Introduction

Conventions. ($\hbar = c = 1$) $\quad x^\mu ; \quad \mu = 0, 1, 2, 3.$

Metric $\quad \eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \quad x^0 \equiv t$

i.e. $\quad ds^2 = dt^2 - \vec{dx}^2. \quad \vec{x} = (x^1, x^2, x^3)$

$\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu} \quad \Box = \cancel{\partial_t^2}\ \partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2.$

$[M] = 1$

$[L] = -1.$

$$S = \int d^4x \ \mathcal{L}(\varphi, \partial_\mu \varphi) \quad [\mathcal{L}] = 4$$

since $[S] = 1 = [\hbar]$

$[\partial_\mu] = +1$

$[\varphi] = 1 \qquad [\psi] = 3/2. \qquad [A] = 1$

since $\quad \mathcal{L}_\varphi \sim \partial_\mu \varphi \partial^\mu \varphi \qquad \mathcal{L}_\psi \sim \bar{\psi} \not{\partial} \psi$

$\mathcal{L}_A \sim (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} = 0$$

Scalar $\quad \mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4$

Free Dirac $\quad \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \qquad$ Free photon $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

Q.E.D $\quad \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - e A_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad A_\mu \to A_\mu + \partial_\mu \chi$

I. ①

Overview,

$\hbar = c = 1$. 4 - fundamental interactions.

$\underbrace{\text{E-M, Weak, Strong}}$, Gravity.

$\downarrow$

SM Effective Field Theory

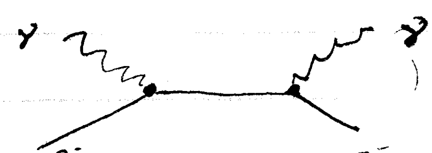
Valid below $E \lesssim 1\,\text{TeV}$?

Strength of Interactions.

E-M Low energy γ, e interaction (Thompson scattering)

$\gamma + e^- \to \gamma + e^-$; $E \ll m_e = 0.5\,\text{MeV}$.

$$\sigma_{Th} = \frac{2}{3}\alpha^2 \underbrace{4\pi R_e^2}_{\text{Effective Area.}} \qquad R_e = \frac{1}{m_e}. \text{ - Compton radius}$$



$$\Rightarrow \frac{e^2}{4\pi} \equiv \alpha \simeq \frac{1}{137}$$

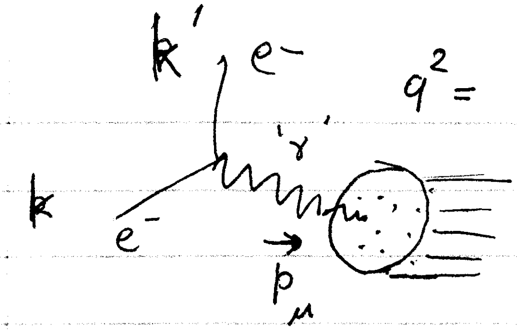
(other low energy determinations: A.C.-Josephson - Quantum Hall, .. g-2).

Typical Strong Interaction cross section eg $\pi(p, n)$ πN

$$\sigma_S \sim \alpha_s^2\, 4\pi R_p^2 \qquad \Rightarrow \alpha_s(\pi N) \sim O(1) \sim 10^2 \alpha$$

Also unlike electron (elementary up to 1 TeV scale!) - Nucleon - substructure - Deep Inelastic Scattering $\Rightarrow$ partons

Evidence for quarks.

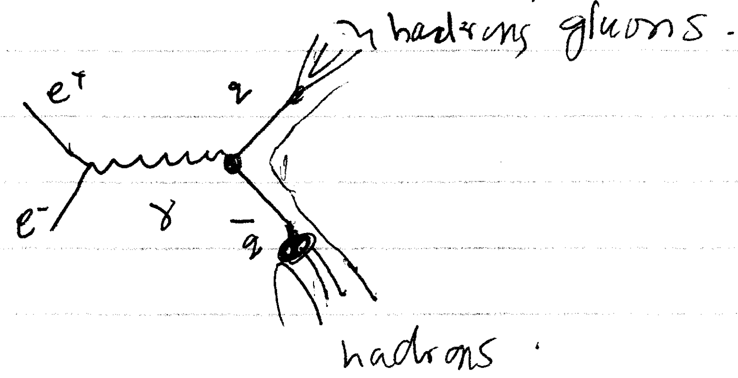


$q^2 = (k-k')^2 \qquad \nu = P\cdot q = M_p \cdot E_\gamma$ Lab.

$q^2 \sim \infty \; E_\gamma \gg m_p \quad q^2 \gg$

Deep Inelastic scattering - Evidence for partons / quarks gluons.

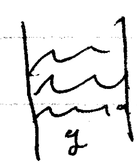
<u>Jet structure.</u>

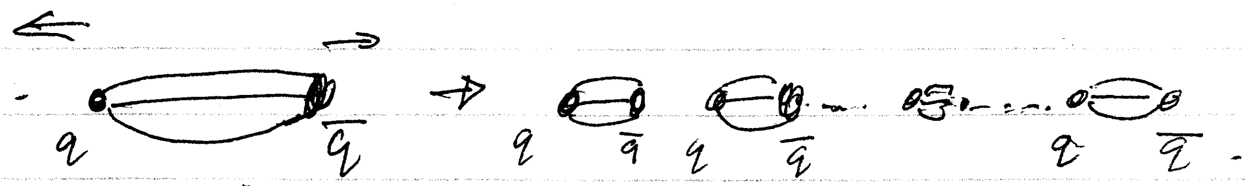


Individual quarks not seen.

Hadrons qqq (Baryons)

$q\bar{q}$ (mesons).





"Colored" gluons.

			Q
Proton / Neutron		u - up	$+2/3$
uud	udd	d - down	$-1/3$.

$\{u, d) = \{u^i, d^j\} \quad i, j = R, G, Y$ - 3 - colors.

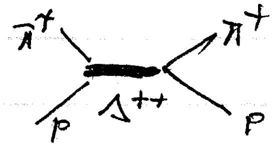
Gluons $\stackrel{3}{\underset{\bar{3}}{\rightleftarrows}}$ $3\times 3 - 1 = 8$ colors.

quarks - spin 1/2 gluons - spin one.

1 F =

Lifetimes Strong Interaction Resonance

eg $\Delta^{++} \sim uuu$



$\tau_{\Delta^{++}} \sim 10^{-23}$ sec

$10^{18}\,\text{GeV} = (10^{-33}\,\text{cm})^{-1} = (10^{-43}\,\text{s})^{-1}$

energy scale $\sim 1\,\text{GeV} \simeq$ length scale $1\,F = 10^{-15}\,\text{m}$

Time scale $= 10^{-25}\,\text{s}$.

Proton stable (?) $\tau_p \gtrsim 10^{34}$ yrs.

E-M. $\pi^0 \to \gamma\gamma$. $\tau_{\pi^0} = 10^{-16}\,\text{s}$.

