①

S-Matrix $\Rightarrow$ Cross sections.

Given $|\alpha, t_i\rangle$ unitary evolution

in Q.M $\Rightarrow$ $|\beta, t_f\rangle = U(t_f, t_i)|\alpha, t_i\rangle$.

$\hat{\alpha}$ a Complete set of commuting observables.

α - eigen values.

$t_i \to -\infty$ in basis

$\hat{S} \doteq U(+\infty,-\infty) = T \exp\left(-i\int_{-\infty}^{+\infty} H_I\right)$

$t_f \to +\infty$ out basis.

Assume in/out ~~qu~~ particles described by well-seperated wave packets $\rightarrow$ simplicity replace by mom^m eigenstates - described by free "in" or "out" theory - with the physical masses.

Asymptotic completeness.

$$\sum_\alpha |\alpha; in\rangle\langle\alpha; in| = \sum_\beta |\beta; out\rangle\langle\beta; out|$$

$$= \mathbb{1}.$$

$\Leftrightarrow$ $SS^\dagger = S^\dagger S = \mathbb{1}$ - unitarity of S-operator

$|I; in\rangle = |p_1, p_2; in\rangle$ — 2-particle state $t \to \infty$.

$|F; in\rangle = |p_3, \ldots p_n; in\rangle$.

(we've suppressed all other quantum #'s ~~pol~~ polarizations of spinning particles charges etc)

$P_i = p_1 + p_2$

$P_f = \sum_{i=3}^{n} p_i$

$$S_{fi} \equiv \langle f; in| \hat{S} |i; in\rangle = \delta_{FI} + i(2\pi)^4 \delta^4(P_I - P_F) \times \langle F; in| \hat{T} |I; in\rangle$$

In a box $L^3 = V$ momenta are quantized. $p_i = \frac{2\pi i}{L}$

$$\Delta i_x \Delta i_y \Delta i_z = \frac{V}{(2\pi)^3} \Delta p_x \Delta p_y \Delta p_z$$

Completeness relation for final states

$$\sum_{F, n} \sum_{\underset{\uparrow}{i_1} \ldots \underset{\uparrow}{i_n}} |F, \vec{i}_1 \ldots \vec{i}_n\rangle \langle F, \vec{i}_1 \ldots \vec{i}_n| = \sum_{F, n} \prod_{r=3}^{n} \int \frac{d^3 p_r V}{(2\pi)^3} |F, p_1 \ldots p_n\rangle \times \langle p_1 \ldots p_n, F| = 1.$$

Integer Mom^m labels. phase space factor.

$\langle i|j\rangle = \delta_{ij}$

$$S_{fi} = \langle F; in| \hat{S} |I; in\rangle = \delta_{FI} + i(2\pi)^4 \delta^4(P_I - P_F) \times \langle F; in| \hat{T} |I; in\rangle$$

Transition probability ~~per unit time~~ in all space-time $I \neq F$

$$P_{FI} = \left[(2\pi)^4 \delta^4(P_I - P_F)\right]^2 \langle F; in| \hat{T} |I; in\rangle \langle I; in| \hat{T}^\dagger |F; in\rangle$$

Interpret in a box. $(2\pi)^4 \delta^4(p) = \int e^{ip\cdot x} d^4x \to VT$

$L^3 \times$ time interval

Transition prob / unit time P_{FI}/T

From LSZ.

$$\Omega_{FI} = V(2\pi)^4 \delta^4(P_I - P_F) \prod_{i\in I} \frac{1}{2E_i V} \prod_{f\in F} \frac{1}{2E_f V} |M_{fi}|^2$$

$(2\pi)^4 \delta^4(P_i - P_f)\, M_{fi} = i^{I+F} \int \prod_i d^4x_i \prod_f d^4x_f \, e^{-i\sum_i p_i\cdot x_i} e^{+i\sum_f p_f\cdot x_f}$

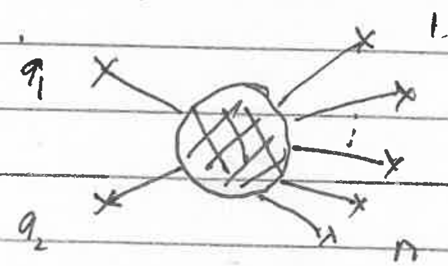
$[\Omega_{fi}] = \ldots$

$$\prod_{i\in I} (\Box_i + m_i^2) \cdot \prod_{f\in F} (\Box_f + m_f^2) \langle 0| T\{\prod \varphi_i(x_i) \prod \varphi_f(x_f)\} |0\rangle$$

$= +1$

– Lorentz invariant $2 + (n-2)$ particle correlation function.

Amputated correlation function – residue of multi-particle pole.



$P_I = q_1 + q_2$

$P_F = \sum_1^n p_i$

$$\lim_{q_i^2 \to m_i^2} \lim_{p_f^2 \to m_f^2} \prod (q_i^2 - m_i^2) \prod (q_f^2 - m_f^2) \tilde{G}(q_1, q_2; p_1, \ldots p_n)$$

Transition probability / unit time into mom^m range $\prod \Delta p_i$

One incident particle in volume $V \Rightarrow$ flux density.

(a relative to b) $\frac{v_{12}}{V} = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$

Differential cross-section transition probability / target particle / unit time/unit flux to mom^n range $\prod_{i=1}^{n} \Delta p_i$

$$d\sigma = \Omega_{si} \prod_{r=1}^{n} \frac{V d^3 p_r}{(2\pi)^3} \Big/ \frac{1}{V} |\vec{v}_1 - \vec{v}_2|.$$

$V \cdot \frac{1}{V^2 \cdot V^n} \cdot V \cdot V^n = V^0$

$$= (2\pi)^4 \delta^4(P_I - P_F) \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} \prod_{r=1}^{n} \frac{d^3 p_r}{(2\pi)^3 2E_r} \cdot |M_{FI}|^2$$

All volume factors cancelled.

$$= \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|} |M_{FI}|^2 \cdot d\Pi_{LIPS},$$

$$d\Pi_{LIPS} = (2\pi)^4 \delta^4(P_I - P_F) \prod_{r=1 \in F}^{n} \frac{d^3 p_r}{2E_r (2\pi)^3}.$$

Lorentz invariant phase space.

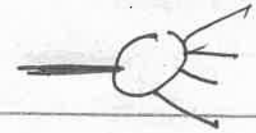
Note. $$\int_{\Delta^3 p} \frac{d^3 p_r}{2E_r (2\pi)^3} = \int_{\Delta^4 p} \pi\, d^4 p_r \, \underbrace{\delta^+(p_r^2 - m_r^2)}_{\text{manifestly Lorentz invariant}}.$$

as is M_{FI}.

Also in an arbitrary frame

$$2E_1 2E_2 |\vec{v}_1 - \vec{v}_2| \Rightarrow \sqrt{(q_1 \cdot q_2)^2 - m_1^2 m_2^2}$$

Decay rates.

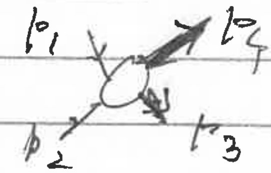


Probability of decay/time of 1 (unstable) particle to several particles. eg $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$,
$Z \to e^+ e^-, \gamma\gamma, \mu^+\mu^-$; $n^0 \to p^+ + e^- + \bar{\nu}_e$
etc.

