HW4 - Phys 7810-001

Problem 1 [80 pts]

Generalized BRST quantization of gauge theories. In class we obatained the path integral for a gauge theory based on the deWitt-Faddeev-Popov method. This showed us that the ratio $Z[\Phi]/Z[1]$ with Φ a product of gauge invariant operators, can be defined by fixing the gauge so that the volume of the gauge group cancels between the numerator and denominator leaving us with well-defined path integrals. The resulting expression is (after factoring out the gauge group volume and an irrelevant constant),

$$Z = \int [d\phi][dc][dc^+]\Omega[\phi]e^{ic^{+\alpha}.M_\alpha{}^\beta.c_\beta - \frac{i}{2\xi}f_\alpha.\kappa^{\alpha\beta}.f_\beta},$$

where I've used $\Omega[\phi] = e^{iS[\phi]}\Phi[\phi]$, with ϕ being the full set of physical fields (those that define external states - may be gauge fields, matter fields both bosons and fermions but not ghosts). The only property of Ω that we need is that it's gauge invariant i.e, $\Omega[\phi^g] = \Omega[\phi]$ as is the case with the measure (defined as we did in class). f_α is the gauge fixing function and $M_\alpha{}^\beta(x,y)$ is the matrix we defined in class. We've also used a condensed notation so that repeated indices are summed over the discrete values of the gauge index as well as integrated over space-time. Thus for instance

$$c^{+\alpha}.M_\alpha{}^\beta.c_\beta \equiv \int_x \int_y c^{+\alpha}(x)M_\alpha{}^\beta(x,y)c_\beta(y),$$

etc. $\kappa_{\alpha\beta}$ is an invariant metric on the gauge group which for compact groups may be taken to be $\delta^{\alpha\beta}\delta^4(x-y)$ but again all we need here is that it is gauge

invariant. Observe that up to an irrelevant (field independent) constant we can write

$$Z = \int [d\phi][dc][dc^+][dh]\Omega[\phi]e^{ic^{+\alpha}.M_\alpha^\beta.c_\beta - ih_\alpha.f^\alpha[\phi] + i\frac{\xi}{2}h_\alpha.\kappa^{\alpha\beta}.h_\beta} \tag{1}$$

Ghost number: Assign $+1$ to c_α, -1 to c_α^+ and zero to all other fields. Clearly the gauge fixed action

$$S[\phi] + ic^{+\alpha}.M_\alpha^\beta.c_\beta - ih_\alpha.f^\alpha[\phi] + i\frac{\xi}{2}h_\alpha.\kappa^{\alpha\beta}.h_\beta \tag{2}$$

conserves ghost number

0.1 [30 pts] Killing vectors

a) [10 pts] Define the Killing vector field $\overrightarrow{K}_\alpha[\phi] = K_\alpha^i[\phi]\frac{\delta}{\delta\phi^i}$. Show that the commutator

$$[\overrightarrow{K}_\alpha[\phi], \overrightarrow{K}_\beta[\phi]] = (K_\alpha^i[\phi]K_{\beta,i}^j[\phi] - K_\beta^i[\phi]K_{\alpha,i}^j[\phi])\frac{\delta}{\delta\phi^j}.$$

Here $K_{\alpha,i}^j = \frac{\delta}{\delta\phi^i}K_\alpha^j$. Note that this diffential operator implements an infinitesimal transformation on the fields:

$$\delta_\theta\phi^i = \delta\theta^\alpha K_\alpha^i.$$

b) [10 pts] If this is an invariance of the action then we must have:

$$\overrightarrow{K}_\alpha S[\phi] = 0.$$

Show that then

$$(K_\alpha^i[\phi]K_{\beta,i}^j[\phi] - K_\beta^i[\phi]K_{\alpha,i}^j[\phi])\frac{\delta}{\delta\phi^j}S[\phi] = 0$$

Show that if this is true off-shell (i.e. is even when $\frac{\delta}{\delta\phi^j}S[\phi] \neq 0$) and the set $\overrightarrow{K}_\alpha$ are a maximal set of symmetries which leave the action invariant (and are independent of the action) then the algebra is closed:

$$[\overrightarrow{K}_\alpha[\phi], \overrightarrow{K}_\beta[\phi]] = c_{\alpha\beta}{}^\gamma[\phi]\overrightarrow{K}_\gamma[\phi]. \tag{3}$$

i.e. there is a non-linear constraint on the K's,

$$(K_\alpha^i[\phi] K_{\beta,i}^j[\phi] - K_\beta^i[\phi] K_{\alpha,i}^j[\phi]) = c_{\alpha\beta}{}^\gamma[\phi] K_\gamma^j[\phi]. \tag{4}$$