Weak decays (β-decay)

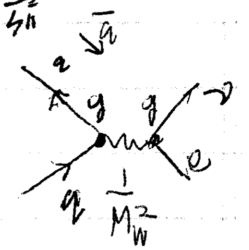
$n \to p + e^- + \bar{\nu}$ $\tau_n \sim 15$ minutes

small phase space

$\pi^- \to e^- + \bar{\nu}$ $\tau \sim 10^{-12}\,\text{s}$.

Characterized by weak coupling.

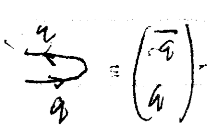
$\frac{g^2}{4\pi} \sim \alpha$



$$\alpha_W \simeq \frac{\alpha}{M_W^2/M_p^2} \sim 10^{-4}\alpha$$

$M_W \sim 80\,\text{GeV}$

$M_p \sim 1\,\text{GeV}$



Gravitational $M_p^2 G_N \simeq \frac{(1\,\text{GeV})^2}{(10^{18}\,\text{GeV})^2} = 10^{-36}$. !

Weakest interaction

Can ignore at scales $\ll 10^{18}\,\text{GeV}$. !

Effective Field Theory - Std Model.

~~All~~ SM - Effective Field Theory

- Describes degrees of Freedom (d o f)

accessible by Expts up to 1 TeV scale.

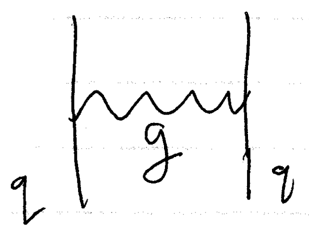
Strong + Electro Wk.

QCD. $SU(3)$

quarks + Gluons

↓

$B = \pm 1/3$ $\quad$ $B = 0$



$E \lesssim$ few GeV

↓

Strongly interacting

Effective Field Theory chiral Lagrangian

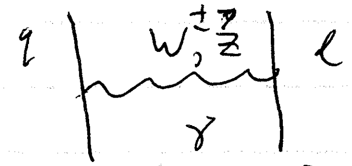
π, K, η, N

$SU(2) \times U(1)$.

quarks + leptons + $W^{\pm}, Z, \gamma$.

↓

$B = 0$ $\quad$ $l = \pm 1$



Weak: $1/q^2 = M_W^2$ $\quad$ $M_W = 80$ GeV

$E \ll 100$ GeV

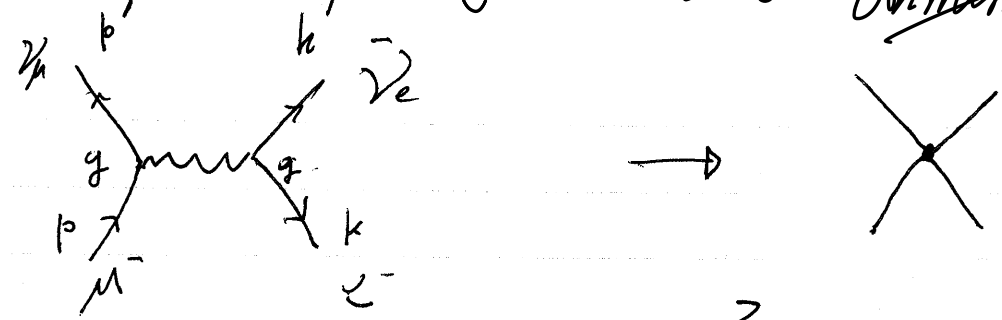
↓

Fermi-Theory.

$G_F \bar{\psi}\psi\bar{\psi}\psi$.

Break down of EFT. ~~Unitarity~~ 5.

$q_\mu \equiv (p-p')_\mu$



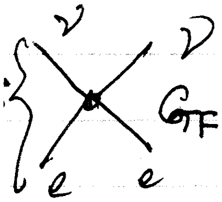
$$-g^2 \frac{1}{q^2 - M_W^2} \sim + \frac{g^2}{M_W^2}\left(1 + O\left(\frac{q^2}{M_W^2}\right)\right)$$

$$q^2 \ll M_W^2 .$$

Fermi Theory $\mathcal{L}_I \sim G_F \bar{\psi}_\mu \psi_{\nu_\mu} \bar{\psi}_e \psi_{\nu_e}$ (1930's?)

$$G_F \sim \frac{g^2}{M_W^2} \simeq 10^{-5}\,\mathrm{GeV}^{-2}$$

Fermi Theory violates unitarity.



$$\sigma \sim G_F^2 s .$$

$s = (p_e + p_\nu)^2 = (E_e + E_\nu)^2_{CM}$

Froissart Bound $\sigma \lesssim (\ln^2 s)$.

~~to~~ Effective Field Theory

valid for $s \lesssim 10^5\,\mathrm{GeV}^2$ $E_{CM} \lesssim 300\,\mathrm{GeV}$

Above this need the

W / Z Boson picture.

Turns out even this breaks unitarity

at a still higher scale — need — Higgs!

SM culmination of Expt^l + Theoretical work. spanning $\sim$ 70 - 80 yrs.

Color index suppressed

Matter. $\{q\}, \{L\}$ (Q)

Quarks ~~Baryons~~.
$B = \pm\frac{1}{3}, \ell = 0.$

$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$ $\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$

Leptons
$B = 0.$ $\ell_{e,\mu,\tau}$

$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$ $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Radiation.

E.M $\gamma \to A_\mu$

Weak $W^\pm_\mu, Z_\mu$.
$Q = 0.$

Strong. g -(gluon)

+ Anti-particle (all charges reversed),

B, ℓ, Q - conserved in all ~~std~~ SM interactions (ignoring non-perturbative tiny effects).

quarks - all interactions

$q \to \{q^R, q^g, q^r\} = \{q^i\}$ $i = 1, 2, 3.$

$g \to \{g^r\}$ $r = 1 \ldots 8,$

Leptons only weak + E-M + gravity.

Only 1st family observed in everyday life (i.e all of science except HEP!).

Why the other two?.

Key principle of SM is Gauge Invariance and Lorentz Invariance, and Q.M.

Expt

Weak Interactions ⇒ chirality.

Left and Right handedness distinguished.

Turns out consistency ~~ne~~ gives only massless particles.

⇒ Need Higgs - sector.

spontaneous symmetry breaking.

Higgs Mechanism (Abelian).