View as scattering $1 \to n$ (though an unstable particle is clearly not an asymptotic state!) (incident $p^0 \equiv E$).

$$d\Gamma = \Omega_{si} \prod_{r=1}^{n} V \frac{d^3 p_r}{(2\pi)^3} \overset{V\to\infty}{=} \frac{1}{2E} |M_{1\to n}|^2 \, d\Pi_{LIPS}$$

$2 \to 2$ Scattering



$$s = (p_1 + p_2)^2 \quad t = (p_2 - p_3)^2 \quad u = (p_3 - p_1)^2$$

$$\delta(\Sigma p) = 0,$$

$$s + t + u = \Sigma m_i^2$$

$$d\Pi_{LIPS} = \cancel{\frac{1}{16\pi^2} d\Omega} \, (2\pi)^4 \delta^4(\Sigma p) \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2E_4}$$

In CM:
$\vec{p}_1 = -\vec{p}_2$
$\vec{p}_3 = -\vec{p}_4$
$E_1 + E_2 = E_{CM} = E_3 + E_4$

$$\Rightarrow \frac{1}{16\pi^2} d\Omega \int_0^\infty dp_f \frac{p_f^2}{E_3 E_4} \delta(E_3 + E_4 - E_{CM})$$

Integrate over $\vec{p}_4$, $p_f = |\vec{p}_3| = |\vec{p}_4|$. $E_i = \sqrt{p_f^2 + m_i^2}$, $i = 3, 4$.

Put $x = E_3(p_f) + E_4(p_f) - E_{CM}$ $\qquad \frac{dx}{dp_f} = \frac{p_f}{E_3} + \frac{p_f}{E_4} = p_f \frac{E_3 + E_4}{E_3 E_4}$

$$d\Pi_{LIPS} = \frac{1}{16\pi^2} d\Omega \int_{m_3 + m_4 - E_{CM}}^{\infty} dx \, \frac{p_f}{E_{CM}} \delta(x) = \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{CM}} \Theta(E_{CM} - m_3 - m_4)$$

Also $|\vec{v}_1 - \vec{v}_2| = \left| \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_2|}{E_2} \right| = p_i \frac{E_{CM}}{E_1 E_2}$

$p_i = |\vec{p}_1| = |\vec{p}_2|$

$$\Rightarrow \frac{d\sigma}{d\Omega}\Big|_{CM} = \frac{1}{64\pi^2 E_{CM}^2} \frac{p_f}{p_i} |M|^2 \, \theta(E_{CM} - m_3 - m_4).$$

For equal masses $p_f = p_i$ $\qquad \frac{\lambda \varphi^4}{4!}$

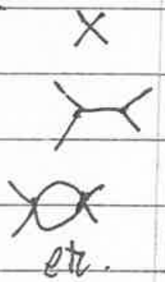
$$\frac{d\sigma}{d\Omega}\Big|_{CM} = \frac{1}{64\pi^2 E_{CM}^2} |M|^2.$$

$M = i\lambda$

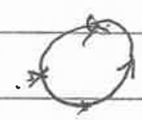
Feynman rules. - Scalars.

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\mu}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4$$

iMc



$\xrightarrow[p]{} \Rightarrow \frac{i}{p^2 - m^2 + i\epsilon}$. $\quad$ vertex $= -i\mu$. $\quad$ X $(-i\lambda)$



q. $\quad \int \frac{d^4 q}{(2\pi)^4}$. over loop mom^m.

Mom^m conserved at each vertex.

etc.

QED.

Photon $\underset{p}{\sim\sim\sim} = -\frac{i}{p^2 + i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$

gauge fixing $\xi = 1$ Feynman

pol^n sum.

$\sum \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k,\lambda) \to -\eta_{\mu\nu}$.

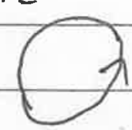
$\sum_s u\bar{u} = m + \not{p}$

spinor $\xrightarrow{p} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$. $\qquad \sim\sim\sim\bullet = \epsilon_\mu(p)$ in, $\quad \bullet\sim\sim\sim = \epsilon^*_\mu(p)$ out

$\sum v\bar{v} = m - \not{p}$

e^- $\begin{cases} \longrightarrow\bullet = u^s(p) \text{ in} \\ \bullet\longrightarrow = \bar{u}^s(p) \text{ out} \end{cases}$

e^+ $\begin{cases} \longleftarrow\bullet = \bar{v}^s(p) \text{ in} \\ \bullet\longrightarrow = v^s(p) \text{ out} \end{cases}$



Fermion loop $\Rightarrow (-1)$ factor.

Review of QFT I. (RJ)

Basic formula.

$$\langle 0;\text{out}|0;\text{in}\rangle_j = \langle 0|T e^{i\int j(x)\hat{\varphi}(x)}|0\rangle$$

– operator form.

$$= Z[j] = e^{iW[j]} = \int [d\varphi]\, e^{\frac{i}{\hbar}S[\varphi] + i\int \varphi(x)j(x)}$$

– func^t integral form.

ϕ can stand for

scalar – fermions – gauge fields etc.

– j – takes appropriate form $j \to \eta$ for $\varphi \to \psi$
fermionic: fermion

$j \to j^\mu$ for $\varphi \to A_\mu$ etc.

For simplicity work with scalars.

$W[J]$ is the generator of <u>connected</u> correlation fns.

i.e
$$\left.\frac{\delta^2 W}{\delta J_1 \delta J_2}\right|_{J=0} = \langle T\varphi_1\varphi_2\rangle_0 - \langle\varphi_1\rangle_0\langle\varphi_2\rangle_0$$

$$\frac{\delta^3 W}{\delta J_1\delta J_2\delta J_3} = \langle T(\varphi_1\varphi_2\varphi_3)\rangle_0 - \sum\langle\varphi_i\varphi_j\rangle\langle\varphi_k\rangle - \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle$$

etc

P2

Write $S[\varphi] = \underbrace{S_0[\varphi]}_{\text{free}} + \underbrace{S_I(\varphi)}_{\text{interaction}}$

For scalars (real) $\quad S_0[\varphi] = \int \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi) - \frac{m^2}{2}\varphi^2$

$$= -\int \frac{1}{2}\varphi(\Box + m^2)\varphi.$$

$$D^2_{xy} = (\Box_x + m^2)\,\delta^4(x-y) \qquad = -\frac{1}{2}\,\underbrace{\varphi_x \cdot D^2_{xy} \cdot \varphi_y}_{\text{integrated over}}$$

Feynman Green's fn.

$$\Delta_F(x,y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik.(x-y)}}{k^2 - m^2 + i\epsilon}.$$

$$\underbrace{D^2_{xy}\,\Delta_F(y,z)}_{\text{repeated index summed (integrated)}} = -\delta_{xz} \qquad (= \delta^4(x-z)).$$

R3.

Free - theory - Gaussian integral

$$\int [d\varphi]\, e^{\frac{i}{\hbar} S_0[\varphi] + i\int \varphi \cdot J}$$

$$= N \quad e^{-\frac{i\hbar}{2} j_x \cdot \Delta_{F\,xy} \cdot j_y}.$$

field independent

Use $-i\frac{\delta}{\delta J} \longrightarrow \varphi$ inside func integral.

