have $\quad$ (P28 - eqn A)

$$K^{-1}[\varphi_c; x, y] = -\int_0^\infty \frac{ds}{16\pi^2 s^2}\, e^{-is(V''(\varphi_c) + \epsilon)}\, e^{-i(x-y)^2/4s}$$

* Sol^n of "heat eq^n" $\quad \frac{\partial H}{\partial(is)} = K\, H(is\,|\,x,y)$

$$K^{-1}[\varphi_c; x, y] = -\int_0^\infty \frac{ds}{16\pi^2 s^2}\, e^{-is(V''(\varphi_c)-i\epsilon)}\, e^{-i(x-y)^2/4s}$$

Rotate contour of integration

i.e. effectively put (no sing at $\tau=0$ for $(x-y)^2<0$)

$$K^{-1}[\varphi_c; x, y] = +i\int_0^\infty \frac{d\tau}{16\pi^2 \tau^2}\, e^{-\tau V''(\varphi_c) + \frac{(x-y)^2}{4\tau}} \qquad \text{— (B)}$$

Integral well defined for $(x-y)^2<0$ but singular at $\tau=0$ if $x\to y$.

Representation of 1-loop effective action.

Integrate P27 - A w.r.t K to get (drop the constant)

$$\ln K[\varphi_c]_{x,y} = -\int_0^\infty \frac{ds}{s}\langle x|\, e^{is\hat{K}(\varphi_c)}\,|y\rangle$$

$$= i\int_0^\infty \frac{d\tau}{\tau}\,\frac{e^{-\tau V''(\varphi_c)}}{16\pi^2\tau^2}\, e^{\frac{(x-y)^2}{4\tau}} \qquad \text{— (C)}$$

But $\underset{\substack{\uparrow \\ \text{1-loop}}}{i\Gamma^{(1)}}[\varphi_c] = -\frac{1}{2}\,\mathrm{Tr}\ln K[\varphi_c]$

$\mathrm{Tr} \Rightarrow$ setting $x\to y$ & $\int d^4x$.

But at $x=y$ both (B) and (C) are divergent (from $\tau\to 0$ end of integral). This is of course the expected short distance singularity of QFT.
Thus need to regularize these expressions.
One obvious way is to cut off the integral at the lower end. So first define.

$$K_\Lambda^{-1}[\phi_c; x, y] = +i\int_{1/\Lambda^2}^{\infty} \frac{d\tau}{16\pi^2\tau^2}\, e^{-\tau V''(\phi_c) + \frac{(x-y)^2}{4\tau}}$$

$$\ln K_\Lambda[\phi_c; x, y] = i\int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau}\frac{e^{-\tau V''(\phi_c)}}{16\pi^2\tau^2}\, e^{\frac{(x-y)^2}{4\tau}}$$

and then take $x\to y$ limit.

This is called proper-time regularization.

Thus the 1-loop term becomes

$$\Gamma^{(1)}_{\Lambda}[\varphi_c] = +\frac{i}{2}\,\mathrm{Tr}\ln K_{\Lambda}[\varphi_c]$$

$$= -\frac{1}{2}\int d^4x \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau}\,\mathrm{tr}\,\frac{e^{-\tau V''}}{16\pi^2\tau^2}$$

— can write as incomplete Γ fn. — (D)

Alternatively use dimensional regularization. In n-dim^{ns} the above is replaced by (see p 27 with $4d \to nd$) and follow the same steps to get

$$\Gamma^{(1)}_{n}[\varphi_c] = -\frac{1}{2}\int d^nx \int_0^{\infty} \underbrace{\frac{d\tau}{\tau}\,\mathrm{tr}\,\frac{e^{-\tau V''}}{(4\pi)^{n/2}\,\tau^{n/2}}}_{} \quad \text{— (E)}$$

well-defined (i.e. no sing from $\tau \Rightarrow 0$ in integral for $n < 0$!!

(E) can be written in terms of Γ fns.

Evaluating this as $n \to 4$ (putting $n = 4-\epsilon$)

$$\Gamma^{(1)}_{4-\epsilon} = -\frac{1}{2}\int d^4x\, \mathrm{tr}\left\{ \frac{(V'')^2}{(4\pi)^2} \cdot \frac{1}{2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi)\right.\right.$$
$$\left.\left. - \ln\frac{V''}{\mu^2} + \frac{3}{2} + O(\epsilon)\right)\right\}$$

Note that an arbitrary scale has appeared again even though we did not cut off the integral (as in (1) where we had the scale Λ)

The renormalization group

Λ (or μ) are arbitrary scales. Γ should be independent of them. i.e $\Lambda \frac{d\Gamma}{d\Lambda} = 0$.

This requires that parameters

i.e coupling constants, mass parameters in $S[\varphi]$ must be Λ dependent

i.e $S \to S_\Lambda[\varphi] = Z[\varphi]\frac{1}{2}(\partial\varphi)^2 - V_\Lambda(\varphi)$

in a scalar field theory

for example. (in φ^4 $V_\Lambda = \frac{m^2}{2}(\Lambda) + \frac{\lambda(\Lambda)\varphi^4}{4!}$)

So $\Gamma[\varphi_c] = S_\Lambda(\varphi) + \sum_n \Gamma_\Lambda^{(n)}(\varphi)$

$\Lambda\frac{d}{d\Lambda}\Gamma = 0 \Rightarrow$ a set of (1^{st} order) diff^l eqns for the parameters of the theory.

β-fn eqns $\quad \Lambda\frac{d\lambda^i}{d\Lambda} = \beta^i(\underline{\lambda})$..

$\forall i$

$\{\lambda^i\}$ complete set of parameters.