c) [5+5 pts] Show that there is another constraint

$$c_{[\alpha\beta}{}^\delta c_{(\delta)\gamma]}{}^\sigma - \vec{K}_{[\alpha} c_{\beta\gamma]}{}^\sigma = 0, \tag{5}$$

where the instruction [...] on the indices means that the expression is summed over cyclic permutaions of the three indices.   . If the symmetry is linearly realized (i.e. $\delta_\theta \phi^i = i\delta\theta^\alpha t^i_{\alpha j}\phi^j$ with $[\mathbf{t}_\alpha, \mathbf{t}_\beta] = if_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma$, find the K's and show that $c_{\alpha\beta}{}^\gamma[\phi] = if_{\alpha\beta}{}^\gamma$. i.e. the c's are field independent. This remains true for Yang-Mills gauge transformations on gauge fields (which have an inhomogeneous piece) as   well as for gravity (diffeomorphisms), but not for example in supergravity

0.2 [50 pts]BRST charge.

a)[10 pts] Define the Slavnov operator,

$$\mathcal{S} = c^\alpha K_\alpha^i \frac{\delta}{\delta\phi^i} - \frac{1}{2} c^\beta c^\gamma c^\alpha_{\beta\gamma}[\phi] \frac{\delta}{\delta c^\alpha} - h^\alpha \frac{\delta}{\delta c^{+\alpha}}. \tag{6}$$

Note that this operator has ghost number +1. A BRST transformation is defined to be the action of $\theta\mathcal{S}$ where θ is a Grassman number (so $\theta^2 = 0$) on the set of fields $\{\phi^i, c_\alpha, c^{+\alpha}, h^\alpha\}$. Show that the gauge fixed action can be written as

$$S_{\mathrm{gf}} = S[\phi] + \mathcal{S}\Psi[\phi, c, c^+, h], \tag{7}$$

where the so-called gauge fixing fermion (with ghost number -1) is given by

$$\Psi = c^{+\alpha} f_\alpha[\phi] + \frac{1}{2}\xi c_\alpha^+ \kappa^{\alpha\beta} h_\alpha. \tag{8}$$

b) [15 pts] It turns out that any gauge theory once it is gauge fixed can be written in the form of eqn. (7) with Ψ an arbitrary (i.e. not necessarily of the form (8)) fermionic functional of ghost no. -1. You don't need to prove this here (for a proof see for example Weinberg Quantum Theory of Fields vol 2

page 40). Using the relations for the Killing vector derived in subsection 0.1 show that the Slavnov operator is nilpotent. i.e.

$$\mathcal{S}^2 = 0$$

c)[05 pts] Use the above to conclude that $\theta\mathcal{S}\mathrm{S}_{\mathrm{g.f.}} = 0$. This shows that the gauge fixed action (although it's not gauge invariant any more) is nevertheless BRST invariant. This is a global (i.e. space time independent since θ is a constant Grassmann number) symmetry of the action. Note that any gauge invariant functional of the physical fields ϕ is automatically BRST invariant i.e.

$$\theta\mathcal{S}\Omega[\phi] = 0.$$

d) [10 pts] Show that the measure in the functional integral (1) is BRST invariant. i.e. if we defined the BRST transformed fields as $\Phi^\theta = \Phi + \theta\mathcal{S}\Phi$ where $\Phi = \{\phi, c, c^+, h)$, show that the Jacobian has determinant unity. i.e.

$$\det \frac{\delta\Phi^\theta}{\delta\Phi} = 1.$$

e) [05 pts] Show that the operator $i\mathcal{S}$ is Hermitian (recall that c, c^+ are real Grassmanian fields) when defined with respect to the functional inner product

$$(\Xi', \Xi) = \int [d\Phi]\Xi'^*(\Phi)\Xi(\Phi).$$

f) [05 pts] The above holds for any gauge theory from Y-M to gravity and supergravity. In other words if one has a set of symmetries $\vec{K}_\alpha$ under which the action is invariant and which form a closed algebra, the gauge fixed action is of the form (7). Define the BRST charge operator $\hat{Q}$ acting on a Dirac ket or bra state as $< \Phi|\hat{Q}|\Xi >= i\mathcal{S}\Xi(\Phi)$ where $|\Phi >$ is an eigenstate of the set of field operators $\hat{\Phi}$, in other words $\Xi(\Phi) =< \Phi|\Xi >$ is the representation in the field space basis of the ket vector $|\Xi >$. Clearly the nilpotency of $\mathcal{S}$ implies that $\hat{Q}^2 = 0$. Physical states may now be defined as states in the BRST cohomology, i.e. if $|\Xi >$ is a physical state $Q|\Xi >= 0$ i.e. it is BRST closed. Note that this is an equivalence class i.e. you can add any BRST exact state (i.e. a state