So

$$e^{iW[j]} = \langle 0 | T\, e^{-i j_x \varphi_x} | 0 \rangle = e^{\frac{i}{\hbar} S_I\left(-i\frac{\delta}{\delta j}\right)}$$

$$\times\, e^{-\frac{i\hbar}{2} j_x \Delta_{F\,xy}\, j_y} \quad - \text{(A)}$$

Note if $S_I \to 0$

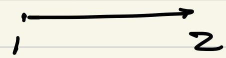
$$G^{(2)\,F}_{12} \equiv \langle T(\varphi_1 \varphi_2)\rangle_c \Big|_{J=0} \overset{\text{Free}}{=} i\hbar\, \Delta_{F12}$$

$$G^{(4)\,F}_{1234} \equiv \langle T(\varphi_1 \varphi_2 \varphi_3 \varphi_4)\rangle = i\hbar\, \Delta_{F12}\, i\hbar \Delta_{F34}$$

+ permutations.

etc - Wick's thm.

$G_{12} = i\Delta_{F\,12}$ — (Feynman) propagator.



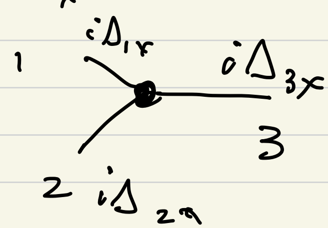
$$S_I = \int_x \mathcal{L}_I(\varphi)$$

- contains vertices.

e.g. $\mathcal{L}_I = \frac{\lambda_3}{3!}\varphi^3 + \frac{\lambda_4}{4!}\varphi^4$

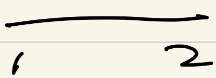
$$\langle T(\varphi_1\, \varphi_2\, \varphi_3 \left(-\frac{i\lambda_3}{3!}\int \varphi^3\right))\rangle_0 = \int_x (i\Delta_{1x})(i\Delta\ \ (i\Delta_{3x})(-i\lambda_3)$$

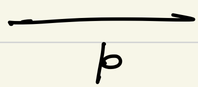
 $= (-i\lambda_3)$



In Fourier space

$$i\Delta_{12} \longrightarrow i\tilde{\Delta}(p) = \frac{i}{p^2 - m^2 + i\epsilon}.$$



$\longrightarrow$


$= \frac{i}{p^2 - m^2 + i\epsilon}$.

If vac. translationally invariant

$$\langle \varphi(x)\, \varphi(y)\rangle = \langle \varphi(x-y), \varphi(0)\rangle \text{ etc.}$$

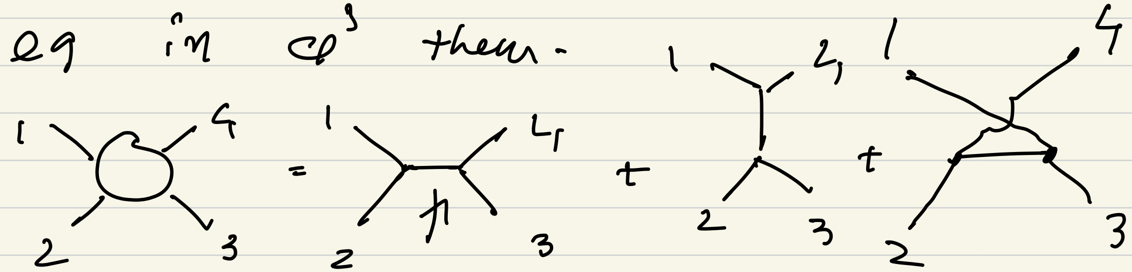
R3 - A

$$e^{iW[J]/\hbar} = e^{\frac{i}{\hbar}\int d^4x\, h_I\left(-i\frac{\delta}{\delta J}\right)} e^{-\frac{i\hbar}{2} J_x \Delta_{xy} J_y}$$

$\Rightarrow$ in pert^n exp^n each propagator has a power $\hbar$ while each vertex has a $\hbar^{-1}$ factor.

In pert^n expansion (in mom^m space) mom^m is conserved at each vertex and all internal momenta a integrated over. Left over interations = # of loops.

eg in ϕ^3 theory.



$O(\hbar^{-2}\cdot\hbar) = O(\hbar^{-1})$ $\quad q = p_1 + p_2 = p_3 + p_4$ $\quad$ Tree graphs.

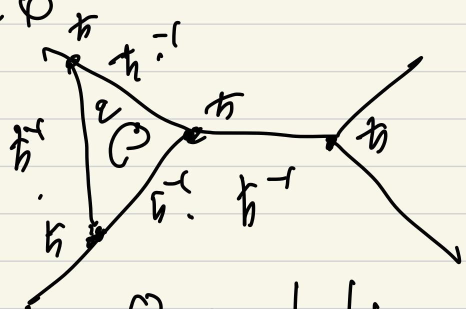
(R6)

So $\quad \langle T\, \Pi\, \tilde{\varphi}_p \rangle_{\text{tree}} = \hbar \cdot \left. \dfrac{\delta^4 W}{\delta \tilde{J}_1 \cdots \delta \tilde{J}_4} \right|_{\text{tree}} = O(\hbar^0)$

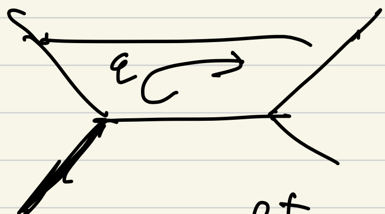
i.e. classical.

All tree graphs are $O(\hbar^0)$

At next order have terms like



$+$



etc.

$+ \cdots$

One-left one integration. $\int \dfrac{d^4 q}{(2\pi)^4}$

$O\left((\hbar^{-1})^4\, O(\hbar^4)\right) = O(\hbar^0)$

↑ from propagators $\qquad$ ↑ vertices.

$\Rightarrow$

$\langle T\, \Pi\, \tilde{\varphi}_p \rangle_{1\text{-loop}} = O(\hbar)$

First quantum correction.

R7

$L =$ # loops $\quad I =$ # internal lines.

$V =$ # vertices. $\quad$ i.e propagators.

$$L = I - V + 1$$

I: # internal 4-momenta

V: # 4-mom conservation conditions

1: overall momentum conservation

$=$ # unconstrained (integrated over.) 4-momenta.

So power of $\hbar$ in each diagram

$$(\hbar)^{I-V} = (\hbar)^{L-1}$$

i.e $\quad \hbar^{-1} W[j] = \hbar^{-1} \sum_{L=0}^{\infty} \hbar^{L} F_L(j).$

$L=0 \quad =$ Sum of tree graphs.

Loop exp^n $\Longleftrightarrow$ semi-classical exp^n.

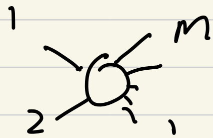
Feynman rules useful for calculating

$O(\hbar^N)$ contribution to $\tilde{G}^{(n)}(p_1, \dots p_n)^{(A)}_{C}$.
(i.e. amputated corr^n funct^n) $\quad \sum_1^n p_i = 0$.

i.e no propagators for external lines. – This is essentially M of LSZ formula.

$$M_{LSZ} = (i)^n \prod_{\substack{i=1 \\ p_i^2 \to m_i^2}}^{n} (p_i^2 - m_i^2)\, \tilde{G}^{(n)}(p_1, \dots, p_n).$$

In terms of which.



$$\frac{d\sigma}{d\Omega} = \frac{|M_{LSZ}|^2 (2\pi)^4 \delta^4\left(\sum_1^{2+m} p_i\right) d\Pi_{LIPS}}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

Feynman rules (n-pt fn at N^{th} order)

1. Draw all distinct connected diagrams with n external lines and N internal vertices joining all external lines

2. To each external line insert

a : 1 - scalars

in : $\epsilon_\mu(k;\lambda)$ - photon, out : $\epsilon^*_\mu(k,\lambda)$

Polarization sum

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda) \to \eta_{\mu\nu}$$

if $k^\mu M_\mu(k,\dots) = 0$

e^- : $u^s(p)$ in, $\bar{u}^s(p)$ out

e^+ : $\bar{v}^s(p)$ in, $v^s(p)$ out

(-ve charge in/out)

Spin sum

$$\sum_s u\bar{u} = m+\not{p}$$

$$\sum_s v\bar{v} = m-\not{p}$$

3. To each internal vertex

$= -i\lambda_{(3)}$, $= -i\lambda_{(4)}$, fer: ie.

and impose momentum conservation.