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \eta^{\mu\nu}(\partial_\mu - igA_\mu)\varphi\,(\partial_\nu + igA_\nu)\varphi^\dagger$$

$$+ V(\varphi.\varphi^\dagger) \quad ; \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu .$$

$$V = \lambda\left(\varphi^\dagger\varphi - \frac{\mu^2}{2\lambda}\right)^2 .$$

$$V_{min} = 0 \quad \text{at} \quad |\varphi| = \sqrt{\frac{\mu^2}{2\lambda}} .$$

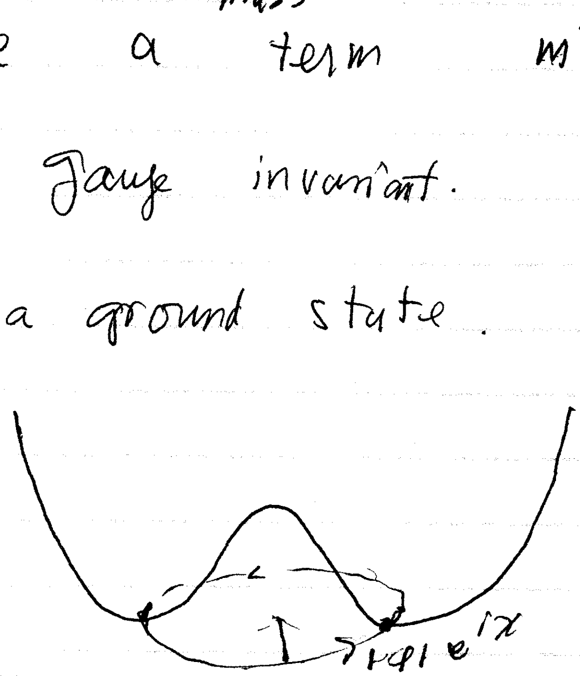
$\mathcal{L}$ invariant under.

$$\varphi(x) \to e^{i\theta(x)}\varphi(x) \qquad \left(\varphi^\dagger \to \varphi^\dagger e^{-i\theta(x)},\ \theta \text{ real}\right).$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \frac{\theta}{g}$$

Note a mass term $m^2 A_\mu A^\mu$ would not be gauge invariant.

Pick a ground state. $\phi_0 \equiv \sqrt{\frac{\mu^2}{2\lambda}}$.



Flat direction

$\chi \to \chi + \theta$

massless.

I

Rewrite $\mathcal{L}$ with $\varphi \to (\varphi_0 + \rho) e^{i g \chi}$

(gauge transformation $\chi \to \chi + \theta$)

χ - disappears from $\mathcal{L}$.

A_μ acquires a mass term.

$$g^2 A_\mu A^\mu \varphi_0^2 = \frac{g^2 \mu^2}{2\lambda} A_\mu A^\mu.$$

In the Standard Model

$W^\pm$, Z (as well as quarks and leptons) get masses from a generalization of this mechanism.

In order to get right value for the weak interaction coupling (i.e $G_F = 10^{-5} M_p^{-2}$) need $M_W \approx 80$ GeV

$\Rightarrow \varphi_0 = \sqrt{\frac{\mu^2}{2\lambda}} \sim 174$ GeV

$\Rightarrow$ (with $\lambda \sim O(1)$) $\mu \simeq 100$ GeV

Problems with S. M.

1) Cannot acount for dark matter.
2). Gauge Hierarchy problem.
   What protects Higgs mass?
3). What drives $\mu^2 < 0$? $V \sim -\mu^2 |\varphi|^2$
   ~~rise to ondent~~
4) Generating neutrino masses naturally.
5) Why 3 - generations?
6) Gauge group and particle reps.
   — is there a more fundamental explanation.
7) What fixes Yukawa couplings CKM matrix.
   (i.e. $\sigma_{ij}\ \varphi\ \bar{\psi}_i\ \psi_j$)
8). How to couple to gravity
   in Q.M. theory?
9) C.C.

5–9 Have no solution yet.

SUSY gives a solution to 1) to 3)
and also gives grand Unification.

Lec. 1. (Lie) ①

Elements of ~~Y~~ Group Theory.

Simplest example (continuous) group

Rotations in 2 d.

$$x \to x' = x\cos\theta + y\sin\theta$$
$$y \to y' = -x\sin\theta + y\cos\theta.$$

i.e. $\underline{x} \to \underline{x}' = \underline{O}\,\underline{x}$.

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \begin{array}{l} OO^T = 1 \\ \det O = 1 \end{array}$$

Clearly $O(\theta_1)O(\theta_2) = O(\theta_1+\theta_2)$

Product of rot^{ns} is also a rot^{n}.

Also $O(0) = \underline{1}$ (identity) $\qquad O(-\theta) = O^{-1}(\theta)$. (inverse)

and $O(\theta_1)(O(\theta_2)O(\theta_3)) = (O(\theta_1)O(\theta_2))O(\theta_3)$

– associativity,

This is the group $SO(2)$

Equivalently write $z = x+iy$.

Above transformation are $z \to e^{i\theta}z$.

This is the group $U(1)$

Clearly $U(1) \simeq SO(2)$ Leaves $x^2+y^2 = zz^*$ invariant.

Def^n of a Group.

A Group G is a set of elements $(g_1, g_2 \ldots)$ with a composition (multiplication) law which satisfies the following conditions.

a) $g_i, g_j \in G$ then $g_i g_j \in G$.

b) $(g_i \cdot g_j) g_k = g_i (g_j g_k)$

c) $\exists$ an identity element e s.t $e.g = g.e = g \ \forall g \in G$

d) $\exists$ $g_i^{-1} \in G$ for every $g_i \in G$ s.t $g_i^{-1} g_i = g_i g_i^{-1} = e$.

Clearly our examples $SO(2)$, $U(1)$ satisfy these.

Simplest <u>discrete</u> group $(-1, +1)$ under multiplication. $\mathbb{Z}_2$.

Cyclic group of order n. $a, a^2, \ldots a^n = e$ $\mathbb{Z}_n$.
(eg $a = e^{2\pi i/n}$

All these examples are <u>Abelian</u>.

i.e $g_i g_j = g_j g_i$ for any pair of elements.

Permutation group S_n - non-abelian finite gp.

<u>Subgroup</u> $H \subset G$ is a (proper) subgroup if H is a group.

If G, G' are two groups

$G = \{g_i\}$ $\quad G' = \{g'_i\}$

<u>defn</u>: Then $G \times G' = \{g_i, g'_j\}$ Direct product.

Such that $g_i g'_j \cdot g_k g'_l = (g_i g_k)(g'_j \cdot g'_l)$

Standard model group $SU(3) \times SU(2) \times U(1)$

Invariant subgroup $N \subset G$ s.t
(normal)

if $n \in N$ then $g n g^{-1} \in N \;\; \forall g \in G$

If a group has no invariant subgroups then it is a <u>simple group</u>

eg $SU(n)$.
($U(n) \simeq SU(n) \times U(1)$ is not ~~un~~ simple)

A group with no <u>Abelian</u> invariant subgroups is <u>semi-simple</u>.

Lie Groups.

We will be interested for the most part in <u>continuous</u> groups. i.e group elements are paramatrized by a set of (real) parameters which vary continuously over a region in $\mathbb{R}^n$.