$\hat{Q}|any >$ to a physical state and it still satisfies the above condition). Show that the S-matrix for the scattering of physical states, is independent of the gauge fixing, i.e.

$$\delta_\Psi < \alpha; \text{out}|\beta; in >= 0.$$

g) To see the meaning of the physical state condition one could consider its effect on states in a gauge theory. See for example the discussion on pp 33-34 in Weinberg Quantum Theory of Fields vol 2 or on pp 453-455 in Sredinicki QFT. One can also show using BRST methods that the quantum effective action in background gauge invariant gauges (see Lec5) not only satisfies background gauge invariance but also is gauge fixing independent at its extrema i.e solutions of $\delta\Gamma/\delta\phi = 0$ are independent of ξ.

Problem 2 [20 pts]

In class we derived the form

$$U_n^{(1)} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} e^{-\mathbf{V}''(\phi_c)s} \frac{1}{(4\pi s)^{n/2}}$$

for the one-loop contribution to the effective potential in n dimensions. a) [10 pts] By rewriting the integral in terms of a Gamma function show that in the limit $\epsilon \equiv 4 - n \to 0$ this expression takes the form

$$\lim_{\epsilon \to 0} U_{4-\epsilon}^{(1)} = -\frac{1}{2} \text{tr} \frac{(\mathbf{V}'')^2}{(4\pi)^2} \frac{1}{2} \left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) - \ln \frac{\mathbf{V}''}{\mu^2} + \frac{3}{2} + O(\epsilon) \right).$$

where μ is an arbitrary scale factor (called the renormalization scale). In the so-called $\overline{\text{MS}}$ subtraction scheme, one adds the counter term

$$\delta S = \frac{1}{2} \int d^4 x \sqrt{g} \, \text{tr} \frac{(\mathbf{V}'')^2}{(4\pi)^2} \frac{1}{2} \left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) \right),$$

to the original action (with couplings defined at the mass scale μ) so that we have for the one-loop corrected quantum effective action (1PI action to one-

loop)

$$\Gamma_{\text{1PI}} \simeq S_{\text{classical}(\mu)} + \lim_{\epsilon \to 0} \left(\delta S + \Gamma^{(1)}_{4-\epsilon} \right)$$

$$= S_{\text{cl}(\mu)} + \frac{1}{2} \int d^4x \sqrt{g} \, \text{tr} \frac{(\mathbf{V''})^2}{(4\pi)^2} \frac{1}{2} \left(\ln \frac{\mathbf{V''}}{\mu^2} - \frac{3}{2} \right) \tag{9}$$

b) [10 pts] Consider a scalar field theory with two fields, one a light field with mass m and another with a heavy field with mass M. The potential for the theory is

$$V(\phi, \Phi; \mu) = \Lambda_{\text{cc}}(\mu) + \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} M^2(\mu) \Phi^2 + \frac{\lambda(\mu)}{4!} \phi^4 + \frac{\eta(\mu)}{4} \phi^2 \Phi^2 + \frac{\sigma(\mu)}{4!} \Phi^4 + \dots$$

The β-functions are obtained by demanding that U is independent of the arbitrary scale μ. Derive the flow equations for the cosmological constant and the purely ϕ dependent couplings:

$$\mu \frac{d\Lambda_{\text{cc}}}{d\mu} = \frac{1}{32\pi^2} \left(m^4 + M^4 \right) \tag{10}$$

$$\mu \frac{dm^2}{d\mu} = \frac{1}{16\pi^2} \left(\lambda m^2 + \eta M^2 \right) \tag{11}$$

$$\mu \frac{d\lambda(\mu)}{d\mu} = \frac{3}{16\pi^2} \left((\lambda^2 + \eta^2) \right) \tag{12}$$

From an effective field theory point of view if $M \gg m$ we would have expected that effect of the heavy field to drop out of the light field beta functions in the regime $\mu \ll M$. However we see in the above that this does not happen. Can you explain this?

The following three problems will not be graded. I'm including them since you might find it useful to try them out. The first two are essentially from Sredicicki and the last one is from Peskin and Schroeder. I will discuss a generalized version of the last one when we discuss anomalies.