4. To each internal line insert

$$\underset{p}{\text{———}} = \frac{i}{p^2 - m^2 + i\epsilon} \qquad {}^{\mu}\underset{k}{\sim\sim\sim}{}^{\nu} = -\frac{i}{k^2 + i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

scalar

photon. ξ gauge fixing parameter.

$$\underset{p}{\longrightarrow} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

Fermion. (spin 1/2).

and integrate $\int \frac{d^4 q}{(2\pi)^4}$ over

each unconstrained momentum

with a factor (-1) for a closed fermion loop.

1

Quantization gauge theory

E of M $\quad \partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - i c^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu) = J$

$$= \partial^\mu(\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + O(A^2)) = J$$

Already in Abelian case $\quad A \to A + \partial\chi$ - redundancy

$\Rightarrow$ diff^l operator $\quad \Box\eta_{\mu\nu} + \partial_\mu\partial_\nu$ has zero eigen vector $\partial_\nu\varphi$

[Note think of diff op. as matrix =

$$D(x,y)_{\mu\nu} = (\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu)\,\delta^4(x-y) \;]$$

matrix indices

$\int d^4y\, D(x-y)_{\mu\nu}\, A^\nu(y)$ $\quad$ matrix multiplication

$\underline{A} = A^\alpha \underline{T}^\alpha$

$$"\langle \prod O(A_i)\rangle_0 = \frac{1}{Z}\int [dA]\, e^{iS[A]} \prod_i O(A_i)"$$

Redundancy $\quad \underline{A} \Leftrightarrow \underline{A}^g = g A g^{-1} + \frac{i}{g}\partial_\mu g g^{-1}$

$g \equiv e^{i T^a \theta^a}$ $\quad$ Redundancy gives infinite # of copies of gauge equivalent configuration in indep.

$\Rightarrow$ ambiguous $\frac{\infty}{\infty}$ factor.

Inv^{nt} measure $\quad$ derived from invariant metric

$$\|\delta\underline{A}\|^2 = \int d^4x\, \mathrm{tr}\, \delta\underline{A}_\mu \delta\underline{A}_\nu \eta^{\mu\nu}$$

$$\|\delta A^g\|^2 = \int d^4x\, \mathrm{tr}\, g\delta A_\mu g^{-1} g\delta A_\nu g^{-1} \eta^{\mu\nu} = \|\delta A\|^2$$

$\Rightarrow \quad [dA^g] = [dA]$

① $SO(2)$ invariant integral.

$\underline{r} = (r,\theta)$

$$Z = \int d\underline{r}\; e^{iS[\underline{r}]} \qquad d\underline{r} = dr\,d\theta$$

θ: $\underline{r} \to r^{\varphi} = (r, \theta+\varphi) \Rightarrow d\underline{r}^{\varphi} = d\underline{r}$.

If $S(\underline{r}^{\varphi}) = S[\underline{r}]$.

Write $Z = \int d\varphi \int d\underline{r}\; e^{iS[\underline{r}]}\, \delta(\theta-\varphi)$ ← insert 1.

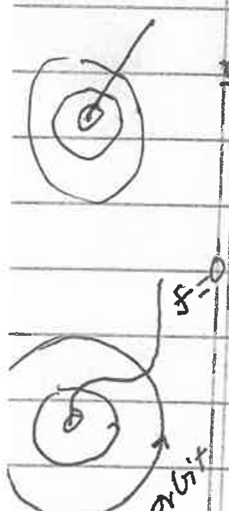
$\underline{r} \to r^{\varphi}$

$$= \int d\varphi \int d\underline{r}^{\varphi} e^{iS[r^{\varphi}]}\; \delta(\theta+\varphi-\varphi)$$

use invariance

$$= \int d\varphi \int d\underline{r}\; e^{iS[r]}\, \delta(\theta) = 2\pi \underbrace{\int d\underline{r}\, e^{iS[r]}\, \delta(\theta)}_{\text{gauge fixed integral}}$$

2π ↑ Vol of $SO(2)$



More general gauge fixing.

$\mathcal{F}(\underline{r}) = 0$ $f(\underline{r}^{\varphi})$ should intersect orbits ~~$\theta \to \theta+\varphi$~~ only once.

$\underline{r} \to r^{\varphi}$

Use $\Delta(\underline{r}) \int d\varphi\, \delta\left[\mathcal{F}(\underline{r}^{\varphi})\right] = 1$ $\Delta(r) = \left.\dfrac{\partial \mathcal{F}(r^{\varphi})}{\partial\varphi}\right|_{\mathcal{F}=0}$

$\because \cdots \delta(\mathcal{F}(r^{\varphi})) = \dfrac{1}{\left|\partial\mathcal{F}/\partial\varphi\right|_{\mathcal{F}=0}}\, \delta(\theta-\varphi)$

$$\Delta^{-1}(\underline{r}^{\varphi'}) = \int d\varphi\, \delta\left[\mathcal{F}(\underline{r}^{\varphi'+\varphi})\right] = \int d\varphi''\, \delta\left[\mathcal{F}(\underline{r}^{\varphi''})\right]$$

$d\varphi'' = d(\varphi'+\varphi) = d\varphi$

$= \Delta^{-1}(\underline{r})$ inv of measure

Δ $SO(2)$ invt.

③

$$S_v \quad Z = \int d\varphi \int d\underline{r}\, e^{iS[r]} \Delta(\underline{r})\, \delta[f(\underline{r}^{\varphi})]$$

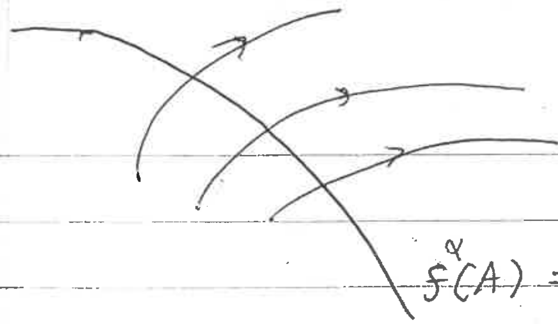
change int. var $r \to r^{\varphi}$

$$= \int d\varphi \cdot \int dr^{\varphi}\, e^{iS[r^{\varphi}]} \Delta(r^{\varphi})\, \delta[f(\underline{r})]$$

$$= \underbrace{\int d\varphi}_{2\pi\,(\text{Vol gauge gp})} \underbrace{\int d\underline{r}\, e^{iS[r]} \Delta(r)\, \delta(f(r))]}_{\text{gauge fixed integral.}}$$

$= 2\pi$ (Vol gauge gp) gauge fixed integral.

④



$f^\alpha(A) = 0$ gauge fixing function: $\alpha = 1 \ldots \dim$

must cross each orbit once.