Eg. $SO(2) \simeq U(1)$ - Abelian. One angle
$SO(3) \simeq SU(2)$ non-abelian.
- 3 Euler angles.

$$g(\underline{\theta}) \qquad \underline{\theta} = (\theta^1, \ldots, \theta^n); \quad g(0) = e.$$

$$g(\underline{\theta})\, g(\underline{\varphi}) = g(\underline{\xi}(\underline{\theta}, \underline{\varphi}))$$

$$(\underline{\theta}, \varphi) \to \xi \doteq \xi(\underline{\theta}, \underline{\varphi}) \qquad \text{map } S \times S \to S$$

S is parameter space $\quad \underline{\theta} \in S$.

n - dimension of parameter space
— ~~dimension of Lie group~~

Note $\xi(\theta, 0) = \xi(0, \theta) = \theta$

Associativity $\Rightarrow \quad \xi(\theta, \xi(\varphi, \psi)) = \xi(\xi(\theta, \varphi), \psi)$

A <u>Lie Group</u> is a continuous group with the composition Law defined by a continuously differentiable map. (ie ξ is continuously differentiable)

<u>def:</u> <u>Generator</u>. $X_i(\underline{\theta}) \equiv i^{-1} g^{-1}(\underline{\theta})\, \partial_i\, g(\theta) \quad i=1,\dots n.$

$g = e^{iX_i\theta^i}$

$g^{-1} = e^{-iX_i\theta^i}$

$\partial_i g = e^{-iX\cdot\theta} X_i e^{+iX\cdot\theta}$

$\{X_i\}$ generators of Lie Group.

For $\underline{\theta}$ infinitesimal (i.e near the identity)

write $\quad g(\underline{\theta}) = e + i \sum_j \theta^j X_j(0) \Rightarrow g^{-1}(\theta) = e - i \sum_j \theta^j X_j(0)$

(for $SO(2)$ $\quad X = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$)) $\quad$ Clearly $X_i = i^{-1} g^{-1} \partial_i g$

Product $\quad g(\underline{\varphi})\, g(\underline{\theta})\, g^{-1}(\underline{\varphi})\, g^{-1}(\underline{\theta})$ — <u>commutator</u> of two elements,

Must be a group element $g(\xi)$ $\quad \xi^i = f^i(\theta, \varphi)$.

with $f^i(0,\varphi) = f^i(\theta, 0) = 0$

Near identity $\quad \xi^i = A^i + B^i{}_j\theta^j + B'^i{}_j\varphi^j + C'^i{}_{jk}\theta^j\varphi^k + C''^i{}_{jk}\theta^j\theta^k + C'''^i{}_{jk}\varphi^j\varphi^k$

From boundary conditions $\quad A = B = B' = C' = C'' = 0$

ie. $\quad \xi^i = c^i{}_{jk}\, \theta^j \varphi^k$

$$g(\varphi)\, g(\theta)\, g^{-1}(\varphi)\, g^{-1}(\theta) = (e + i\varphi^j X_j + \dots)(e + i\theta^k X_k \dots)$$
$$\times\ (e - i\varphi^l X_l)(e - i\theta^m X_m \dots)$$

$$= (e + i\varphi^j X_j + i\theta^k X_k - \varphi^j\theta^k X_j X_k + \dots)$$
$$(e - i\varphi^j X_j - i\theta^k X_k - \varphi^j\theta^k X_j X_k + \dots)$$

$$= (e + \varphi^j\theta^k X_j X_k + \varphi^j\theta^k X_k X_j$$
$$- 2\varphi^j\theta^k X_j X_k \quad \dots)$$

$$= e + \theta^j\varphi^k [X_j, X_k] + \dots$$

$$= g(\xi) = e + i c^l{}_{jk}\, \theta^j\varphi^k X_l$$

So

$$[X_j, X_k] = i c^l{}_{jk} X_l \qquad — (A)$$

$$\boxed{c^l{}_{jk} = -c^l{}_{kj}}$$

structure constants.

From the Jacobi identity

$$[X_j,[X_k,X_l]] + [X_k,[X_l,X_j]] + [X_l,[X_j,X_k]] = 0$$

$$c^m{}_{jk} c^n{}_{lm} + c^m{}_{kl} c^n{}_{jm} + c^m{}_{lj} c^n{}_{km} = 0. \qquad —(J)$$

The vector space ~~whose basis~~ spanned by $\{X^i\}$ satisfying (A) is called the Lie algebra h. of the group G.

* The structure constants $c^l{}_{jk}$ specify the Lie group

Introduce a symmetric bilinear form $h \times h \to R$

$$ -\langle X_i, X_j \rangle = g_{ij} \qquad g_{ij} = g_{ji} $$

$$ C_{ijk} \equiv g_{il} C^l{}_{jk} = C^l{}_{jk} \langle X_i, X_l \rangle $$

$$ = \langle X_i, [X_j, X_k] \rangle $$

The dimension of a Lie group (~~Algebra~~) is the dimension of its Algebra. (i.e # of linearly independent generators)

The rank of a Lie group (Algebra) is the ~~maximal~~ # of commuting generators $[X_i, X_j] = 0$.

This Abelian sub algebra is called the Cartan subalgebra of the group

Representations (faithful).

Realization of group elements in terms of matrices $\quad g \to D(g) \quad$ $n \times n$ matrix.

s.t. $\quad g_i g_j \to D(g_i g_j) = D(g_i) D(g_j)$.

$D(g^{-1}) = D^{-1}(g) \qquad D(e) = \underset{\substack{\uparrow \\ \text{unit } n\times n \\ \text{matrix}}}{I}$.

If $\exists$ M a non singular matrix such that
$$M D M^{-1} = \begin{bmatrix} D_1(g) & & 0 \\ 0 & D_2(g) & \\ & & \ddots \end{bmatrix}$$
for all $g \in G$. $\quad$ block diagonal.

then rep is <u>reducible</u>.

$$D(g) = D_1(g) \oplus D_2(g) \oplus \cdots$$

If this cannot be done — rep is <u>irreducible</u>.

$D(g)$ — linear <u>ops</u> on a vector space (of $n\times 1$ column vectors) $\quad \underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$\underline{v} \to D(g)\,\underline{v}$. $\qquad \underline{v} \in V$

dim^n of V (n here) is the dim^n of <u>rep</u>.

A <u>unitary rep</u> is such that

$$D(g)\,D^{\dagger}(g) = D^{\dagger}(g)\,D(g) = 1 .$$

<u>Unitary groups</u> are defined in terms of their <u>defining</u> unitary ~~d~~ <u>rep</u>. eg $SU(n)$ <u>defined</u> as the group of special (i.e $\det = 1$) unitary matrices $(n \times n)$.

If $D(g)$ is rep so is $D^*(g)$

$$D(g_1)\,D(g_2) = D(g_1 g_2) \quad \Rightarrow \quad D^*(g_1)\,D^*(g_2) = D^*(g_1 g_2)$$

T - rep matrix in D rep. Write $D(g(\theta)) = e^{i\,\underline{\theta}.\underline{T}} \simeq 1 + i\,\underline{\theta}.\underline{T}$ for $\theta \ll 1$.

