HW1 - Phys 7810-001

due 02/04/21

Problem 1

[40=2+3+5+5+5+20 pts] The composition law of a Lie group is given by $g(\theta)g(\phi) = g(\xi(\theta,\phi)$ where $\theta = \{\theta^i\}, \phi = \{\phi^i\}$ are n-dimensional parameter vectors and $\xi = \{\xi^i\}$. Show that a) $\xi(\theta, 0) = \xi(0,\theta) = \theta$. b) $\xi(\theta, \xi(\phi,\psi)) = \xi(\xi(\theta,\phi),\psi)$. c) Write $g(\phi)g(\theta)g^{-1}(\phi)g^{-1}(\theta) = g(\xi(\theta,\phi))$. Show that near the identity element $\xi^i = c^i_{jk}\theta^j\phi^k$. d) By evaluating the commutator $g(\phi)g(\theta)g^{-1}(\phi)g^{-1}($ show that the generators satisfy the commutation relations $[X_j, X_k] = ic^l_{jk}X_l$. e) Deduce that $c^l_{jk} = -c^l_{kj}$, and that $c^m_{jk}c^n_{lm} + c^m_{kl}c^n_{jm} + c^m_{lj}c^n_{km} = 0$. f) For a matrix group we define the Cartan-Killing metric on the Lie algebra by $g_{ij} = \mathrm{tr}(X^iX^j)$. i) Show that $c_{ijk} \equiv g_{il}c^l_{jk}$ is totally anti-symmetric in i, j, k. ii) If $U = e^{iH}$ is a unitary matrix with $\det U = 1$, show that $\mathrm{tr}H = 0$. g) Let $\psi, \phi, \ldots$ be vectors in the space of n-dimensional column vectors ($\psi = \{\psi_a\}$ etc.) which carry an n-dimensional unitary representation of some Lie group. Suppose the group elements $\{g\}$ of a group of unitary transformations on this vector space are given in some unitary representation by the matrices $D(g)$. i) Show that the totally anti-symmetric tensor $\epsilon_{i_1 \ldots i_N} = \pm 1$ (with upper(lower) sign for even(odd) permutations of $1, 2, \ldots, N$) is an invariant of the group $SU(N)$. ii) Suppose the vector $\psi = \{\psi_i,\, i = 1, \ldots, N\}$ is in the fundamental (defining) representation of $SU(N)$. Then the tensor ψ_{ij} transforms as the direct product of $\psi \times \psi \equiv \{\psi_i\psi_j\}$. Define the permutation operator P so that $P\psi_{ij} = \psi_{ji}$. Show that P commutes with the group transformation law. Show that ψ_{ij} is a reducible tensor representation by demonstrating that the symmetric and anti-symmetric combinations $\psi^{\pm}_{ij} \equiv \frac{1}{2}(\psi_{ij} \pm \psi_{ji})$ do not mix under the group transformations.

Problem 2

[30=5+10+10+5 pts] For the Lie algebra of $SU(N)$ show that a) $C_{ijk}T_jT_k = \frac{i}{2}C_2(G)T_i$ b) Prove the completeness relation (for the generators in the fundamental representation)

$$(T_i)_{\gamma\beta}(T_i)_{\alpha\lambda} = \frac{1}{2}(\delta_{\beta\alpha}\delta_{\gamma\lambda} - \frac{1}{N}\delta_{\beta\gamma}\delta_{\alpha\lambda})$$

c) Show that in any IR r the generators satisfy the relation

$$T_iT_jT_i = \left[C_2(r) - \frac{1}{2}C_2(G)\right]T_j.$$

d) Using this (or otherwise?) show that if the generators in the fundamental are normalized with $C(N) = \frac{1}{2}$ then $C(G) = C_2(G) = N$.

Problem 3

[30=5+5+5+5+5+5 pts] a) Starting from the Lorentz algebra and defining $J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}$, $K_i \equiv -M_{0i}$ and $\mathcal{J}_i^{\pm} \equiv \frac{1}{2}(J_i \pm iK_i)$, show that

$$\begin{aligned} [\mathcal{J}_i^{\pm}, \mathcal{J}_j^{\pm}] &= i\epsilon_{ijk}\mathcal{J}_k^{\pm}, \\ [\mathcal{J}_i^{\pm}, \mathcal{J}_j^{\mp}] &= 0. \end{aligned}$$

b)Show that if χ_L is in the $(\frac{1}{2}, 0)$ representation, $\epsilon\chi_L^*$ (here $\epsilon = i\sigma_2$) is in the $(0, \frac{1}{2})$ representation (i.e. it transforms like χ_R). c) Show that $\mathcal{L} = -\frac{1}{2}m\bar{\psi}_M\psi_M = m(\psi_L^T\epsilon\psi_L - \psi_L^\dagger\epsilon\psi_L^*)$ where ψ_M is the four component Majorana spinor and ψ_L is a left chiral Weyl spinor. d) Show that $(\psi_D^c)^c = \psi_D$. e) Show that $\mathcal{L} = -m[(\psi_D^c)_L^T C(\psi_D)_L + h.c.)$ is a Dirac mass term and that $\mathcal{L} = -m[\psi_{DL}^T C\psi_{DL} + h.c.)$ is a Majorana mass term. Here ψ_D is a four component Dirac spinor and C is the charge conjugation matrix. .e) Show that if $\psi_D^c = \psi_D^c$ then $\psi_D = \psi_M$. f) Show that $\mathcal{L} = -\frac{1}{2}m\overline{(\psi_D^c)_R}\psi_{DL} + h.c.)$ is a Majorana mass.