Gauge fixing

$$A^g = A + (\partial_\mu + [T, A])\,\delta\theta.$$

$$f[A^g] = 0 \quad \text{has unique solution}$$

i.e. $\delta f^\alpha = \delta\theta^\beta \dfrac{\delta f^\alpha}{\delta\theta^\beta} = 0 \Rightarrow \delta\theta^\beta = 0$

$\Rightarrow M_{\alpha\beta}$ non-sing.

$g = g(\theta)$ $\quad \theta = \{\theta^\alpha\}$

$= 1 + \delta\theta^\alpha T_\alpha$ $\quad \alpha = 1 \ldots \dim G$

Identity

$$\Delta[A_\mu] \int [dg]\, \delta\{f^\alpha[A^g]\} = 1.$$

$$\Delta^{-1}[A_\mu] = \det \left.\frac{\partial f^\alpha}{\partial\theta^\beta}\right|_{f=0} \qquad M^f_{\alpha\beta} = \left.\frac{\partial f^\alpha}{\partial\theta^\beta}\right|_f$$

$A^g = g^{-1} A_\mu g$

$+ \frac{1}{ig} \partial_\mu g^{-1}$

measure invt

$$\Delta^{-1}[A^{g'}_\mu] = \int [dg]\, \delta\{f^\alpha[A^{g'g}]\} = \int [d(g'g)]\, \delta\{f^\alpha[A^{g'g}]\}$$

$(A^{g'})^g$

$g'g \equiv g''$

$$= \int [dg'']\, \delta\{f^\alpha[A^{g''}]\} = \Delta^{-1}[A_\mu].$$

$= g(g' A_\mu g'^{-1}) g^{-1}$

i.e. $\Delta[A_\mu]$ is gauge invt.

$+ g(\partial_\mu g')g'^{-1} g^{-1}$

$+ \partial_\mu g\, g^{-1}$

$= (g'g) A_\mu (g'g)^{-1}$

$+ \partial_\mu(g'g)(g'g)^{-1}$

Note $\quad f_\alpha[A^g(x)] = f_\alpha[A(x)] + \int d^4y\, M^f_{\alpha\beta}(x,y)\, \theta^\beta(y) + O(\theta^2)$

for inf^l θ.

A

ie M is the first variation of f under gauge transf^n.

Invariant vol^n on group

$\mathrm{Tr}\ g^{-1}\delta g \wedge g^{-1}\delta g$

- Haar measure.

$g^{-1}g'^{-1}\partial_\alpha g$

Inv^{nt} metric $\|\delta g\|^2 = \int_X \mathrm{tr}\ g^{-1}\partial_\alpha g\, g^{-1}\partial_\beta g\ \delta\theta^\alpha \delta\theta^\beta = g_{\alpha\beta}\,\delta\theta^\alpha\delta\theta^\beta$

Inv^{nt} under $\cancel{g \to g' g}$ $\quad g^{-1}g'^{-1}$ $\quad$ ~~Right~~ inv^{nt} $\quad g \to g g'$

$$\|\delta g^{g'}\|^2 = \int_X \mathrm{tr}\ g'^{-1}g^{-1}\partial_\alpha(g g')\, g'^{-1}g\, \partial_\beta(g g')\,d\theta$$

$$= \int_X \mathrm{tr}\left(g^{-1}\partial_\alpha g\ g\,\partial_\beta g\right) d\theta^\alpha d\theta^\beta.$$

$$[dg] = \int_X \sqrt{g}\,\sqrt{\det g_{\alpha\beta}}\ d\theta^{\alpha_1} d\theta^{\alpha_2} \ldots d\theta^{\alpha_n} = \frac{1}{n!}\mathrm{tr}\ g^{-1}\delta g \wedge g^{-1}\delta g \ldots$$

equivalently ~~right~~ Left invariant measure

derived from $\|\delta g\|^2 = \int_X \mathrm{tr}\left(\partial_\alpha g\, g^{-1}\, \partial_\beta g\, g^{-1}\right)\delta\theta^\alpha\delta\theta^\beta$.

invariant under $g \to g' g$.

$$Z_{\mathcal{O}} = \int [dA]\, e^{iS[A]}\, \mathcal{O}(A) \qquad \text{where } \mathcal{O}(A^g) = \mathcal{O}(A)$$

$$= \int [dg] \int [dA]\, e^{iS[A]}\, \Delta[A_\mu]\, \delta[f^\alpha(A^g)]\, \mathcal{O}(A)$$

$$= \int [dg] \int [dA^{g^{-1}}]\, e^{iS[A^{g^{-1}}]}\, \Delta[A_\mu^{g^{-1}}]\, \delta[f^\alpha(A)]\, \mathcal{O}(A^{g^{-1}})$$

$$\downarrow \text{Inv}^{ce}$$

$$= \int [dg] \int [dA]\, e^{iS[A]}\, \Delta[A_\mu]\, \delta[f^\alpha(A)]\, \mathcal{O}(A)$$

$$= (\text{Vol of gauge gp}) \times \text{gauge fixed PI.}$$

The delta fn restricts integral to one gauge slice (ie to a surface which intersects each orbit once).

The redundancy factors out as vol^{me} of gauge gp.

In Abelian gauge theory: M and hence Δ ind. of A_μ since $\delta A_\mu = \frac{1}{g}\partial_\mu \theta(x)$ (at least for linear gauges!)

Using Grassmann integration we may write

$$\Delta[A_\mu] = \det M^f = \int [dc][dc^\dagger]\, e^{i\int dx\,dy\, c^\dagger_\alpha(x) M_{\alpha\beta}(x,y) c_\beta(y)}$$

Where $c, c^\dagger$ are complex Grassmann valued fields ie. they are scalars obeying Fermi statistics

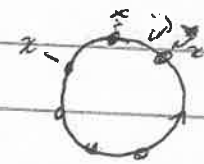
Quick perturbation theory argument

Complex scalar det. $(\varphi^*, \underline{K}\,\varphi) = \int \varphi^* \underline{K}\, \varphi \, d^4x$

Say $\underline{K} = K_0 + \lambda\chi$ $\qquad K_0 = \Box$

$$e^{iW[\chi]} = \int e^{i(\varphi, \underline{K}\varphi)}\, d\underline{\varphi} = \det[iK]^{-1} = e^{-i\,\mathrm{tr}\ln K}.$$

$\mathrm{Tr}\ln K$
$= \mathrm{Tr}\ln\left(1 + \frac{\lambda\chi}{K_0}\right)$
$+ \mathrm{Tr}\ln K_0$

$W = \Sigma$  $+$ cnt

Fermions. factor (-1) $\Rightarrow$

$$\int d\psi\, d\bar\psi\; e^{\int d^4x\, (\bar\psi, K^{1/2}\psi)} = \det iK^{1/2}$$

eg $\quad K^{1/2} = (i\partial\!\!\!/ - m)$

Path Integrals for Fermions.

Since fermions (operators) obey anti-comm^n relations (and so the T ordered product must be defined with a relative − sign) the corresponding path integrals are taken over Grassmann variables.

These are objects ψ_i such that

$$\psi_i \psi_j = -\psi_j \psi_i \qquad \text{so that} \qquad \psi_i^2 = 0 .$$

Suppose there is only one of ~~these~~ ψ. Then

$$f(\psi) = f_0 + \psi f_1 \quad , \; f_0, f_1 \in \mathbb{C}$$

(ie ordinary complex numbers).