Problem 3

[30 pts] In class we discussed the quantization of Non-Abelian gauge theories and in particular derived the Feynman rules for a class of Lorentz invariant

gauges. Here you should proceed in the same manner (i.e. using the Faddeev-Popov (FP) method) to quantize in a class of non-covariant gauges - defined by the gauge fixing condition $n^\mu A_\mu = 0$ where n^μ is a fixed but arbitrary 4-vector (for example in the so-called axial gauge a.k.a. Arnowitt-Finkler gauge $n^\mu = \eta^{\mu 3}$). Derive the Feynman rules in this gauge (you only need to work out those which are different from what we had for the covariant gauges in class). Show that the FP ghosts are non propagating and that only physical degrees of freedom of the gauge field propagate. (Although the gauge fixing is non-covariant the S-matrix is still Lorentz invariant. In this sense it is similar to the Coulomb gauge in QED.

Problem 4

(40 pts)

Wilson line:

Consider the Wilson line integral i.e. the path ordered line integral

$$\mathbf{W}_P(x_1, x_0) = P \exp\left\{ ig_S \int dt \frac{dx^\mu}{dt} \mathbf{A}_\mu(x(t)) \right\}. \tag{13}$$

Here $\mathbf{A}_\mu = A_\mu^i T_i$ (with T_i the generators of some gauge group in some representation R), and path ordering implies that in the expansion of the exponential the products of $A's$ is ordered in the t - i.e. it is "time ordered" in t which parametrizes some curve in space time. Find the solution to the "parallel propagation" equation

$$\mathbf{D}_t \psi(x(t)) = \frac{dx^\mu}{dt} \mathbf{D}_\mu \psi(x(t))$$

where $\mathbf{D}_\mu$ is the covariant derivative in the representation R under which ψ transforms. Hence show that the Wilson line transforms under gauge transformation $\mathbf{A} \to \mathbf{A^g}$ as

$$\mathbf{W}_P(x_1, x_0) \to g(x_1)\mathbf{W}_P g^{-1}(x_0)$$

where $g(x)$ is the gauge group element in the representation R at the point x.

Deduce that the Wilson loop integral

$$\mathbf{W}_C = \mathrm{tr} P \exp\left\{ ig_S \oint_C dt \frac{dx^\mu}{dt} \mathbf{A}_\mu(x(t)) \right\}$$

where the integral is taken over a closed path is gauge invariant.

For $U(1)$ gauge theory show that the vacuum expectation value of the Wilson loop is

$$< 0|\mathbf{W}_C|0 >= \exp\left[ig_s^2 \oint_C dx^\mu \oint_C dy^\nu \Delta_{\mu\nu}(x - y) \right]$$

where $\Delta_{\mu\nu}(x - y) = \eta_{\mu\nu}/\left[4\pi^2(x - y)^2\right]$ is the propagator for the gauge field A_μ in Feynman gauge. The integral is divergent because of the singularity as $x \to y$. By cutting the integral off at a length scale a show that the integral takes the form

$$< 0|\mathbf{W}_C|0 >= \exp\left[-\frac{cg_s^2}{a} L \right]$$

where L is the length of C.

By taking C to be a circle of radius $R = L/2\pi$ evaluate the constant c. Remember that the integral should be cut off when $|x - y| < a$.

Problem 5

[30 pts]

The (Abelian) anomaly equation is

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}.$$

a) By integrating this over space time show that

$$\Delta N_R - \Delta N_L = -\frac{g^2}{2\pi^2} \int d^4x \mathbf{E}.\mathbf{B}. \tag{14}$$

Here $\Delta N_{R,L}$ is the change in the number of (right/left handed) fermions in a time interval. b) Show that the Hamiltonian for massless fermions may be written in the form

$$H = \int d^3x \left[\psi_R^\dagger(-i\sigma.\mathbf{D})\psi_R - \psi_L^\dagger(-i\sigma.\mathbf{D})\psi_L \right],$$

with $D^i = \nabla^i - ieA^i$. c) Consider the eigenvalue problme for ψ_R i.e. the eqn. $-i\sigma.\mathbf{D})\psi_R = E\psi_R$. Choose the E-M potential to be $A^\mu = (0, 0, Bx^1, A)$ with A, B being constants. Then the eigen vectors can be written as

$$\psi_R = \begin{pmatrix} \phi_1(x^1) \\ \phi_2(x^1) \end{pmatrix} e^{i\left(k_2 x^2 + k_3 x^3\right)}.$$

Show that the functions ϕ_i obey the harmonic oscillator equation. d) I the system of fermions is in a box with side of length L and periodic boundary conditions, the momenta k_2, k_3 will be quantized $k_i = 2\pi n_i/L$. From the eqn derived in part c) show that the condition that the center of the oscillation is inside the box leads to the condition $k_2 < gBL$ and that each energy level has a degeneracy $eL^2B/2\pi$. e) Consider the effect of changing the background by $\Delta A = 2\pi/gL$. Show that the vacuum loses right-handed fermions and by repeating the analysis for left-handed fermions that the vacuum gains the same number of left handed fermions and that the net change is in accord with (14).