HW2 - Phys 7810-001

due 02/18/21

Problem 1

1) The kinetic term $\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i$ for N real scalar fields is inariant under a symmmetry $\phi_i \to O_{ij}\phi_j$, $i,j = 1,\ldots,N$, where $O^T O = \mathbf{I}$ i.e. the symmetry group is $O(N)$. When N is even this group contains the subgroup $U(N/2)\times U(N/2)$. For $N = 4$ define the comple basis of fields $\varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\psi = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4)$, and construct the 2×2 complex matrix

$$\mathbf{\Phi} = \begin{pmatrix} \varphi & \bar{\psi} \\ \psi & -\bar{\varphi} \end{pmatrix}$$

Here $\bar{\varphi} = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$ and similarly for $\bar{\psi}$. In terms of this observe that the reality condition on the fields ϕ_i translates to the condition $(i)\bar{\mathbf{\Phi}} \equiv \varepsilon\mathbf{\Phi}^*\varepsilon = \mathbf{\Phi}$ ($\varepsilon = i\sigma_2$) (where $*$ means complex conjugation) and the $O(N)$ invariant form is $(ii)\boldsymbol{\phi}^T\boldsymbol{\phi} = -2\det\mathbf{\Phi}$. Show that these conditions $(i),(ii)$ are preserved by

$$\begin{aligned} a)\,\mathbf{\Phi} &\to U\mathbf{\Phi}, \\ b)\,\mathbf{\Phi} &\to \mathbf{\Phi}V. \end{aligned}$$

for arbitrary unitary matrices $U.V$. This shows by explicit construction that $O(4)$ is actually equivalent to $SU(2)\times SU(2)$. Sometimes these are referred to as left/right $SU(2)$'s since the first/second acts by multiplication on the left/right. 2) Consider a theory with one Majorana fermion ψ and two real scalar fields φ,χ subject to the transformations

$$\delta\psi \to i\omega\gamma_5\psi,\ \delta\varphi \to 2\omega\chi,\ \delta\chi \to -2\omega\varphi$$

for ω an infinitesimal constant parameter. Write down the most general renormalizable (i.e. with operator dimension less than equal to four) for this set of fields. Identify the vacuum (i.e. potential minimum) field configuration and mass spectrum both in the broken and unbroken phases (i.e. for both choices of sign for the coefficient of the the quadratic term of the potential). Couple a vector field to this system by constructing the appropriate covariant derivatives and find the action for this system that is invariant under local gauge transformations (i.e. ω such that $\partial_\mu \omega \neq 0$). Identify the spectrum in both the broken and the unbroken phases.

Problem 2

a) If $\mathbf{H}$ is an $SU(2)$ doublet show that so is $\epsilon \mathbf{H}^*$. b) Show the equivalence of the two forms of the standard model kinetic terms. i.e. show that $\overline{(\psi^c)_L}\gamma^\mu D_\mu (\psi^c)_L = \overline{\psi_R}\gamma^\mu D_\mu \psi_R$. c) Derive from the gauge invariant kinetic terms of the Higgs Lagrangian after spontaneous symmetry breakdown, the mass terms for the W and the Z bosons. d) Define the four by four matrix of Higgs fields

$$\mathbf{\Phi} = \frac{1}{\sqrt{2}}(\epsilon \mathbf{H}^*, \mathbf{H}) = \frac{1}{\sqrt{2}} \begin{pmatrix} H^{0*} H^+ \\ -H^- H^0 \end{pmatrix}.$$

Show that we can rewrite the Higgs Lagrangian as

$$\begin{aligned} \mathcal{L}_{Higgs} &= \mathrm{tr}(D_\mu \mathbf{\Phi})^\dagger (D_\mu \mathbf{\Phi}) - V(\mathbf{\Phi}) \\ V(\mathbf{\Phi}) &= \lambda(\mathrm{tr}\mathbf{\Phi}^\dagger \mathbf{\Phi} - \frac{\mu^2}{2\lambda})^2 \\ D_\mu \mathbf{\Phi} &= \partial_\mu \mathbf{\Phi} + i\frac{g}{2}\sigma.W_\mu \mathbf{\Phi} - i\frac{g'}{2} B_\mu \mathbf{\Phi} \sigma_3 \end{aligned}$$

c) The action of $SU(2)_L \times U(1)_Y$ on $\mathbf{\Phi}$ is then (with $U_L \varepsilon SU(2)_L$, $SU(2)_L : \mathbf{\Phi} \to U_L \mathbf{\Phi}$, $U(1)_Y \to \mathbf{\Phi} e^{-i\sigma_3 \theta/2}$. Check directly the invariance of $\mathcal{L}_{Higgs}$ in the above form under this group action. e) Show that in the limit $g' \to 0$ this Lagrangian has a *global* symmetry $SU(2)_R$: $\mathbf{\Phi} \to \mathbf{\Phi} U_R^\dagger$, $U_R \varepsilon SU(2)_R$. In other words in this limit the Higgs sector has the approximate accidental *global* symmetry $SU(2)_L \times SU(2)_R : \mathbf{\Phi} \to U_L \mathbf{\Phi} U_R^\dagger$. f) Show that after spontaneous symmetry breakdown this global symmetry broken to $SU(2)_{L+R}$: $\mathbf{\Phi} \to U_L \mathbf{\Phi} U_L^\dagger$. g) Show that W_μ^i transforms as a triplet under global $SU(2)_L$ and a singlet under $SU(2)_R$. How does W transform under $SU(2)_{L+R}$? What does this tell us about M_W, and M_Z? Note: This global symmetry is called the "custodial symmetry". g) Show that the t, b Yukawa couplings have a custodial symmetry in the limit $m_t = m_b$.

Problem 3

a) Show that in the standard model there is no non-trivial CKM like matrix for the lepton sector. Hence deduce that there are two additional $U(1)$ symmetries for the leptons - i.e. a lepton number associated with each generation. b) Let us now extend the standard model to include a (right-handed) Dirac field N_R and add the Yukawa interaction $\Delta\mathcal{L}_{Yukawa} = -f_\nu^{AB} \overline{L_L^A} \epsilon \mathbf{H}^* N_R^i$. How should N_R transform under $SU(2)_L \times U(1)_Y$? What is its lepton number? b) Given that neutrinos actually do have mass one may want to add this field and this term to the Lagrangian. But since neutrino masses are of $O(10^{-3} eV)$ how big can the above Yukawa coupling be? Do you think it is OK to have such a value in your Lagrangian? c) Show that gauge invariance allows a Majorana mass term $-\frac{1}{2} M_R^{AB} (N_R^A)^T C N_R^B + h.c.$. However note that it violates lepton number. d) As an alternative to adding a new field, consider looking at higher dimension operators to generate neutrino masses. So we introduce some high scale M (this could be a scale at which new physics appears). There is then a dimension 5 operator that will contribute to giving a neutrino mass term

$$\mathcal{L}_5 = \frac{c^{AB}}{M} (L_L^A)^T \epsilon \mathbf{H} C \mathbf{H}^T \epsilon L_L^B + h.c.$$

i) Show that $\mathcal{L}_5$ is gauge invariant, and that c^{AB} is a symmetric matrix and that this term violates lepton number. ii) Find the effective neutrino mass term coming from the Higgs effect on $\mathcal{L}_5$. Assuming that the dimensionless coupling $c \sim O(1)$ how big must M be in order to generate neutrino masses at the observed values. Can you associate this value with some other physics that you may have heard of? iii) Show that the analogue of the CKM matrix in the lepton sector has six physically relevant parameters (remember that c^{AB} is a complex symmetric matrix).