Derivative defined by

$$\frac{d}{d\psi} f(\psi) = f_1 .$$

Integral by $\int d\psi = 0 \quad , \quad \int \psi \, d\psi = 1 .$

With these definition it ~~is~~ is easy to see that

$(f_0 + f_1\psi)(g_0 + g_1\psi)$

$= f_0 g_0 + f_1 g_0 \psi + f_0 g_1 \psi$

$f_1 g_0 + f_0 g_1$

$f_1 g_0 + f_1 g_0 \psi$

$_0 - f_1 \psi) g_1$

$$\frac{d}{d\psi}\left[f(\psi) g(\psi)\right] = \left(\frac{d}{d\psi} f(\psi)\right) g(\psi) + f(-\psi) \frac{d}{d\psi} g(\psi)$$

or $\displaystyle \frac{d}{d\psi} \cdot \psi = 1 - \psi \frac{d}{d\psi}$. Not $\left\{ \frac{d}{d\psi}, \frac{d}{d\psi} \right\} = 0$

also $\displaystyle \int f(\psi) \, d\psi = f_1 = \frac{d}{d\psi} f(\psi)$

Note $\displaystyle \int f(\psi + \chi) \, d\psi = \int f(\psi) \, d\psi$ (cmp to $\displaystyle \int_{-\infty}^{+\infty} f(x+y)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$)

$$\int \frac{d}{d\psi} f(\psi) \, d\psi = 0 .$$

We may take ψ to be a complex Grassmann variable and its conjugate $\bar\psi$ as an independent Grassmann variable

$$\psi\bar\psi = -\bar\psi\psi$$

$$\frac{d}{d\psi}\bar\psi = -\bar\psi\frac{d}{d\psi}$$

* Actually $\bar\psi$ need not be complex conjugate of ψ

$$F(\psi,\bar\psi) = F_0 + \psi F_1 + \bar\psi\bar F_1 + \psi\bar\psi F_2$$

$$\int F(\bar\psi,\psi)\,d\bar\psi\,d\psi = F_2 .$$

where F_i are ordinary (complex) numbers.

For many ~~Grass~~ Grassmann variables ψ_i easy to check that

$$\left\{\frac{\partial}{\partial\psi_i},\frac{\partial}{\partial\psi_j}\right\} = 0 \qquad \left\{\frac{\partial}{\partial\psi_i},\psi_j\right\} = \delta_{ij}$$

$$\int d\psi_i = 0 \qquad \int d\psi_i\,\psi_j = \delta_{ij}$$

$i,j=1,2.$

$$e^{\frac{1}{2}\psi_i A_{ij}\psi_j} = e^{\psi_1 A_{12}\psi_2} = 1 + A_{12}\psi_1\psi_2 .$$

$$\int d\psi_1 d\psi_2\, e^{\frac{1}{2}\psi_i A_{ij}\psi_j} = A_{12} = \sqrt{\det A} \qquad A=\begin{pmatrix}0 & A_{12}\\ -A_{12} & 0\end{pmatrix}$$

$A_{ji}=-A_{ij}.$

$$\int d\psi_{2n}\cdots d\psi_1\, e^{\frac{1}{2}\psi_i A_{ij}\psi_j} = \sqrt{\det A} = \text{Pfaff } A .$$

$$\text{Pfaff } A = \frac{1}{2^n n!}\epsilon_{i_1 i_2 i_3\ldots i_{2n}} A_{i_1 i_2}A_{i_3 i_4}\cdots A_{i_{2n-1}i_{2n}} .$$

Contrast with $\int dx_1\cdots dx_n\, e^{-\frac{1}{2}x_i A_{ij}x_j} = \pi^{n/2}(\det A)^{-1/2}$

If we had "complex" Grassmann variables.

$$\int \prod_{i=1}^{N} d\bar{\psi}_i \, d\psi_i \; e^{-\bar{\psi}_i A_{ij} \psi_j} = \det A$$

(comp. $\int \cdots \int e^{\bar{z}_i \Omega_{ij} z_j} \prod_{i=1}^{N} \left(\frac{d\bar{z}_i \, d z_i}{\pi} \right) = (\det \Omega)^{-1}$.

Now consider Fermion path integral with sources (corresponding to forced harmonic oscillator).

$$\langle 0|0 \rangle_{\eta, \bar{\eta}} = \int [d\psi][d\bar{\psi}] \; e^{i \int d^4x \, [\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi]}$$

η, $\bar{\eta}$ are Grassmann sources (external currents corresponding to the J's in bosonic case)

(Note ψ, η column ~~vector~~ matrix & $\bar{\psi}$, $\bar{\eta}$ row matrices)

$$\{\eta, \bar{\eta}\} = \{\eta, \eta\} = \{\bar{\eta}, \bar{\eta}\} = \{\psi, \eta\} = \{\bar{\psi}, \eta\} = \{\psi, \bar{\eta}\} = \{\bar{\psi}, \bar{\eta}\} = 0$$

Consider the solution ψ_c of the classical fermion equations with source i.e.

$$(i\partial\!\!\!/ - m)\psi_c + \eta = 0$$

and

$$\bar{\psi}_c(-i\overleftarrow{\partial\!\!\!/} - m) + \bar{\eta} = 0.$$

From VI-22 eq^n (4) we have

(Note like ψ, η is a 4×1 (column) matrix & $\bar{\psi}$, $\bar{\eta}$ - - - - - - 1×4 (row) matrix).

$$\psi_c(x) = -\int d^4x'\, S_F(x-x')\,\eta(x').$$

$$\bar\psi_c(x) = -\int d^4x'\, \bar\eta(x')\, S_F(x'-x).$$

Now in functional integral shift variables so that

$$\psi(x) = \psi_c(x) + \psi'(x)$$
$$\bar\psi(x) = \bar\psi_c(x) + \bar\psi'(x).$$

Then

$$\langle 0|0\rangle_{\eta,\bar\eta} = \int [d\psi'][d\bar\psi']\, e^{i\int dx\,[\bar\psi_c(i\partial\!\!\!/-m)\psi_c + \bar\psi_c\eta + \underbrace{\bar\eta\psi_c}_{\bar\eta\psi_c} + \bar\psi'(i\partial\!\!\!/-m)\psi']}.$$

$$= e^{iS_{cl}}\,\langle 0|0\rangle_{\eta=\bar\eta=0}.$$

But

$$iS_{cl} = i\int d^4x\,[\bar\psi_c(i\partial\!\!\!/-m)\psi_c + \underbrace{\bar\psi_c\eta + \bar\eta\psi_c}_{\bar\eta\psi_c}]$$

$$= -i\int d^4x\,d^4x'\;\bar\eta(x)\,S_F(x-x')\,\eta(x').$$

So $$Z[\eta,\bar\eta] = e^{-i\int d^4x\,d^4x\,\bar\eta(x)S_F(x-x')\eta(x')}\,Z[0].$$

Comp with
$$Z[j] = e^{-\frac{i}{2}\int d^4x\,d^4x'\,j(x)\,\Delta_F(x-x')\,j(x')}$$

If we had a theory of free fermi fields and free boson (K-G) fields with Lagrangian

$$\mathcal{L}(\varphi,\psi) = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 + \bar\psi(i\partial\!\!\!/ - m)\psi.$$

Then

$$Z[j,\eta,\bar\eta] = \int[d\varphi][d\psi][d\bar\psi]\, e^{i\int[\mathcal{L} + j\varphi + \bar\eta\psi + \bar\psi\eta]}$$

$$= e^{-\frac{i}{2}\int d^4x\, d^4x'\, j(x)\Delta_F(x-x')j(x') - i\int d^4x\, d^4x'\, \bar\eta S_F \eta} \times Z[0,0,0].$$

$$= \langle 0|T\, e^{i\int[j\hat\varphi + \bar\eta\hat\psi + \hat{\bar\psi}\eta]}|0\rangle \times Z[0,0,0]$$

This then gives Wick's thm for Fermions + bosons.