Note if D is <u>unitary</u> T is <u>Hermitian</u> (i.e $T^{\dagger} = T$).

From $\exp^n$ near identity it is clear that $X_i^{\phi} \to T_i^{\phi}$

and so $[T_i, T_j] = i\,C^k{}_{ij}\,T_k$

The structure constants $c^i{}_{jk}$ define the adjoint rep.

$$c^i{}_{jk} = i\,(T_j)^i{}_k .$$

From the Jacobi identity p 6 - 7.

we see that these T's satisfy the Lie Alg.

The metric can be defined by

$$g_{ij} = \langle T_i , T_j \rangle = C(1)\, \mathrm{tr}\, T_i T_j$$

for T_i in the defining rep.

g_{ij} is a real symmetric + ve definite matrix. It can be diagonalised by an orthogonal transformation - eigen values are + ve.

$A' \subset A$ is a subalgebra of the Lie algebra if A' is also an Algebra.

eg. $SU(2) \subset SU(3)$ is a subalgebra of $SU(3)$.
(we use same not^n for group and algebra).

Classification of Lie Algebras.

(compact simple finite dimensional)

Parameter space is compact and finite dimen^l.

We have finite dimensional Unitary reps of such Lie Algebras (groups).

Algebra has finite # of linearly independent generators - represented by Hermitian matrices.

If there is no generator which commutes with all other generators then the algebra is semi-simple.

If in addition the generators $\{T_i\}$ cannot be split into two mutually commuting sets - the algebra is simple.

Cartan-Killing classification
All simple compact finite dimensional Lie algebras fall into one of following:

Cartan Classification of Lie Groups/Algebras (compact finite dim^l)

i) Unitary groups. $G = SU(n)$ $n \in \mathbb{Z}^+$.

dim

$n \times n$ Unitary matrices $\det \underline{U} = 1$.

$\underline{U} \in G$

$\dim G = n^2 - 1$

$$\underline{U}(\theta) = e^{i \sum_{i=1}^{n^2-1} \theta^i T_i}$$

Action on $\underline{\psi} \xrightarrow{G} \underline{U}\,\underline{\psi}$.

Leaves $\underline{\psi}^\dagger \underline{\psi}$ Invariant $\underline{\psi} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$

Classical groups

Complex n-vector.; Note phase transformation $\psi \to e^{i\phi}\psi$ also leaves $\psi^\dagger\psi$ invariant

2) Special Orthogonal groups $SO(2n)$, $SO(2n+1)$, $n \in \mathbb{Z}^+$

$\dim G = \frac{m(m-1)}{2}$.

$\underset{m \times m}{\underline{O}} \in G$ $\underline{O}\,\underline{O}^T = \underline{O}^T \underline{O} = 1$ $\det O = +1$.

$m = 2n, 2n+1$

Leaves $\phi^T \phi$ Invariant under $\phi \to O\phi$.

$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix}$ ϕ - real

3) Simplectic Groups. $Sp(2n)$. $n \in \mathbb{Z}^+$

$\dim G = \frac{n(2n+1)}{2}$

- Leaves invariant simplectic form

$$\sum_{i=1}^{n} (p_i q_i' - q_i p_i') = (p_i, q_i) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_i' \\ q_i' \end{pmatrix}$$

4) Exceptional Groups/Algebras

\# refer to rank.

G_2, F_4, E_6, E_7, E_8 ← occurs in string theory. (E_6 - GUT group!)

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Note

$u^\dagger u = 1$

$\Rightarrow T_i^\dagger = T_i$

Note

$\det U = 1$

$\Rightarrow \mathrm{Tr}\, T_i = 0$

T_i – generators

Examples

$SU(2)$ – 2×2 Unitary matrices $\{u\}$.

with $\widehat{\det}\, u = 1$

$$U(\theta_1, \theta_2 \theta_3) = e^{i \sum_{i=1}^{3} \theta^i T_i} \qquad T_i = \frac{\sigma_i}{2}.$$

$$[T_i, T_j] = i\epsilon_{ijk} T_k \qquad \mathrm{Tr}\, \sigma_i \sigma_j = 2\delta_{ij}$$

$SU(3)$:

3×3 unitary ... $\det U = 1$.

$i = 1 \ldots 8$. $$U(\theta^i) = e^{i \sum_{i=1}^{\infty} \theta^i T_i} \qquad T_i = \lambda_i / 2.$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Normalisation $\quad \mathrm{Tr}\, \lambda_i \lambda_j = 2\delta_{ij}$.

$$[T_i, T_j] = i f_{ijk} T_k \qquad T_i = \lambda_i/2.$$

Rank: # of commuting generators

$SU(2)$ rank 1; $SU(3)$ rank 2. (T_3, T_8).

Normalization of generators.

For a compact group there are finite dimensional unitary irreps.

T_i - generator in such a rep.

$$D_{ij} \equiv \mathrm{tr}\,(T_i T_j)$$

is an symmetric positive definite matrix. By an orthogonal transformation and rescaling can find a basis of generators such that

$$D_{ij} = C(r)\, \delta_{ij} \equiv C(r)\, g_{ij}$$

$C(r)$ is a representation dependent constant.

i.e choose metric $g_{ij} = \delta_{ij} = \frac{1}{C(r)} \mathrm{tr}(T_i T_j)$

to raise lower indices. ~~So~~ ~~position~~

Note once basis chosen such that $D_{ij} = \mathrm{tr}\, T_i T_j \propto \delta_{ij}$ in any irrep $D_{ij} \propto \delta_{ij}$ in all irreps

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$$[T_i^{(r)}, T_j^{(r)}] = i\, C_{ij}{}^{\ell}\, T_\ell^{(r)}.$$

$$C_{ijk} \equiv \delta_{k\ell}\, C_{ij}{}^{\ell}$$

$$\mathrm{tr}\left\{ [T_i^{(r)}, T_j^{(r)}]\, T_k^{(r)} \right\} = i C_{ij}{}^{\ell}\, \mathrm{tr}\left(T_k T_\ell \right)$$

$$= i\, C_{ij}{}^{\ell}\, C(r)\, \delta_{k\ell} = i\, C_{ijk}\, C(r)$$

i.e $$C_{ijk} = -i\, \mathrm{tr}\left\{ [T_i^{(r)}\, T_j^{(r)}]\, T_k^{(r)} \right\} / C(r).$$

– Note C_{ijk} is totally anti-symmetric

Conjugate rep. for every irrep r $\exists$

a conjugate rep $\bar{r}$ ~~(sometimes $r = \bar{r}$ real rep.) or $r = S\bar{r}S^{-1}$ pseudo real)~~

$$\underline{\phi} \Rightarrow (1 + i\theta^i T_i^{(r)})\, \underline{\phi} \qquad \underline{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$$

$$\underline{\phi}^* \rightarrow (1 - i\theta^i (T_i^{(r)})^*)\, \phi^*$$

$$T_i^{(\bar{r})} = -\left(T_i^{(r)}\right)^* = -T_i^{(r)T}$$

Since $\underline{\phi}^\dagger \underline{\phi} = (\underline{\phi}^*)^T \underline{\phi}$ is invariant.