Note in particular

$$\frac{1}{Z[0]}\frac{\delta}{\delta\eta(x')}\frac{\delta}{\delta\bar\eta(x)} Z[j,\eta,\bar\eta]\Big|_{j,\eta,\bar\eta} = \langle 0|T(\hat\psi(x)\,\hat{\bar\psi}(x'))|0\rangle$$

$$= i\,S_F(x-x').$$

i.e. Feynman rule $\underset{x}{\bullet}\!\longrightarrow\!\underset{x'}{} = i\,S_F(x-x')$

Should write δ fn term in different way.

* (This makes Feynman exp^n possible)

First write gauge fixing as

$$f_\alpha(A_\mu) = B_\alpha(x)$$

B is some fixed arbitrary fn. This just amounts to changing def^n of f!

~~Z[A]~~ =

$$Z = \int [dA]\, e^{iS[A]} \underset{\Delta_F[A]}{\det M}\; \delta(f_\alpha(A) - B_\alpha)$$

(We've removed Vol G factor).

We've used $\int [dg]\, \Delta_F(A)\, \delta[f_\alpha(A^g) - B] = 1$

~~As we~~ We may change this by a const.

$$\int [dB]\, e^{-\frac{i}{2\xi}\int d^4x\, B^2(x)} \int [dg]\, \Delta_F(A)\, \delta[f_\alpha(A^g) - B]$$

$$= \int [dB]\, e^{-\frac{i}{2\xi}\int d^4x\, B^2} = \text{const.}$$

So we may redefine Z by a const to unity.

$$Z = V_{G}^{const} \int [dA]\, e^{iS[A]}\, \Delta_F(A)\, e^{-\frac{i}{2\xi}\int d^4x\, f_\alpha^2(A_\mu(x))}$$

$$= V_{G}^{const} \int [dA][dc][dc^\dagger]\, e^{iS[A] + iS_{gf} + iS_{gh}}$$

$$S_{gf} = -\frac{1}{2\xi}\int d^4x\, \left(f_\alpha(A_\mu(x))\right)^2 \qquad (i.e. \textstyle\sum_\alpha f_\alpha f_\alpha)$$

$$S_{gh} = \int d^4x\, d^4y \sum_{\alpha,\beta} c_\alpha^\dagger(x)\, M^f_{\alpha\beta}(x,y)\, c_\beta(y).$$

In order to derive Feynman rules we need to introduce sources.

$$S_{source} = \int J_\alpha^\mu(x) A_\mu^\alpha(x) + \int \eta^\dagger(x) c(x) + \int c^\dagger(x) \eta(x)$$

$\eta, \eta^\dagger$ are grassmann valued sources for ghosts.

$$Z[J, \eta, \eta^\dagger] = \int [dA][dc][dc^\dagger]\, e^{iS_{tot}[A, c, c^\dagger; J, \eta, \eta^\dagger]}$$

$$S_{tot} = S[A] + S_{gf} + S_{gh} + S_{source}.$$

Covariant gauges.

Lorentz $\qquad f^\alpha(A_\mu) = \partial^\mu A_\mu^\alpha \qquad \cancel{(=0 \text{ inside P.I.})}$

For θ inf^{d}.

$$g(\theta) = 1 + i\theta\cdot T + O(\theta^2)$$

$A_\mu^g = A_\mu +$

$$f^\alpha(A^g) = f^\alpha(A_\mu + i\theta^\beta \underbrace{[T^\beta, A_\mu]}_{i c^{\beta\gamma\delta} T^\delta A^\gamma} + \tfrac{i}{g}\partial_\mu \theta T)$$

$$= \partial^\mu A_\mu + \partial^\mu(-c^{\alpha\beta\gamma}\theta^\beta A_\mu^\gamma(x) + \tfrac{1}{g}\partial_\mu \theta^\alpha)$$

$$= f^\alpha(A) + \int d^4y\, M^f(x,y)_{\alpha\beta}\, \theta^\beta(y),$$

ie $\quad M^f(x,y)_{\alpha\beta} = \overbrace{+\tfrac{1}{g}\partial^\mu(\delta^{\alpha\beta}\partial_\mu - g c^{\alpha\beta\gamma}A_\mu^\gamma)}^{\partial^\mu [D_\mu]_{\alpha\beta}}\delta(x-y)$

$$S_{gf} = -\frac{1}{2\xi}\int d^4x\, (\partial^\mu A_\mu)^2$$

$$S_{gh} = \int d^4x\; c_\alpha^\dagger(x)\, \partial^\mu[(\delta^{\alpha\beta}\partial_\mu - g c^{\alpha\beta\gamma}A_\mu^\gamma)c_\beta(x)$$

$$\left(\text{rescaled } \begin{array}{l} c \to \sqrt{g}\, c \\ c^\dagger \to \sqrt{g}\, c^\dagger \end{array}\right)$$

To do pert. th. write $S_{tot} = S_0 + S_1$
where the free act^n is

$$S_0 = \int d^4x \Big[-\tfrac{1}{4}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^\alpha)^2$$
$$+ c_\alpha^\dagger \partial^2 c_\alpha + J_\mu^\alpha A^{\mu\alpha} + \eta^{\alpha\dagger} c^\alpha + \eta^\alpha c^{\alpha\dagger}\Big]$$

and the interaction action

$$S_1 = \int d^4x \, [-\tfrac{1}{2}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha) g\, c^{\alpha\beta\gamma} A^{\beta\mu} A^{\gamma\nu}$$

$$+ \tfrac{1}{4} g^2 c^{\alpha\beta\gamma} c^{\alpha\delta\epsilon} A_\mu^\beta A_\nu^\gamma A^{\delta\mu} A^{\epsilon\nu} + i g \partial^\mu \bar{c}^{\alpha}_\epsilon{}^{\alpha\beta\gamma} A_\mu^\gamma c^\beta$$

Propagators.

Now that we have fixed the gauge the kinetic is invertible.

For A_μ we have from partial integration the kinetic term

$$\int d^4x \left[-\tfrac{1}{4}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)(\partial^\mu A^{\nu\alpha} - \partial^\nu A^{\mu\alpha}) - \tfrac{1}{2\xi}\partial^\mu A_\mu^\alpha \partial^\nu A_\nu^\alpha\right]$$

$$= \int d^4x \left[\tfrac{1}{2} A_\mu^\alpha \left[\partial^2 \eta^{\mu\nu} - \frac{\xi - 1}{\xi} \partial^\mu \partial^\nu\right] A_\nu^\alpha\right]$$

$$= \int d^4x\, d^4y \left[\tfrac{1}{2} A_\mu^\alpha(x) K^{\mu\nu}_{\alpha\beta}(x-y) A_\nu^\beta(y)\right]$$

$$K^{\mu\nu}_{\alpha\beta}(x-y) = \left[\eta^{\mu\nu}\partial^2 - (1 - \tfrac{1}{\xi})\partial^\mu\partial^\nu\right] \delta_{\alpha\beta}\, \delta(x-y)$$

The inverse of this operator (ie Green fn) with Feynman bdy condns is

$$\Delta^{\mu\nu}_{\alpha\beta}(x-y) = \delta_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[-\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) - \xi \frac{k^\mu k^\nu}{k^2}\right] \frac{1}{k^2 + i\epsilon}$$

$\Delta \equiv \Delta_F$
we drop the F for not^l convenience!

Check:

$$\int d^4y\, K^{\mu\nu}_{\alpha\beta}(x-y)\, \Delta^{\beta\gamma}_{\nu\lambda}(y-z) = \eta^{\mu}_{\lambda}\, \delta^{\gamma}_{\beta}\, \delta^4(x-z).$$

Similarly we invert the ghost kinetic operator.

$$\int d^4x\, d^4y\; c^{\dagger}_{\alpha}(x)\delta^{\alpha\beta}\partial^2\delta^4(x-y)\, c_{\beta}(y) = \int d^4x\, d^4y\; c^{\dagger}_{\alpha}(x)\, K^{\alpha\beta}(x-y)\, c(\ldots$$

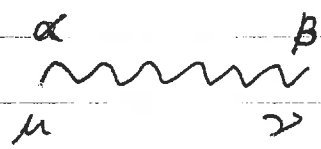
$$K^{-1}_{\alpha\beta}(x-y) = \Delta_{\alpha\beta}(x-y) = -\int \frac{d^4k}{(2\pi)^4}\, \frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\, \delta_{\alpha\beta}.$$

This is essentially the scalar propagator

$$\int d^4y\, K_{\alpha\beta}(x-y)\, \Delta_{\beta\gamma}(y-z) = \delta_{\alpha\gamma}\, \delta^4(x-z)$$

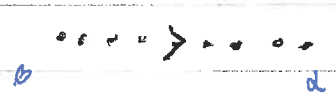
Momentum space Feynman rules.

i) Gauge boson propagator



$$i\Delta^{\alpha\beta}_{\mu\nu}(k) = -i\delta_{\alpha\beta}\frac{\left[\eta^{\mu\nu} - (1-\xi)\frac{k^{\mu}k^{\nu}}{k^2}\right]}{k^2+i\epsilon}$$

ii) FP ghost propagator



$$i\Delta_{\alpha\beta} = i\,\delta_{\alpha\beta}\,\frac{1}{k^2+i\epsilon}$$

(c). Gauge Field couplings.

Tri linear. $\mathcal{L}_I^3 = -\frac{1}{2}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)\, g\, \epsilon^{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma$

If $\epsilon^\mu(k)$ is the polarization vector
(recall quantization of A_μ!)
amplitude for 3-Gauge boson coupling is

$\tilde{A}$ - mom^m space A !

$$\tilde{A}^{\alpha_1\mu_1}(k_1)\,\tilde{A}^{\alpha_2\mu_2}(k_2)\,\tilde{A}^{\alpha_3\mu_3}(k_3)\ \Gamma^{\alpha_1\alpha_2\alpha_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$$

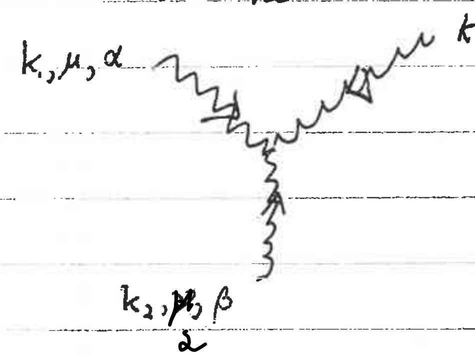
Clearly Γ must be symmetric under permutation of 1, 2, 3 $\quad k \leftrightarrow k_2,\ \mu \leftrightarrow \nu,\ \alpha \leftrightarrow \beta$ etc

Group structure already fixed.

$$\Gamma^{\alpha_1\alpha_2\alpha_3}_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = C^{\alpha_1\alpha_2\alpha_3}\,\Gamma_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$$

Since $C^{\alpha\beta\gamma}$ totally anti-symm. $\Gamma_{\mu\nu\lambda}$ must be A. Sym under $\begin{matrix}\mu\leftrightarrow\nu \\ k_1\leftrightarrow k_2\end{matrix}$ etc. By inspection of Lagrangian we have terms like $-i k_\mu \eta$. So symmetric we have

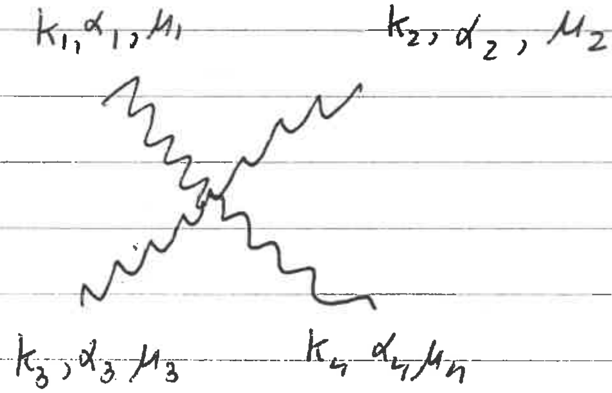
iii)



$$= i\,\Gamma^{\alpha\beta\gamma}_{\mu\nu\lambda} = C^{\alpha\beta\gamma}\big[(k_1-k_2)_{\mu_3}\eta_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1}\eta_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2}\eta_{\mu_3\mu_1}\big]$$

with $k_1+k_2+k_3=0$

Quadrilinear

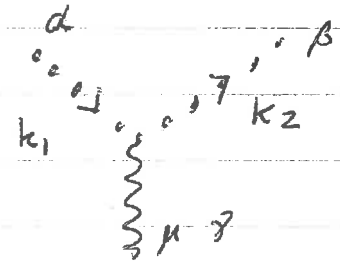


$$= \quad i\,\Gamma^{\alpha_1 \dots \alpha_4}_{\mu_1 \dots \mu_4}(k_1, \dots k_4)$$

$$\begin{aligned}
= & -i g^2 C^{\alpha_1 \alpha_3 \beta} C^{\alpha_2 \alpha_4 \beta} \left(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right) \\
& + i g^2 C^{\alpha_2 \alpha_3 \beta} C^{\alpha_1 \alpha_4 \beta} \left(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_2\mu_4}\eta^{\mu_1\mu_3}\right) \\
& + i g^2 C^{\alpha_1 \alpha_2 \beta} C^{\alpha_3 \alpha_4 \beta} \left(\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right)
\end{aligned}$$

with $\quad k_1 + k_2 + k_3 + k_4 = 0.$

Ghost - gauge field coupling.



$$i\,\Gamma^{\alpha\beta\gamma}_{\mu} = g\,\epsilon^{\alpha\beta\gamma} k_{1\mu}$$

(asymmetric - convention either left or right mom^m consistently)

Only enters in closed loops.
For every diagram with gauge field loop $\exists$ one with ghost loop in same place.
Also minus sign for each closed loop as for fermions.

Matter couplings

(D). Fermions

$$\mathcal{L}_f = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$

$$D_\mu \psi = \partial_\mu \psi - ig T^\alpha A_\mu^\alpha \psi .$$

Fermion prop.

$$i\,\Delta_{nm}(k) = \delta_{nm}\,\frac{i}{\not{k} - m + i\epsilon} =$$

n,m gp indices.

Vertex

$$i\,\Gamma^{\alpha\mu}_{nm} = ig\left(T^\alpha\right)_{nm}\gamma^\mu$$