$r \times \bar{r} \supset I$: identity rep.

If there is $\exists$ U such that $T^{(\bar{r})} = U T^{(r)} U^\dagger$

Then representation is (pseudo) real.

If $q, \chi \in r \Leftrightarrow \bar{r}$ $\exists$

a matrix G_{ab} (either symm or anti-symm $G_{ab} = \pm G_{ba}$).

such that $q^a G_{ab} \chi^b$ is invariant.

In SU(2) If $\underline{U}, \underline{V}$ = spin one (vector) ($s=1$) rep

Then $u^i \delta_{ij} v^j$ invariant.

If ψ, χ are in spinor rep.

$\psi^\alpha \epsilon_{\alpha\beta} \chi^\beta$ — invariant. $\epsilon_{\alpha\beta} = \pm 1$, $\epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}$.

Fundamental (defining) rep.

For SU(n) $\underline{\psi}_n = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ complex vector

$u \in G$ are $n \times n$ unitary matrices, all $\det u = 1$

$\psi \to u\psi$. $u \neq u^*$ so $r \neq \bar{r}$

Representation complex for $n > 2$.

For SU(2) the $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ rep is actually pseudo real.

Casimir Operator

$$T^2 \equiv \sum_{i=1}^{\dim G} T_i T_i$$

$$[T^2, T_i] = 0 \quad \text{for any generator.}$$

So $\quad T^{2\,(r)} = C_2(r)\, I \quad$ — (A)

unit matrix. ($\dim r \times \dim r$).

$C_2(r)$ - quadratic Casimir - depends on rep.

For the adjoint rep. $\quad c^i{}_{jk} = \delta^{il} c_{ljk} = i(T_j)^i{}_k$

$$c_{ijk}\, c_{ljk} = C_2(G)\, \delta_{il}$$

Casimir of G. i.e of adj rep!

Recall $\quad \mathrm{tr}(T_i T_j) = C(r)\, \delta_{ij} \quad$ — (B).

$i, j = 1, \ldots \dim G$.

(A) & (B) $\Rightarrow \quad d(r)\, C_2(r) = \dim G\; C(r) \quad$ — (C)

In SU(N) groups you always embed the SU(2) generators as elements of SU(N) basis

$\begin{pmatrix} \sigma^i/2 & 0 \cdots 0 \\ \vdots & 0 \end{pmatrix}$ So take in fundamental

$$\mathrm{tr}\, T_i^{(n)} T_j^{(n)} = \frac{1}{2}\delta_{ij}$$

$\Rightarrow C(n) = \frac{1}{2}$.

So from (C) $\quad C_2(n) = \frac{(n^2-1)}{2n}$

Basics of Spinors - $d=3+1$.

$c=1$ $\quad \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. $\eta_{\mu\nu} dx^\mu dx^\nu \equiv ds^2 = dt^2 - dx^{1\,2} - dx^{2\,2} - dx^{3\,2}$

$V_\mu = \eta_{\mu\nu} V^\nu$ $\quad$ Inhomogeneous Lorentz (Poincare) transformations.

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Inf: $x^\mu \to x'^\mu = (\underbrace{\delta^\mu_\nu + \omega^\mu{}_\nu}_{\text{Lorentz}}) x^\nu + \underbrace{a^\mu}_{\text{translations}}$

$\Lambda^\mu{}_\nu \Lambda^\nu{}_\sigma = \delta^\mu_\sigma \Rightarrow$ $\quad \omega_{\mu\nu} = -\omega_{\nu\mu}$.

Translations generated by / Rotations + Boosts

$$U(a) = e^{ia^\mu P_\mu} \simeq I + ia^\mu P_\mu + O(a^2); \quad U(\Lambda) = e^{i\omega^{\mu\nu} M_{\mu\nu}} \simeq I + i\omega^{\mu\nu} M_{\mu\nu} + \cdots$$

Lorentz transformations. $\quad \Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \omega^\mu{}_\nu + O(\omega^2)$

Poincare algebra.

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\lambda] = i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho})$$

Represented on scalar fields

$$(\varphi(x) \to \varphi'(x') = \varphi(x) \text{ for } x \to x' = \Lambda x)$$

$\partial_\mu \equiv \partial/\partial x^\mu$.

by $\quad P_\mu = i\partial_\mu \qquad M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$.

On vector fields. $A_\mu \to A'_\mu(x) = \Lambda_\mu{}^\nu A_\nu(x)$.

$$M_{\mu\nu} = -i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}$$

$$(\Sigma_{\mu\nu})^{\lambda\sigma} = \delta_\mu^\lambda \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\lambda .$$

On spinor fields.

$$M_{\mu\nu} = -i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu},$$

$$\Sigma_{\mu\nu} \equiv \frac{i}{4}\,[\gamma_\mu, \gamma_\nu].$$

$\gamma^{0^2} = I$, E'v's ± 1

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I .$$

$\gamma^{i^2} = -I$, E'v's $\pm i$ (Anti-Hermitian)

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Explicit Representation (Weyl).

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

$\mu = 0,1,2,3$.
$i = 1,2,3$

$$\sigma^\mu \equiv (I, \sigma^i) \qquad \bar\sigma^\mu \equiv (I, -\sigma^i)$$

Pauli: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$

$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$$\sigma^{\mu\nu} \equiv \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right), \quad \bar\sigma^{\mu\nu} \equiv \frac{i}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right) = (\sigma^{\mu\nu})^\dagger$$

$\dagger$ = Hermitian conjugation

Note $\sigma_\mu \equiv \eta_{\mu\nu}\sigma^\mu = \bar{\sigma}^\mu$, $\bar{\sigma}_\mu = \eta_{\mu\nu}\bar{\sigma}^\nu = \sigma^\mu$

$$\mathrm{Tr}\left(\sigma^\mu \bar{\sigma}^\nu\right) = 2\eta^{\mu\nu}$$

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} I = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu.$$

$$\mathrm{Tr}\left(\sigma^{\mu\nu}\sigma^{\alpha\beta}\right) = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}\right) + \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}.$$

$$\epsilon_{\mu\nu\alpha\beta} = \pm 1 \qquad \epsilon_{0123} = +1.$$

$$\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} = 2i\sigma^{\mu\nu} \qquad \epsilon^{\mu\nu\alpha\beta}\bar{\sigma}_{\alpha\beta} = -2i\bar{\sigma}^{\mu\nu}$$

$$\Sigma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & \\ & \bar{\sigma}_{\mu\nu} \end{pmatrix}$$

$$\sigma^{ij} = \frac{i}{4}\left(\sigma^{i}\bar{\sigma}^{j} - \sigma^{j}\bar{\sigma}^{i}\right)$$

$$= -\frac{i}{4}\left(\sigma^{i}\sigma^{j} - \sigma^{j}\sigma^{i}\right)$$

$$= -\frac{i}{4}\cdot 2i\,\epsilon^{ijk}\sigma^{k}$$

$$= \frac{1}{2}\,\epsilon^{ijk}\sigma^{k}$$

$$\sigma^{0i} = \frac{i}{4}\left(\sigma^{0}\bar{\sigma}^{i} - \sigma^{i}\bar{\sigma}^{0}\right)$$

$$= \frac{i}{4}\left(I\,(-\sigma^{i}) - \sigma^{i} I \right)$$

$$= \frac{i}{4}\left(-2\sigma^{i}\right) = -\frac{i}{2}\sigma^{i}.$$

$$\mathrm{Tr}\left(\sigma^{ij}\sigma^{kl}\right) = \mathrm{Tr}\left[\frac{1}{2}\epsilon^{ijm}\sigma^{m}\ \frac{1}{2}\epsilon^{kln}\sigma^{n}\right]$$

$$= \frac{1}{4}\,\epsilon^{ijm}\,\epsilon^{kln}\ \mathrm{Tr}\,\sigma^{m}\sigma^{n}$$

$$= \frac{1}{4}\,\epsilon^{ijm}\,\epsilon^{kln}\ 2\,\delta^{mn}$$

$$= \frac{1}{2}\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right) = \frac{1}{2}\left(\eta^{ik}\eta^{jl} - \eta^{il}\eta^{jk}\right)$$

$$\mathrm{Tr}\left(\sigma^{0i}\sigma^{kl}\right) = \mathrm{Tr}\left(\left(-\frac{i}{2}\sigma^{i}\right)\frac{1}{2}\epsilon^{klm}\sigma^{m}\right)$$

$$= -\frac{i}{4}\,\epsilon^{klm}\ 2\delta^{im} = -\frac{i}{2}\,\epsilon^{ikl}$$

$$= \frac{i}{2}\,\epsilon^{0ikl}$$

$$\mathrm{Tr}\left(\sigma^{0i}\sigma^{0j}\right) = \mathrm{Tr}\left[\left(-\frac{i}{2}\sigma^{i}\right)\left(-\frac{i}{2}\sigma^{j}\right)\right] = -\frac{1}{4}\,2.\delta^{ij} = \frac{1}{2}\eta^{ij}$$

Note: $SO(3,1) \simeq SU_+(2) \times SU_-(2)$ Locally.

$\vec{J}$ - A.M

$J_i \equiv \frac{1}{2}\epsilon_{ijk} M_{jk}$. $K_i \equiv -M_{0i}$

check!

$\mathcal{J}_i^{\pm} \equiv \frac{1}{2}(J_i \pm i K_i)$. (HT notation $\mathcal{J}_i^+ = S_i$, $\mathcal{J}_i^- = T_i$)

$[J_i, J_j] = i\epsilon_{ijk} J_k$

$[J_i, K_j] = i\epsilon_{ijk} K_k$

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

Casimirs $\mathcal{J}^{\pm 2}$

$\to j_\pm (j_\pm + 1)$

Note $\mathcal{J}^2$ not a Casimir! since $[K, J^2] \neq 0$

Can write Lorentz algebra ($\mathcal{L}$) as

$$[\mathcal{J}_i^{\pm}, \mathcal{J}_j^{\pm}] = i\epsilon_{ijk}\mathcal{J}_k^{\pm} \qquad \pm \to {}^R_L ?$$

$$[\mathcal{J}_i^+, \mathcal{J}_j^-] = 0.$$

i.e it is (as a Lie Algebra) the direct sum of two SU(2) Algebras.

So any Rep of ($\mathcal{L}$) can be written as (j_-, j_+)

j_1 "A.M" of $SU_-(2)$ "A M" of $SU_+(2)$.

$j_{+,-} = (0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ etc).

Lowest non trivial Reps $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$

$(\frac{1}{2}, 0)$ is Left chiral (ie $P_L = \frac{1}{2}(1-\gamma_5)$) projection of the Dirac spinor

$(0, \frac{1}{2})$ is Right chiral (ie $P_R = \frac{1}{2}(1+\gamma_5)$) proj^n.

<u>check</u>

$$g^+_i = \tfrac{1}{2}(J_i + iK_i) = \begin{pmatrix} \sigma_i/2 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$g^-_i = \tfrac{1}{2}(J_i - iK_i) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i/2 \end{pmatrix}.$$

$$J_i = \tfrac{1}{2}\epsilon_{ijk} M_{jk} \qquad K_i = -M_{0i}$$

$$= \tfrac{1}{2}\epsilon_{ijk} \begin{pmatrix} \sigma_{jk} & \\ & \bar{\sigma}_{jk} \end{pmatrix} \qquad \cancel{\tfrac{1}{2}\tfrac{i}{4}\epsilon_{ijk}\epsilon_{jkl}\sigma^l} \quad \cancel{= \tfrac{i}{2}\sigma}$$

$$= \begin{pmatrix} \sigma_i/2 & \\ & \sigma_i/2 \end{pmatrix} \qquad \bar{\sigma}_i = -\sigma_i$$

$$K_i = -M_{0i} = -\tfrac{i}{4}\begin{pmatrix} \bar{\sigma}_i - \sigma_i & 0 \\ 0 & \sigma^i - \bar{\sigma}^i \end{pmatrix}$$

$$= +\tfrac{i}{2}\begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix}$$

$$g^+ = \tfrac{1}{2}(J_i + iK_i) = \tfrac{1}{2}\begin{pmatrix} \sigma_i/2 & \\ & \sigma_i/2 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} \sigma_i/2 & \\ & -\sigma_i/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i/2 \end{pmatrix}$$

$$g^- = \begin{pmatrix} \sigma_i/2 & 0 \\ 0 & 0 \end{pmatrix}. \qquad \sigma_{0i} = \tfrac{i}{4}(\sigma_0\bar{\sigma}_i - \sigma_i\bar{\sigma}_0)$$
$$= -\tfrac{i}{2}\sigma_i$$

$$\sigma_{\mu\nu} = \tfrac{i}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu) \qquad \sigma_{jk} = -\tfrac{i}{4}(\sigma_j\sigma_k - \sigma_k\sigma_j) = -\tfrac{i}{4}\, 2i\epsilon_{jkl}\sigma_l$$
$$= +\tfrac{1}{2}\epsilon_{jkl}\sigma_l, \quad \tfrac{1}{2}\epsilon_{ijk}\epsilon_{jkl}\left(\tfrac{1}{2}\right)\sigma_l = \frac{\sigma_i}{2}$$

Restricting to
In two components (2×2 matrices)

We may represent (in $(\frac{1}{2},0)$ rep),
(j_-, j_+).

$$J_{-i} = \frac{1}{2}\sigma_i \qquad J_{+i} = 0. \quad (\text{left}, \text{right}).$$

$$\Rightarrow \quad J_i = \frac{1}{2}\sigma_i \qquad K_i = \frac{i}{2}\sigma_i$$

So a (2-component) Weyl spinor ψ_{LA}

transforms as

A: $$\psi_L \to e^{i\frac{\vec{\sigma}}{2}\cdot\vec{\theta}}\psi_L \qquad \psi_L \to e^{-\frac{1}{2}\vec{\sigma}\cdot\vec{\eta}}\psi_L.$$

($\vec{\eta}$ is rapidity $|\vec{\eta}| = \tanh^{-1}\beta$
$\beta = V$)

(ψ_L has spin $\frac{1}{2}$).

Majorana mass term (~~Min~~ simplest fermion mass term).
(single Weyl spinor ψ_L).

$$\Delta\mathcal{L} = \frac{1}{2}m\left(\psi_L^T \epsilon \psi_L + h.c.\right) \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \frac{1}{2}m\left(\psi_A \epsilon^{AB}\psi_B + h.c.\right) \qquad = [\epsilon^{\alpha\beta}]$$

Let L-transform be $\psi_L \to M\psi_L$ (see (A)).

$$\psi_L^T\epsilon\psi_L \to \psi_L^T M^T \epsilon M \psi_L$$
$$= \psi_L^T \epsilon \psi_L$$

~~Since~~ $M^T_{\alpha\beta}\,\epsilon^{\beta\gamma}\,M_{\gamma\delta}$ $\det M = 1$
$= M_{\beta\alpha} M_{\gamma\delta}\epsilon^{\beta\gamma} = \det M\,\epsilon_{\alpha\delta}$

Note $\quad \psi_A \to M_A{}^B \psi_B$.

$$\psi_A \epsilon^{AB} \psi_B \to M_A{}^C \psi_C \epsilon^{AB} M_B{}^D \psi_D.$$

$$= \epsilon^{AB} M_A{}^C M_B{}^D \psi_C \psi_D.$$

$$= \epsilon^{CD} \det M \quad \psi_C \psi_D = \psi_C \epsilon^{CD} \psi_D.$$

since $\det M = 1$.

So Δh is invariant.

Note such a term cannot be written down if ψ_L is in an complex Rep of an internal symm. gp. (eg. - a $U(1)$ charge)

i.e if $\psi_L \to U \psi_L \qquad U^* \neq U$,

$$\psi_L^T \epsilon \psi_L \to \psi_L^T U^T \epsilon U \psi_L = \psi_L^T \epsilon U^T U \psi_L.$$

$$U^T = U^{+*} = U^{-1*} \neq U^{-1} \text{ if } U^* \neq U.$$

Note ϵ and U act on different indices in ψ $\psi_A^{\dot{i}}$

If Rep is real i.e $U^* = U$ then $\exists$ a Majorana mass.

If a fermion has a Majorana mass - then its anti-particle is equivalent to itself

- Majorana fermion.

- cannot be in a complex Rep

in particular cannot have an E.M. charge.

~~Suppose~~

Dirac Mass.

Suppose $\psi_L \to U \psi_L \qquad U^* \neq U.$

and. χ_L (in $(\frac{1}{2}, 0)$) transforms

under $\chi_L \to U^* \chi_L$.

Then clearly $\Delta \mathcal{L} = m (\chi_L^T \epsilon \psi_L + h.c.).$

is an Lorentz invariant mass term invariant

under the internal symmetry group.

$\psi_L \to U \psi_L \qquad \chi_L^T \to \chi_L^T U^{*T} = \chi_L^T U^+ \, , \quad U^+ U = 1.$

If a fermion can have a Dirac mass - then it is a Dirac fermion - but it consists of two Weyl fermions (ψ_L, χ_L) - hence 4 - complex - components (before imposing Dirac eqn!)

We can now write the Dirac fermion in - in 4-component notation as $\psi = \begin{pmatrix} \psi_L \\ \epsilon \chi_L^* \end{pmatrix}$. $\epsilon = i\sigma_2$

* Check: $\epsilon \chi_L^*$ - transforms as ψ_R.

$$\Delta \mathcal{L} = -m\bar{\psi}\psi = -m(\psi_L^\dagger, \chi_L^{*\dagger}\epsilon^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \epsilon\chi_L^* \end{pmatrix}$$

$\epsilon^\dagger = -\epsilon$

$$= m\left(\chi_L^T \epsilon \psi_L - m\,\psi_L^\dagger \epsilon \chi_L^*\right) \qquad \begin{pmatrix} \epsilon\chi_L^* \\ \psi_L \end{pmatrix}$$

$$= m\left(\chi_L^T \epsilon \psi_L + h.c.\right).$$

$$P_{\substack{L\\R}} = \tfrac{1}{2}(1 \mp \gamma_5)$$

Note $\quad P_L\psi = \begin{pmatrix}\psi_L\\ 0\end{pmatrix} \qquad P_R\psi = \begin{pmatrix}0\\ \epsilon\chi_L^*\end{pmatrix}$

or $\begin{pmatrix}0\\ \chi_R\end{pmatrix}$.

(Sometimes we may simply identify in the 4-component notation $\psi_{\substack{L\\R}} \equiv P_{\substack{L\\R}}\psi$;

and $\quad \psi = \psi_L + \psi_R$.)

A Majorana spinor is a 4-component spinor built of one Weyl spinor.

$$\psi_M = \begin{pmatrix}\psi_L\\ \epsilon\psi_L^*\end{pmatrix}$$

$$\Delta\mathcal{L}_M = -\frac{m}{2}\bar{\psi}_M\psi_M = \frac{m}{2}\left(\psi_L^T\epsilon\psi_L + h.c.\right)$$

as before. $\qquad$ check!

Charge conjugation matrix

$\epsilon = i\sigma_2$

$\epsilon^2 = -\sigma_2^2 = -1$

$\psi = \begin{pmatrix} \psi_L \\ \epsilon \chi_L^* \end{pmatrix}$

$$C \equiv \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

$$\psi^c \equiv C \gamma^0 \psi^* = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L^* \\ \epsilon \chi_L \end{pmatrix}$$

$$= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \epsilon \chi_L \\ \psi_L^* \end{pmatrix} = \begin{pmatrix} \chi_L \\ \epsilon \psi_L^* \end{pmatrix}$$

or $\begin{pmatrix} \psi_L^c \\ (\epsilon \chi_L^*)^c \end{pmatrix} = \begin{pmatrix} \chi_L \\ \epsilon \psi_L^* \end{pmatrix}$ or $\psi_L^c = \chi_L$.

$$P_L \psi^c = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = (P_R \psi)^c .$$

Note if $\psi^c = \psi$. Then $\psi_D = \psi_M$.

This is the Majorana condition.

Note $C^T = -C = C^\dagger = C^{-1}$, $C^2 = -I$.

$C^\dagger C = I$.

$$C \gamma_\mu^T C^{-1} = -\gamma_\mu.$$

and. I